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SOME RESULTS ON k-HYPERGEOMETRIC FUNCTION

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Abstract. In this paper, we establish integral representation and differentiation formulas for k-Gauss hypergeometric function ${}_2F_{1,k}(a, b; c; z)$ and develops a relationship with k-Confluent hypergeometric function ${}_1F_{1,k}(a, b; c; z)$, which are based properties defined by Rao and Shukla. Our study is to identify the integral as well differential representation of ${}_2F_{1,k}(a, b; c; z)$ and also find the inverse Laplace transform on it.

Keywords: k-Gauss hypergeometric function; k-Beta function; k-Gamma Function; Laplace transform.

AMS Subject Classification: 33E12, 44A10, 33B15, 33C05.

1. Introduction, Definitions and Preliminaries

The hypergeometric function plays an important role in mathematical analysis and its application. Apart from hypergeometric function, Many generalization and expansions of the various k-parameter of special function and k-fractional derivatives was considered by many researchers (see; [14, 16, 17, 19, 20]). Several identities for gamma functions, beta functions and Pochhammer symbols have been systematically investigated using the k-parameter. The integral representation of k-gamma function, k-beta function, respectively, is based on literature (see [4]) as:

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}} - 1}{(z)_{n,k}}, \quad k > 0, z \in \mathbb{C}\backslash k\mathbb{Z}^-, \tag{1}$$

where $(z)_{n,k}$ for $(z \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}^+)$, is denoted as under

$$(z)_{n,k} = z(z + k)(z + 2k)\cdots(z + (n - 1)k).$$

For $\Re(z) > 0$, then $\Gamma_k(z)$ define as integral

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}} dt = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)\cdot \tag{2}$$

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It is obvious that $\Gamma_k(z) \to \Gamma(z)$ for $k \to 1$, where $\Gamma(z)$ is the known Gamma function. k-Gamma function satisfies the following properties:

$$\left.\begin{aligned} \Gamma_k(z+k) &= \Gamma_k(z) \\ (z)_{n,k} &= \frac{\Gamma_k(z+nk)}{\Gamma_k(z)} \\ \Gamma_k(k) &= 1 \\ \Gamma_k(z)\Gamma_k(k-z) &= \frac{\pi}{\sin(\frac{\pi x}{k})} \end{aligned}\right\}, \tag{3}$$

$$\Gamma_k\left((2n+1)\frac{k}{2}\right) = k^{\frac{(2n-1)}{2}} \frac{(2n!\sqrt{\pi})}{2^n n!}, \quad k > 0, n \in \mathbb{N}, \tag{4}$$

$$\Gamma_k(2n) = \sqrt{\frac{k}{\pi}}\, 2^{\frac{(2n)}{k}-1}\Gamma_k(n)\Gamma_k\left(n+\frac{k}{2}\right)\cdot \tag{5}$$

In the same paper [4], the k-Beta function $\mathfrak{B}_k(x,y)$ is given as:

$$\mathfrak{B}_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad \Re(x) > 0, \Re(y) > 0. \tag{6}$$

Kokologiannaki [1], Mansour [3], Mubeen et al. [9], Krasniqi [11], Gahlot and Nisar [12] and Rahman et al. [21, 22] develop number of properties for the k-gamma and k-beta functions. The k-Gauss hypergeometric function [6, 7] defined as:

$${}_2F_{1,k}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^k}{k!} = {}_2F_1\left((a,k),(b,k);(c,k);z\right)\cdot \tag{7}$$

It converts into the known Gauss hypergeometric function as $k \to 1$. A detail account of hypergeometric function, the reader may be referred to the earlier extensive works [5, 8, 10, 13, 15, 18, 23, 24, 25, 26]

2. Main Result

The aim of this research article to present the different type of integral and differential properties for the k-Gauss hypergeometric function.

