# Quantum Fisher Information of Several Qubits in The Superposition of A GHZ and Two W States With Arbitrary Relative Phase


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We study the quantum Fisher information (QFI) of a system of several particles which is in a superposition of a GHZ and two W states with arbitrary relative phase. We show that as the number of particles increases from 3 to 4, the behavior of QFI drastically changes. We also show how the dependence of QFI on the relative phase weakens as the number of particles increases. We also analyze the QFI for the state for several instances of N due to the change of the relative phases.



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For the tasks requiring phase sensitivity that includes quantum interferometers, atomic clocks, etc., it was shown that multipartite entanglement itself is not sufficient to outperform the separable states [1], therefore additional constraints are required on the multipartite entangled states to find and study the "useful" ones. Quantum Fisher information (QFI) [2] is a practical tool for such a classification and studied for several quantum systems such as a qubit system inside a dissipative cavity [3], trapped ion in a laser field [4], chaos in Dicke model [5], mixed state qubit [6], two-qubit pure states [7], a three qubit state in the superposition of GHZ and two W states with relative phase [8], a state of several qubits in the superposition of a GHZ and a W state [9], superpositions of spin states [10], four qubits in a state of symmetric state and two W states [11], generalized one-axis twisting model [12] mixed Hamiltonian model and spin squeezing [13], etc. Quantum Fisher information under various noise effect has also been recently studied for Bell states [14] and GHZ states [15].

Pezze and Smerzi introduced an elegant way based on Quantum Fisher information [16], i.e. for particle-entanglement, if the condition for a specific input state $\hat{\rho}_{inp}$

$$\chi^2 \equiv \frac{N}{F_Q[\hat{\rho}_{inp}, \hat{J}_{\overrightarrow{n}}]} < 1 \tag{1}$$

holds, then the state is "useful". Here $F_Q[\hat{\rho}_{inp}, \hat{J}_{\overrightarrow{n}}] \leq 4(\Delta\hat{R}^2) = 4(\Delta J_{n_\perp})^2_{max}$ is the quantum information (equality holds for pure states); the Hermitian operator $\hat{R}$ is the solution of the equation $\{\hat{R}, \hat{\rho}_{inp}\} = i[\hat{J}_{\overrightarrow{n}}, \hat{\rho}_{inp}]$; $\hat{J}_{\overrightarrow{n_i}} = \hat{J}_k.\overrightarrow{n_i}$ with three mutually orthogonal unit vectors $\overrightarrow{n_1}$, $\overrightarrow{n_2}$ and $\overrightarrow{n_3}$; and the fictitious angular momentum operator $\hat{J}_k = \Sigma_{l=1}^{N}\hat{\sigma}_k^{(l)}$ where $\hat{\sigma}_k^{(l)}$ is the Pauli matrix operating on the $l$th particle; $k = x, y, z$. Also $n_\perp$ refers to an arbitrary axis, which is perpendicular to mean spin direction.

By "useful" states we mean the states which outperform the separable states in phase sensitivity tasks, i.e. beat the so called "shot-noise" limit of separable states: $\Delta\theta_{SN} \equiv 1/\sqrt{N}$. The fundamental limit is called the Heisenberg limit and found to be $\Delta\theta_{HL} \equiv 1/N$ [16] and shown to be the true upper limit for local phase sensitivity [17]. On the other hand, it was shown how to beat the standard quantum limit using quantum mechanical tricks such as Heisenberg's Uncertainty Principle [18]. In this context, $\chi^2 < 1$ implies that the phase sensitivity of a useful input state ($\Delta\theta_u$) is better than that of classical separable states, i.e. $1/N \leq \Delta\theta_u < 1/\sqrt{N}$.

In this work we extend the work on the quantum Fisher information (QFI) of a three qubit system (in the superposition of GHZ and W states with relative phases) [8], to a many qubit system. Although entanglement dynamics of a GHZ state or a W state do not essentially change as the number of the particles of the state increases, we find that


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for a state of superpositions of GHZ and two W states with relative phase, as the number of particles of the state increases from $N = 3$ to $N = 4$, the behavior of quantum Fisher information of the state drastically changes. But no such change is observed as $N$ increases up to 10.

