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SPHERICAL FUZZY MATRICES

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Abstract. In this paper, we introduce spherical fuzzy matrices (SFMs) which is an advanced tool of the fuzzy matrices, intuitionistic fuzzy matrices and picture fuzzy matrices. We investigate the basic properties of SFMs and compare the idea SFMs with picture fuzzy matrices. Then some algebraic operations, such as max-min, min-max, complement, algebraic sum, algebraic product are defined and investigated their algebraic properties. Further, scalar multiplication (nA) and exponentiation (A^n) operations of a SFM A using algebraic operations are constructed, and their desirable properties are studied. Finally, we define a new operation(@) on spherical fuzzy matrices and discuss distributive laws in the case where the operations of $\oplus_s, \otimes_s, \wedge_s$ and $\vee_s$ are combined each other.

Keywords: Intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, Picture fuzzy matrix. Spherical fuzzy matrix, Algebraic sum, Algebraic product, Scalar multiplication, Exponentiation operations.

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1. Introduction and Preliminaries

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. [4] and simultaneously Im et al. [3] to generalize the concept of Thomason's [17] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0,1]$ and $0 \leq \mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$. Since the presence of IFM, a few analysts have significantly added to the improvement of IFM hypothesis and its applications [2, 5, 6, 7, 8, 9, 10, 11]. Khan et al. [4] development of IFMs is of excellent notoriety however chiefs are some way or another limited in relegating esteems because of the condition on $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$. Now and again a portion of their participation degrees are better than 1. In such a circumstance, to accomplish a sensible result IFM falls flat. In this way, managing such circumstance [9] in 2020, established the concept of Pythagorean fuzzy matrices (PyFM) by assigning membership degree say $\zeta_{a_{ij}}$ along with non-membership degree say $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$. In [12, 13, 14, 15], the developed of PyFM theory.

In this paper we extend the concept of Pythagorean fuzzy matrix to Spherical fuzzy natrix by assigning neutral membership degree say $\eta_{a_{ij}}$ along with positive and negative

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membership degrees say $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$.

Dogra and Pal [1] construction of picture fuzzy matrices (PFM) is of exceptional reputation but decision makers are some how restricted in assigning values due to the condition on $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$. In [16], some algebraic operations of Picture fuzzy matrices are defined and their desirable properties are proved. Sometimes sum of their membership degrees are superior then 1. In such situation, to attain reasonable outcome PFM fails. To describe this situation, we take an example, for provision and in contradiction of the membership degrees. The alternatives are $0.2, 0.6$ and 0.6 respectively. This gratifies the situation that their sum is superior then 1 and PFM fails to deal such type of data. Dealing with such kind of circumstances, we proposed new structure by defining spherical fuzzy matrices (SFMs) which enlarge the space of membership degrees $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ somehow bigger than that of picture fuzzy matrices. In SFM, membership degrees are satisfying the condition $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$.

The part of this paper is as follows. In section 2, spherical fuzzy matrices and its algebraic operations are defined and their desirable properties are developed. In section 3, we define a new operation(@) on Spherical fuzzy matrices and their algebraic properties are investigated. In section 4, spherical fuzzy matrix and algebraic structure on this matrix, the results are applicable. We write the conclusion of the paper in the last section 5.

Definition 1.1. [4] *An intuitionistic fuzzy matrix (IFM) of order $m \times n$ is defined as $A = \left(\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ where $\zeta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition*

$$0 \leq \zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1$$

for all i, j.

Definition 1.2. [4] *A Pythagorean fuzzy matrix (PFM) of order $m \times n$ is defined as $A = \left(\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ where $\zeta_{a_{ij}} \in [0,1]$ and $\delta_{a_{ij}} \in [0,1]$ are the membership and non-membership values of the ij^{th} element in A satisfying the condition*

$$0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$$

for all i, j.

Definition 1.3. [1] *A Picture fuzzy matrix (PFM) A of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition*

$$0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$$

for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

2. Spherical fuzzy matrices and their basic operations

In this section, spherical fuzzy matrix and their algebraic operations are defined. Then some algebraic properties, such as idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement are proved.

Now, we are going to define Algebraic operations of Spherical fuzzy matrices by restricting the measure of positive, neutral and negative membership but keeping their sum in the interval $[0,1]$.