Theorem 2.1. *If* $a,b,c,\phi \in \mathbb{C}$; $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(\phi) > 0, |\,(\eta z)\,| < 1$, $k > 0$, *then*

$$\frac{\Gamma_k(c+\phi)}{\Gamma_k(c)\Gamma_k(\phi)} \int_0^z t^{\frac{c}{k}-1}(z-t)^{\frac{\phi}{k}-1}\, {}_2F_{1,k}(a,b;c;\eta t)dt = kz^{\frac{c+\phi}{k}-1}\, {}_2F_{1,k}(a,b;c+\phi;\eta z). \tag{8}$$

Proof. Taking left hand side of equation (8) denoted by I_1 and using equation (7) and then the changing order of summation and integration, we have

$$I_1 = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+\phi)}{\Gamma_k(c+\phi)\Gamma_k(b)\Gamma_k(\phi)} \frac{\eta^n}{n!} \int_0^z t^{\frac{c}{k}+n-1}(z-t)^{\frac{\phi}{k}-1}dt. \tag{9}$$

By taking $t = zu$ and $dt = zdu$ in equation (9), we get

$$I_1 = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+\phi)}{\Gamma_k(c+\phi)\Gamma_k(b)\Gamma_k(\phi)} \frac{(\eta z)^n}{n!} z^{\frac{c+\phi}{k}-1} \int_0^1 u^{\frac{c}{k}+n-1}(1-u)^{\frac{\phi}{k}-1}du$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+\phi)}{\Gamma_k(c+\phi)\Gamma_k(b)\Gamma_k(\phi)} \frac{(\eta z)^n}{n!} z^{\frac{c+\phi}{k}-1} k\mathfrak{B}_k(c+nk,\phi)\cdot$$

Now, using equation (6), we obtain desired result. □

Theorem 2.2. *If* $a,b,c,\in\mathbb{C}$; $\Re(a)>0,\Re(b)>0,\Re(c)>0,k>0,\mid z\mid<1$, *then*

$$c\int_0^z t^{\frac{c}{k}-1}\,{}_2F_{1,k}(a,b;c;t)\left(1+\frac{2}{k}(z-t)\right)dt-\int_0^z t^{\frac{c}{k}}\,{}_2F_{1,k}(a,b;c+k;t)dt$$
$$= kz^{\frac{c}{k}}\,{}_2F_{1,k}(a,b;c+k;z) \tag{10}$$

Proof. To prove the above theorem, we consider the integral

$$\int_0^z {}_2F_1(a,b,c+k;t)\left(1+\frac{(z-t)}{\Gamma_k(2k)}\right)t^{\frac{c}{k}}dt$$
$$=\int_0^z \sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+k)}{\Gamma_k(c+nk+k)\Gamma_k(b)}\left(1+\frac{(z-t)}{\Gamma_k(2k)}\right)t^{\frac{c}{k}+n}\frac{1}{n!}dt. \tag{11}$$

Changing order of summation and integration in equation (11), we have

$$=\sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+k)}{\Gamma_k(c+nk+k)\Gamma_k(b)}\frac{1}{n!}\left(\int_0^z t^{\frac{c}{k}+n}dt+\frac{1}{\Gamma_k(2k)}\int_0^z t^{\frac{c}{k}+n}(z-t)dt\right),$$

Taking $t=zu$ and $dt=zdu$ in second integral part of above equation, we get

$$=\sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c+k)}{\Gamma_k(c+nk+k)\Gamma_k(b)}\frac{1}{n!}$$
$$\times\left(k\frac{z^{\frac{(c+nk+k)}{k}}}{c+nk+k}+\frac{1}{\Gamma_k(2k)}z^{\frac{(c+nk+2k)}{k}}k\mathfrak{B}_k(c+nk+k,2k)\right)\cdot$$

Using equation (6) in above and simplifying, we get

$$=\frac{k}{(c+k)}z^{\frac{c}{k}+1}\,{}_2F_1(a,b;c+2k;z)+\frac{k}{(c+k)(c+2k)}z^{\frac{c}{k}+2}\,{}_2F_1(a,b;c+3k;z), \tag{12}$$

If we take $\phi=2k,\eta=1$ in (8) then the first term of the equation (12) becomes

$$\frac{k}{(c+k)}z^{\frac{c}{k}+1}\,2F_{1,k}(a,b;c+2k;z)=\frac{c}{k}\int_0^z t^{\frac{c}{k}-1}(z-t)\,{}_2F_{1,k}(a,b;c;t)dt\cdot \tag{13}$$

Similarly if $\phi=3k,\eta=1$ in the equation (8), then the second term of equation (12) becomes

$$\frac{k}{(c+k)(c+2k)}z^{\frac{c}{k}+2}\,{}_2F_{1,k}(a,b;c+3k;z)=\frac{c}{k^2}\int_0^z t^{\frac{c}{k}-1}(z-t)^2\,{}_2F_{1,k}(a,b;c;t)dt\cdot \tag{14}$$