We also show the decrease of the effect of the relative phase on the QFI of the state, as the number of qubits increases, and we present our results on QFI of several instances of N due to the change of relative phases, $\mu$ and $\nu$.

The state we study (given in [8] for $N = 3$) is

$$|\Psi^N\rangle = \alpha e^{i\mu}|GHZ^N\rangle + \beta e^{i\nu}|W^N\rangle + \gamma|\overline{W}^N\rangle \tag{2}$$

where $N$ is the number of qubits, $\alpha$, $\beta$ and $\gamma$ are the superposition coefficients and $\mu$ and $\nu$ are the relative phases of the state. Here $\beta = \sqrt{1-(\alpha^2+\gamma^2)}$ and $\overline{W}$ is the $W$ state in which Pauli spin matrix ($\sigma_X$) is applied to each qubit.

In order to find the mean quantum Fisher information per particle, i.e. $\chi^2$ of Eq.(1), we calculate the maximal variance $(\Delta J_{n\perp})^2_{max}$ for our input state $\hat{\rho}_{inp} = |\Psi^N\rangle\langle\Psi^N|$, which is a pure state with respect to the rotations along the $\vec{n}$ direction.

$$(\Delta J_{n\perp})^2_{max} = \frac{1}{2}\langle J^2_{n_1} + J^2_{n_2}\rangle + \frac{1}{2}\sqrt{\langle J^2_{n_1} - J^2_{n_2}\rangle^2 + \langle\{J_{n_1}, J_{n_2}\}\rangle^2} \tag{3}$$

where

$$\begin{aligned} J_{n_1} &= -sin\phi J_x + cos\phi J_y, \\ J_{n_2} &= -cos\theta cos\phi J_x - cos\theta sin\phi J_y + sin\theta J_z \end{aligned} \tag{4}$$

and the polar angle $\theta$ and the azimuth angle $\phi$ can be found as

$$\theta = arccos\left(\frac{\langle J_z\rangle}{Rsin\theta}\right), \tag{5}$$

$$\phi = \begin{cases} arccos\left(\frac{\langle J_x\rangle}{Rsin\theta}\right), & \text{if } \langle J_y\rangle > 0, \\ 2\pi - arccos\left(\frac{\langle J_x\rangle}{Rsin\theta}\right), & \text{if } \langle J_y\rangle \leq 0 \end{cases} \tag{6}$$

and the length of the mean spin can therefore found as

$$R = \sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2}. \tag{7}$$

We first calculate the mean quantum Fisher information per particle, for our state of N qubits, for $3 \leq N \leq 10$, with respect to the superposition coefficient $\gamma$, for a fixed value of $\alpha = 0.6$ and three instances of the relative phase $\nu = -\pi/6$, $\nu = -\pi/8$ and $\nu = -\pi/9$. We present the results in Figure 1 for these three cases for each $N$, from 3 to 10, with solid, dashed and dotted lines, respectively. We also provide the analytic solutions of $R$ and the expectation values of $J_x$, $J_y$ and $J_z$ with respect to $\gamma$ and $\nu$ for $\alpha = 0.6$ and $\mu = \pi$ in Table 1.

As one can easily see from Figure 1, in such a multipartite state, as the number of the qubits increase, the effect of $\nu$ becomes insignificative. Therefore in quantum information tasks requiring phase sensitivity, one can avoid the effect of the change in the relative phase of the state, only by increasing the number of qubits.

Figure 1 also demonstrates an interesting change of the behavior of QFI: Starting from $N = 4$, the behavior flips in a sense that for $N = 3$, maximum value of $\chi^2$ appears for $0.4 \leq \gamma \leq 0.6$, whereas for all $N > 3$, the minimum value of $\chi^2$ appears in the same region.