Definition 2.1. *A Spherical fuzzy matrix (SFM) A of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition*

$$0 \leq \zeta_{a_{ij}}^2 + \eta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leq 1$$

for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Let $S_{m\times n}$ denote the set of all the Spherical fuzzy matrices.

Example

[1] $\mathbf{A} = \begin{bmatrix} (0.2, 0.6, 0.6) & (0.2, 0.4, 0.2) \\ (0.3, 0.4, 0.2) & (0.4, 0.4, 0.2) \end{bmatrix}$ is not a PFM, but it A is a SFM.

This development can be evidently recognized in Fig. 1. Each element in an PFM is expressed by an ordered pair $\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle$ with $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$ and $0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$. It was clearly seen that $0.2 + 0.6 + 0.6 > 1$, and thus it could not be described by PFM. To describe such evaluation in this paper we have proposed spherical fuzzy matrix (SFM)and its algebraic operations. Each element in an SFM is expressed by an ordered pair $\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle$ with $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$ and $0 \leq \zeta_{a_{ij}}^2 + \eta_{a_{ij}}^2 + \delta_{a_{ij}}^2 \leq 1$. Also, we can get $(0.2)^2 + (0.6)^2 + (0.6)^2 = 0.04 + 0.36 + 0.36 = 0.76 \leq 1$, which is good enough to apply the SFM to control it. The order structure of the circular fuzzy matrix is appeared in Fig. 1.

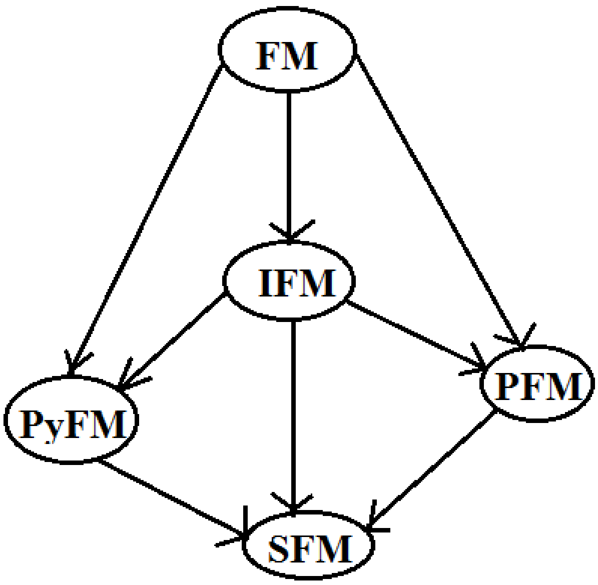

FIGURE 1. The structure between FM, IFM, PyFM, PFM and SFM.

Definition 2.2. *The Spherical fuzzy matrices A and B of the form, $A = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ and $B = (\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle)$. Then*

- $A < B$ *iff* $\forall i, j, \zeta_{a_{ij}} \leq \zeta_{b_{ij}}, \eta_{a_{ij}} \leq \eta_{b_{ij}}$ *or* $\eta_{a_{ij}} \geq \eta_{b_{ij}}, \delta_{a_{ij}} \geq \delta_{b_{ij}}$
- $A^C = (\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \rangle)$
- $A \vee_s B = (\langle \max(\zeta_{a_{ij}}, \zeta_{b_{ij}}), \min(\eta_{a_{ij}}, \eta_{b_{ij}}), \min(\delta_{a_{ij}}, \delta_{b_{ij}}) \rangle)$
- $A \wedge_s B = (\langle \min(\zeta_{a_{ij}}, \zeta_{b_{ij}}), \min(\eta_{a_{ij}}, \eta_{b_{ij}}), \max(\delta_{a_{ij}}, \delta_{b_{ij}}) \rangle)$

- $A \oplus_s B = \left(\left\langle \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right\rangle\right)$
- $A \otimes_s B = \left(\left\langle \zeta_{a_{ij}} \zeta_{b_{ij}}, \sqrt{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2 \eta_{b_{ij}}^2}, \sqrt{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2 \delta_{b_{ij}}^2} \right\rangle\right)$.