Now, combining eqns. (11) and (12) by using eqns.(13) and (14), we have

$$\int_0^z {}_2F_1(a,b;c+k;t)\left(1+\frac{(z-t)}{\Gamma_k(2k)}\right)t^{\frac{c}{k}}dt$$
$$=\frac{c}{k}\left(\int_0^z t^{\frac{c}{k}-1}(z-t)\,{}_2F_{1,k}(a,b;c;t)dt+\frac{1}{k}\int_0^z t^{\frac{c}{k}-1}(z-t)^2\,{}_2F_{1,k}(a,b;c;t)dt\right). \tag{15}$$

Finally, differentiating both sides of (15) with respect to z by Lebnitz rule, we have

$$z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)+\frac{1}{k}\int_0^z t^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;t)dt$$
$$=\frac{c}{k}\left[\int_0^z t^{\frac{c}{k}-1}\ {}_2F_{1,k}(a,b;c;t)dt+\frac{2}{k}\int_0^z t^{\frac{c}{k}-1}(z-t)\ {}_2F_{1,k}(a,b;c;t)dt\right],$$

which is also written as

$$z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b,c+k;z)+\frac{1}{k}\int_0^z t^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;t)dt$$
$$=\frac{c}{k}\left[\int_0^z t^{\frac{c}{k}-1}\left(1+\frac{2(z-t)}{k}\right)\ {}_2F_{1,k}(a,b;c;t)dt\right]$$

After simple simplification, we get desired result.

□

Theorem 2.3. *If* $a,b,c,\in\mathbb{C}$; $\Re(a)>0,\Re(b)>0,\Re(c)>0,k>0,\mid z\mid<1$, *then*

$$\frac{d^m}{dz^m}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]=\frac{1}{k^m}\frac{\Gamma_k(c)}{\Gamma_k(c-mk)}z^{\frac{c}{k}-m}\ {}_2F_{1,k}(a,b;c-(m-1)k;z). \tag{16}$$

Proof. The equation (16) can be proved by the principle of mathematical induction. First we prove the theorem for $m=1$, we have

$$\frac{d}{dz}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]$$
$$=\frac{d}{dz}\left[z^{\frac{c}{k}}\sum_{n=0}^{\infty}\frac{(a)_{n,k},\Gamma_k(b+nk)\Gamma_k(c+k)}{\Gamma_k(c+(n+1)k)\Gamma_k(b)}\frac{z^n}{n!}\right]$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k},\Gamma_k(b+nk)\Gamma_k(c+k)}{\Gamma_k(c+(n+1)k)\Gamma_k(b)}\frac{(c+nk)}{n!}\frac{1}{k}z^{\frac{c}{k}+n-1}. \tag{17}$$

Using k-Gamma property in eq.(17), we get

$$\frac{d}{dz}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]=\frac{c}{k}z^{\frac{c}{k}-1}\ {}_2F_{1,k}(a,b;c;z)$$

Hence the theorem is true for $m=1$. Again suppose that the theorem is true for $m=s$ (a fixed positive integer) that is

$$\frac{d^s}{dz^s}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]=\frac{1}{k^s}\frac{\Gamma_k(c)}{\Gamma_k(c-sk)}z^{\frac{c}{k}-s}\ {}_2F_{1,k}(a,b;c-(s-1)k;z) \tag{18}$$

Now differentiating of equation (18) again with respect to z, we get

$$\frac{d^{s+1}}{dz^{s+1}}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]=\frac{1}{k^s}\frac{\Gamma_k(c)}{\Gamma_k(c-sk)}$$
$$\times\sum_{n=0}^{\infty}\frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c-(s-1)k)}{\Gamma_k(c-(s-1)k+nk)\Gamma_k(b)}\frac{(c+(n-s)k)}{k}\frac{z^{\frac{c}{k}+n-s-1}}{n!} \tag{19}$$