For a more general setting we calculate $\chi^2$ with respect to the changes of both of the relative phases $\mu$ and $\nu$ between $-\pi$ and $\pi$ for several instances of $N$ with equal superposition coefficients. We observe that for $N = 3$, $\chi^2$ highly depends on the changes of the relative phases but for $N = 4$ this dependence weakens sharply and as $N$ increases the weakening keeps going. We present the results for $N = 3, 4, 5$ and 6 in Figure 2.

Note that if the superposition coefficients are not taken equal but very close to each other, for example $\alpha = \gamma = 0.6$, $\beta = \sqrt{1-(\alpha^2+\gamma^2)} \cong 0.53$, there appear regions which $\chi^2$ sharply depends on the change of $\mu$ and $\nu$, as we present in Figure 3.

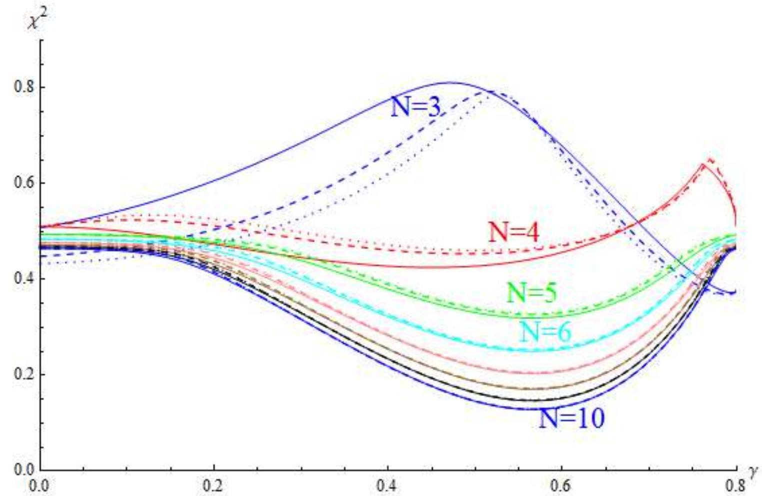


FIG. 1: Mean quantum Fisher information per particle with respect to $\gamma$ (for $\alpha = 0.6$). Solid line represents $\nu = -\pi/6$, dashed line $\nu = -\pi/8$ and dotted line $\nu = -\pi/9$.