On solving equation (19) by using equation (3), we get

$$\frac{d^{s+1}}{dz^{s+1}}\left[z^{\frac{c}{k}}\ {}_2F_{1,k}(a,b;c+k;z)\right]$$
$$=\frac{1}{k^{(s+1)}}\frac{\Gamma_k(c)}{\Gamma_k(c-(s+1)k)}\left(z^{\frac{c}{k}-(s+1)}\sum_{n=0}^{\infty}\frac{(a)_{n,k}\Gamma_k(b+nk)\Gamma_k(c-sk)}{\Gamma_k(c-sk+nk)\Gamma_k(b)}\frac{z^n}{n!}\right)$$

$$= \frac{1}{k^{s+1}} \frac{\Gamma_k(c)}{\Gamma_k(c-(s+1)k)} \left(z^{\frac{c}{k}-(s+1)} \, {}_2F_{1,k}(a,b;c-sk;z) \right) \cdot$$

As the eqn (16) is true for $m = s+1$, so by mathematical induction the theorem is true for every $m \in \mathbb{N}$. □

Theorem 2.4. *If* $a, b, c, \in \mathbb{C}$; $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, k > 0, |(z)| < 1$, *then*

$$\int_0^\infty exp\left(\frac{-1}{4k} \left(\frac{z^2}{t} \right)^k \, {}_2F_1\left((a,k),(b,k);(c,2k);z^{2k}\right) \right) z^{c-1} dz$$

$$= \sqrt{\frac{\pi}{k}} \frac{\Gamma_k(c)}{\Gamma_k(\frac{(c+k)}{2})} t^{\frac{c}{2}} \, {}_2F_{1,k}\left(a,b;\frac{(c+k)}{2};t^k\right). \quad (20)$$

Proof. Using the eqn (7), in the left hand side of equation (20), we have

$$\int_0^\infty exp\left(\frac{-1}{4k} \left(\frac{z^2}{t} \right)^k \right) \left(\frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_0^\infty \frac{(a)_{n,k}\Gamma_k(b+kn)}{\Gamma_k(c+2kn)} \frac{z^{2kn+c-1}}{n!} \right) dz, \quad (21)$$

interchanging order of summation and integration, putting $z^2 = 4^{\frac{1}{k}}tu$ in (21) and simplify, we get

$$= \left(\frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_0^\infty \frac{(a)_{n,k}\Gamma_k(b+kn)}{\Gamma_k(c+2kn)} \frac{1}{n!} \right)$$

$$\times \int_0^\infty exp\left(\frac{(-u)^k}{k} \right) \left(2^{\frac{1}{k}} \sqrt{tu} \right)^{2kn+c-1} 2^{\frac{1}{k}-1} t^{\frac{1}{2}} u^{\frac{-1}{2}} du,$$

after simplification, we get

$$= \left(\frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_0^\infty \frac{(a)_{n,k}\,\Gamma_k(b+kn)}{\Gamma_k(c+2kn)} \frac{1}{n!} 2^{\frac{c+2kn}{k}-1} t^{\frac{c+2kn}{2}} \right) \Gamma_k\left(\frac{c+2kn}{2} \right), \quad (22)$$

using k-Gamma property given in eqn. (5) by substituting $x = \left(\frac{c+2kn}{2}\right)$, we obtain

$$\left[\frac{\Gamma_k\left(\frac{(c+2kn)}{2}\right)}{\Gamma_k(c+2kn)} \right] = \sqrt{\frac{\pi}{k}} \frac{1}{2^{\frac{(c+2kn)}{k}-1}} \frac{1}{\Gamma_k \frac{(c+2kn+k)}{2}}. \quad (23)$$

Using eqn. (22) in eqn. (23), we get

$$= \left(\frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_0^\infty \frac{(a)_{n,k}\Gamma_k(b+kn)}{n!} 2^{\frac{(c+2kn)}{k}-1} t^{\frac{(c+2kn)}{k}-1} \right) \left(\sqrt{\frac{\pi}{k}} \frac{1}{2^{\frac{(c+2kn)}{k}-1}} \frac{1}{\Gamma_k \frac{(c+2kn+k)}{2}} \right)$$

$$= \sqrt{\frac{\pi}{k}} t^{\frac{c}{2}} \left(\frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_0^\infty \frac{(a)_{n,k}\Gamma_k(b+kn)}{n!} t^{kn} \right) \left(\frac{1}{\Gamma_k \frac{(c+2kn+k)}{2}} \right)$$

$$= \sqrt{\frac{\pi}{k}} \frac{\Gamma_k(c)}{\Gamma_k(\frac{(c+k)}{2})} t^{\frac{c}{2}} \, {}_2F_{1,k}\left(a,b;\frac{(c+k)}{2};t^k\right),$$

which is desired result. □

Theorem 2.5. *If* $a, b, c, \lambda \in \mathbb{C}$; $\Re(2b) > \Re(\lambda) > \Re(b) > \Re(k) > 0, \Re(c) > \Re(b) > 0, k > 0, \Re(a) > 0, |z| < 1$, *then*

$${}_2F_{1,k}(a,b;c;z) = \frac{\Gamma_k(c)}{\Gamma_k(\lambda-b+k)\Gamma_k(2b-\lambda)\Gamma_k(c-b)}$$

$$\times \int_0^1 x^{\frac{(b-k)}{k}} \, {}_2F_{1,k}(a, b; 2b-\lambda; zx) \, {}_2F_{1,k}\left(k, (k-c+b); (\lambda-b+k); \frac{x}{k}\right) dx. \tag{24}$$

Proof. The integral representation of k-Gauss hypergeometric function [6], is defined as

$${}_2F_{1,k}(a, b; c; z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 x^{\frac{b}{k}-1}(1-x)^{\frac{(c-b)}{k}-1}(1-kxz)^{\frac{-a}{k}} dx. \tag{25}$$

Now express $x^{\frac{b}{k}-1}(1-kxz)^{\frac{-a}{k}}$ as in following form

$$x^{\frac{b}{k}-1}(1-kxz)^{\frac{-a}{k}} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\, z^n \, x^{\frac{b}{k}+n-1}}{n!} \tag{26}$$

and using the derivative formula

$$\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{(m-n)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{(m-n)} \tag{27}$$

by replacing n by $\frac{(b-\lambda)}{k}$ and m by $\frac{(2b-\lambda+nk-k)}{k}$ in above expression, we have

$$\frac{d^{\frac{b-\lambda}{k}}}{dx^{\frac{b-\lambda}{k}}} x^{\frac{(2b-\lambda+nk-k)}{k}} = \frac{\Gamma\frac{(2b-\lambda+nk)}{k}}{\Gamma\frac{(b+nk)}{k}} x^{\frac{b}{k}+n-1}$$

Since

$$\frac{\Gamma\left(\frac{(2b-\lambda+nk)}{k}\right)}{\Gamma\left(\frac{(b+nk)}{k}\right)} = \frac{1}{k^{\frac{(b-\lambda)}{k}}} \frac{\Gamma_k(2b-\lambda+nk)}{\Gamma_k(b+nk)}$$

Therefore,

$$\frac{d^{\frac{b-\lambda}{k}}}{dx^{\frac{b-\lambda}{k}}} \left(x^{\frac{(2b-\lambda+nk-k)}{k}}\right) = \frac{1}{k^{\frac{(b-\lambda)}{k}}} \frac{\Gamma_k(2b-\lambda+nk)}{\Gamma_k(b+nk)} x^{\frac{b}{k}+n-1} \tag{28}$$

putting the value of $x^{\frac{b}{k}+n-1}$ from eqn. (27) in eqn. (26), we obtain

$$x^{\frac{b}{k}-1}(1-kxz)^{\frac{-a}{k}} = \frac{d^{\frac{(b-\lambda)}{k}}}{dx^{\frac{(b-\lambda)}{k}}} \sum_{n=0}^{\infty} \frac{(a)_{n,k}\, z^n \, k^{\frac{b-\lambda}{k}} \, \Gamma_k(b+nk)}{n! \, \Gamma_k(2b-\lambda+nk)} x^{\frac{(2b-\lambda+nk-k)}{k}}$$

$$= \frac{\Gamma_k(b)}{\Gamma_k(2b-\lambda)} \frac{d^{\frac{(b-\lambda)}{k}}}{dx^{\frac{(b-\lambda)}{k}}} \left(k^{\frac{b-\lambda}{k}} \, x^{\frac{2b-\lambda-k}{k}} \, {}_2F_1(a, b; 2b-\lambda; zx)\right) \tag{29}$$