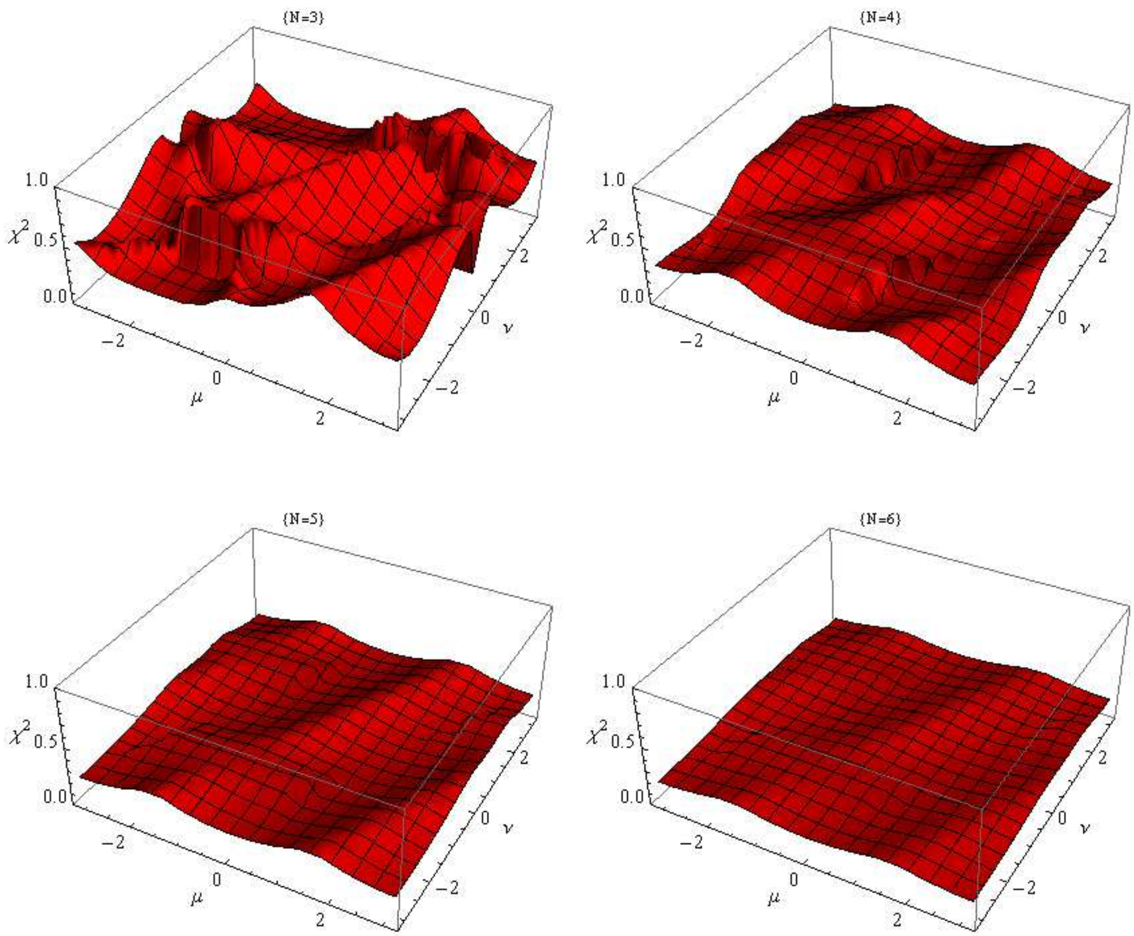


FIG. 2: Mean quantum Fisher information per particle for $N = 3, 4, 5$ and $6$ for $-\pi \leq \mu \leq \pi$ and $-\pi \leq \nu \leq \pi$ and equal superposition coefficients, i.e. $1/\sqrt{3}$.

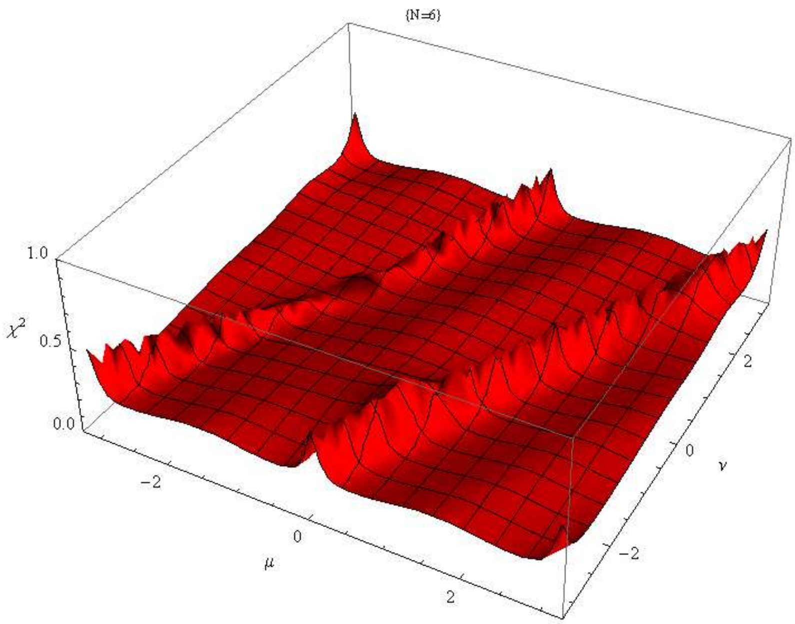


FIG. 3: Mean quantum Fisher information per particle for $N = 6$ for $-\pi \leq \mu \leq \pi$ and $-\pi \leq \nu \leq \pi$ and $\alpha = \gamma = 0.6$, therefore $\beta = \sqrt{1 - (\alpha^2 + \gamma^2)} \cong 0.53$.