Finally, substituting the value of $x^{\frac{b}{k}-1}(1-kxz)^{\frac{-a}{k}}$ from eqn. (28), in the eqn. (25), we get

$${}_2F_{1,k}(a, b; c; z) = \frac{\Gamma_k(c)}{\Gamma_k(b)(c-b)}$$

$$\times \int_0^1 (1-x)^{\frac{c-b}{k}-1} \frac{\Gamma_k(b)}{\Gamma_k(2b-\lambda)} \frac{d^{\frac{(b-\lambda)}{k}}}{dx^{\frac{(b-\lambda)}{k}}} \left(k^{\frac{b-\lambda}{k}} \, x^{\frac{2b-\lambda-k}{k}} \, {}_2F_1(a, b; 2b-\lambda; zx)\right) \tag{30}$$

Fractional integration by parts [2], which is as follows: If $0 < \alpha < 1$ then

$$\int_a^b \left(D^{\alpha}_{a+} q_1\right)(t) q_2(t) dt = \int_a^b q_1(t) \left(D^{\alpha}_{b-} q_2\right)(t) dt + q_1(b) \left(I^{1-\alpha}_{b-} q_2\right)(b) - \left(I^{1-\alpha}_{a+} q_1\right) q_2(a) \tag{31}$$

where q_1 and q_2 possessing fractional Riemann-Liouville derivative. Using eqn. (30) in eqn. (29), we get

$$= \frac{k^{\frac{b-\lambda}{k}}\Gamma_k(c)}{\Gamma_k(c-b)\Gamma_k(2b-\lambda)} \times \int_0^1 x^{\frac{(2b-\lambda-k)}{k}}\ {}_2F_1\left(a,b;2b-\lambda;zx\right)\left(\frac{d^{\frac{b-\lambda}{k}}}{dx^{\frac{b-\lambda}{k}}}(1-x)^{\frac{(c-b)}{k}-1}\right)dx \tag{32}$$

$$= \frac{k^{\frac{b-\lambda}{k}}\,\Gamma_k(c)}{\Gamma_k(c-b)\,\Gamma_k(2b-\lambda)}\int_0^1 x^{\frac{(2b-\lambda-k)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right) \times \left(\frac{d^{\frac{b-\lambda}{k}}}{dx^{\frac{b-\lambda}{k}}}\sum_{n=0}^{\infty}(1-\frac{(c-b)}{k})_n\frac{x^n}{n!}\right)dx$$

$$= \frac{k^{\frac{b-\lambda}{k}}\,\Gamma_k(c)}{\Gamma_k(c-b)\,\Gamma_k(2b-\lambda)}\int_0^1 x^{\frac{(2b-\lambda-k)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right) \times \left(\sum_{n=0}^{\infty}\frac{\left(1-\frac{(c-b)}{k}\right)_n}{n!}\frac{n!}{\left(n-\frac{(b-\lambda)}{k}\right)!}x^{\left(n-\frac{(b-\lambda)}{k}\right)}\right)dx.$$

Using the gamma property given by eqn. (2) and pochhammer symbol property eqn. (3) in the above expression, we have

$$= \frac{k^{\frac{b-\lambda}{k}}\,\Gamma_k(c)}{\Gamma_k(c-b)\,\Gamma_k(2b-\lambda)}\int_0^1 x^{\frac{(b-\lambda)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right) \times\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{(nk-(c-b)+k)}{k}\right)}{\Gamma\left(\frac{(nk-(b-\lambda)+k)}{k}\right)}\left(\frac{1}{\Gamma\left(\frac{(k-(c-b))}{k}\right)}\right)x^n dx \tag{33}$$

Again using k-Gamma function second part of above expression , we have

$$= \frac{(\Gamma_k(b-c)+nk+k)}{k^{\frac{(b-c+nk+k)}{k}-1}}\ \frac{k^{\frac{(k-c+b)}{k}}-1}{\Gamma_k(k-c+b)}\frac{k^{\frac{(nk-(b-\lambda)+k)}{k}-1}}{\Gamma_k((\lambda-b)+nk+k)} \tag{34}$$