In conclusion, we have studied the quantum Fisher information of a state of several qubits in the superposition of GHZ and W states with relative phases. We have found a behavior change of the quantum Fisher information when the number of the particles increase from $N = 3$ to $N = 4$. We have shown that the relative phase of the states in the superposition becomes less effective as the number of particles increase; especially when the superposition coefficients are equal to each other. On the contrary to the bipartite case, since the non-classical dynamics of multipartite entangled states has not been understood well, our work may help understanding the multipartite entanglement. In particular, creating large scale multipartite entangled sates is an interesting and a crucial step for quantum information processing and we have recently studied the limits for the case of polarization based entangled photonic $W$ states in the ideal case where experimental imperfections and photon losses are not taken into account [19, 20]. Therefore we believe the studies on quantum Fisher information of multipartite entangled states may help understanding the limits of creating the states in the non-ideal cases.

| N | $\langle J_x\rangle$, $\langle J_y\rangle$ , $\langle J_z\rangle$ , $R$ |
|---|---|
| N=3 | $\langle J_x\rangle = -0.734847\gamma + (-0.734847 + 2\gamma)\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = (-0.734847 - 2\gamma)\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 0.32 - \gamma^2$<br>$R = \sqrt{(0.32-\gamma^2)^2 + (-0.734847\gamma + (-0.734847+2\gamma)\sqrt{0.64-\gamma^2}Cos[\nu])^2 + (0.734847+2\gamma)^2(0.64-\gamma^2)Sin[\nu]^2}$ |
| N=4 | $\langle J_x\rangle = -0.848528\gamma - 0.848528\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -0.848528\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 0.64 - 2\gamma^2$<br>$R = \sqrt{0.8704 - 2.56\gamma^2 + 4\gamma^4 + 1.44\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |
| N=5 | $\langle J_x\rangle = -0.948683\gamma - 0.948683\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -0.948683\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 0.96 - 3\gamma^2$<br>$R = \sqrt{1.4976 - 5.76\gamma^2 + 9\gamma^4 + 1.8\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |
| N=6 | $\langle J_x\rangle = -1.03923\gamma - 1.03923\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -1.03923\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 1.28 - 4\gamma^2$<br>$R = \sqrt{2.3296 - 10.24\gamma^2 + 16\gamma^4 + 2.16\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |
| N=7 | $\langle J_x\rangle = -1.1225\gamma - 1.1225\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -1.1225\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 1.6 - 5\gamma^2$<br>$R = \sqrt{3.3664 - 16\gamma^2 + 25\gamma^4 + 2.52\gamma\sqrt{0.64-\gamma^2}Cos}[\nu]$ |
| N=8 | $\langle J_x\rangle = -1.2\gamma - 1.2\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -1.2\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 1.92 - 6\gamma^2$<br>$R = \sqrt{4.608 - 23.04\gamma^2 + 36\gamma^4 + 2.88\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |
| N=9 | $\langle J_x\rangle = -1.27279\gamma - 1.27279\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -1.27279\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 2.24 - 7\gamma^2$<br>$R = \sqrt{6.0544 - 31.36\gamma^2 + 49\gamma^4 + 3.24\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |
| N=10 | $\langle J_x\rangle = -1.34164\gamma - 1.34164\sqrt{0.64-\gamma^2}Cos[\nu]$<br>$\langle J_y\rangle = -1.34164\sqrt{0.64-\gamma^2}Sin[\nu]$<br>$\langle J_z\rangle = 2.56 - 8\gamma^2$<br>$R = \sqrt{7.7056 - 40.96\gamma^2 + 64\gamma^4 + 3.6\gamma\sqrt{0.64-\gamma^2}Cos[\nu]}$ |

TABLE I: Analytic solutions for $\langle J_x\rangle$, $\langle J_y\rangle$ , $\langle J_z\rangle$ and $R$ with respect to $\gamma$ and $\nu$, for $\alpha = 0.6$ and $\mu = \pi$.

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