Finally, using eqn.(33) in eqn.(32), we obtain

$$= \frac{\Gamma_k(c)}{\Gamma_k(2b-\lambda)\,\Gamma_k(c-b)}\int_0^1 x^{\frac{(b-k)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right) \times\sum_{n=0}^{\infty}\frac{(k-c+b)_{n,k}}{(\lambda-b+k)_{n,k}}\frac{1}{\Gamma_k(\lambda-b+k)}\frac{\Gamma(n+1)}{n!}x^n dx$$

$$= \frac{\Gamma_k(c)}{\Gamma_k(2b-\lambda)\,\Gamma_k(c-b)}\int_0^1 x^{\frac{(b-k)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right) \times\sum_{n=0}^{\infty}\frac{(k-c+b)_{n,k}}{(\lambda-b+k)_{n,k}}\frac{1}{\Gamma_k(\lambda-b+k)}\frac{\Gamma_k(nk+k)}{(n!)(\Gamma_k(k))}\left(\frac{x}{k}\right)^n dx$$

$$= \frac{\Gamma_k(c)}{\Gamma_k(2b-\lambda)\,\Gamma_k(c-b)\Gamma_k(\lambda-b+k)} \times\int_0^1 x^{\frac{(b-k)}{k}}\ {}_2F_{1,k}\left(a,b;2b-\lambda;zx\right)\ {}_2F_{1,k}\left(k,(k-c+b);(\lambda-b+k)-\lambda;x/k\right)dx\cdot$$

Which is required result. □

Theorem 2.6. *If* $a, b, c \in \mathbb{C}$; $\Re(c) > \Re(b) > 0, \Re(a) > 0, k > 0, \mid z \mid < 1$, *then*

$$ {}_2F_{1,k}(a,b;c;z) = A. \int_0^{\infty} x^{\frac{b}{k}-1}(1-x)^{\frac{-c}{k}}\, {}_2F_{1,k}\left(a,c;c;z\left(\frac{x}{x+1}\right)\right) dx \tag{35} $$

where $A = \frac{\Gamma_k(c)}{k\,\Gamma_k(b)\Gamma_k(c-b)}$.

Proof. Taking right hand side of eqn. (35) and substituting $\frac{x}{(x+1)} = u$, then $dx = \frac{1}{(1-u)^2}$, we have

$$ = A \int_0^1 u^{\frac{b}{k}-1}(1-u)^{\frac{(c-b)}{k}-1}\, {}_2F_{1,k}\left(a,c;c;zu\right) du $$

$$ = A \int_0^1 u^{\frac{b}{k}-1}(1-u)^{\frac{(c-b)}{k}-1} \sum_{n=0}^{\infty} (a)_{n,k} \frac{(zu)^n}{n!} du \tag{36} $$

Changing the order of integration and summation in the eqn. (35) and applying the k-beta property and substituting the value of A, we obtain the required result. □

Theorem 2.7. *If* $a, b, c \in \mathbb{C}$; $\Re(c) > 0, \Re(b) > 0, \Re(a) > 0, k > 0, \mid z \mid < 1$, *then*

$$ L^{-1}\left[s^{-a}\, {}_2F_{1,k}\left(a,c;b;\frac{z}{s}\right)\right] = t^{a-1}\frac{1}{\Gamma(a)}\, {}_1F_{1,k}\left(c;b;zt\right) \tag{37} $$

Proof. The definition of k-Gauss hypergeometric function by eqn. (7) in the left hand side of eqn. (36), we have

$$ L^{-1}\left[s^{-a}\, {}_2F_{1,k}\left(a,c;b;\frac{z}{s}\right)\right] = \frac{\Gamma_k(b)}{\Gamma_k(c)} \sum_{n=0}^{\infty} \frac{(a)_{n,k}\Gamma_k(c+kn)}{\Gamma_k(b+kn)} \frac{(z)^n}{n!}\, L^{-1}\left(\frac{1}{s^{a+n}}\right) $$

Applying the Inverse Laplace transform with further simplification, we get the right hand side of eqn. (2.29). □

3. Special Cases

If we take $k \to 1$ in the k-Gauss hypergeometric function defined in equation (7), the function convert into the classical Gauss hypergeometric function. So the above results are also true for the classical Gauss hypergeometric function. If $a, b, c, \phi \in \mathbb{C}$; $\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(\phi) > 0, k > 0, \mid \eta z \mid < 1$, then

$$ \frac{\Gamma_k(c+\phi)}{\Gamma_k(c)\Gamma_k(\phi)} \int_0^z t^{(c-1)}(z-t)^{(\phi-1)}\, {}_2F_1(a,b;c;\eta t)dt = z^{(c+\phi)-1}\, {}_2F_1(a,b;c+\phi;\eta z). $$

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this paper

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