Definition 2.3. *The scalar multiplication operation over SFM A and is defined by*

$$nA = \left(\left\langle \sqrt{1 - [1 - \zeta_{a_{ij}}^2]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right\rangle\right)$$

Definition 2.4. *The exponentiation operation over SFM A and is defined by*

$$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt{1 - [1 - \eta_{a_{ij}}^2]^n}, \sqrt{1 - [1 - \delta_{a_{ij}}^2]^n} \right\rangle\right).$$

Let $S_{m\times n}$ denote the set of all the Spherical fuzzy matrices.
The following theorem relation between algebraic sum, and algebraic product of SFMs.

Theorem 2.1. *For $A, B \in S_{m\times n}$, then $A \otimes_s B \leq A \oplus_s B$.*

Proof. Let $A \oplus_s B = \left(\left\langle \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right\rangle\right)$ and
$A \otimes_s B = \left(\left\langle \zeta_{a_{ij}} \zeta_{b_{ij}}, \sqrt{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2 \eta_{b_{ij}}^2}, \sqrt{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2 \delta_{b_{ij}}^2} \right\rangle\right)$
Assume that,

$\zeta_{a_{ij}} \zeta_{b_{ij}} \leq \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}$

$(i.e)$ $\quad \zeta_{a_{ij}} \zeta_{b_{ij}} - \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2} \geq 0$

$(i.e)$ $\quad \zeta_{a_{ij}}^2 (1 - \zeta_{b_{ij}}^2) + \zeta_{b_{ij}}^2 (1 - \zeta_{a_{ij}}^2) \geq 0$

which is true as $0 \leq \zeta_{a_{ij}}^2 \leq 1$ and $0 \leq \zeta_{b_{ij}}^2 \leq 1$
And

$\eta_{a_{ij}} \eta_{b_{ij}} \leq \sqrt{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2 \eta_{b_{ij}}^2}$

$(i.e)$ $\quad \eta_{a_{ij}} \eta_{b_{ij}} - \sqrt{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2 - \eta_{a_{ij}}^2 \eta_{b_{ij}}^2} \geq 0$

$(i.e)$ $\quad \eta_{a_{ij}}^2 (1 - \eta_{b_{ij}}^2) + \eta_{b_{ij}}^2 (1 - \eta_{a_{ij}}^2) \geq 0$

which is true as $0 \leq \eta_{a_{ij}}^2 \leq 1$ and $0 \leq \eta_{b_{ij}}^2 \leq 1$
And

$\delta_{a_{ij}} \delta_{b_{ij}} \leq \sqrt{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2 \delta_{b_{ij}}^2}$

$(i.e)$ $\quad \delta_{a_{ij}} \delta_{b_{ij}} - \sqrt{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2 - \delta_{a_{ij}}^2 \delta_{b_{ij}}^2} \geq 0$

$(i.e)$ $\quad \delta_{a_{ij}}^2 (1 - \delta_{b_{ij}}^2) + \delta_{b_{ij}}^2 (1 - \delta_{a_{ij}}^2) \geq 0$

which is true as $0 \leq \delta_{a_{ij}}^2 \leq 1$ and $0 \leq \delta_{b_{ij}}^2 \leq 1$
Hence $A \otimes_s B \leq A \oplus_s B$. □

Theorem 2.2. *For any Spherical fuzzy matrix A, then*
(i) $A \oplus_s A \geq A$,
(ii) $A \otimes_s A \leq A$.

Proof. (i) Let $A \oplus_s A = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus_s (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$
$A \oplus_s A = \left(\left\langle \sqrt{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2}, (\eta_{a_{ij}})^2, (\delta_{a_{ij}})^2 \right\rangle\right)$
$\sqrt{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2} = \sqrt{\zeta_{a_{ij}} + \zeta_{a_{ij}}(1 - \zeta_{a_{ij}})} \geq \zeta_{a_{ij}}$ for all i, j
and $\quad (\eta_{a_{ij}})^2 \leq \eta_{a_{ij}}$ for all i, j
and $\quad (\delta_{a_{ij}})^2 \leq \delta_{a_{ij}}$ for all i, j
Hence $A \oplus_s A \geq A$.
Similarly, we can prove that (ii) $A \otimes_s A \leq A$. □

Theorem 2.3. *For $A, B, C \in S_{m\times n}$, then*
(i) $A \oplus_s B = B \oplus_s A$,
(ii) $A \otimes_s B = B \otimes_s B$,
(iii) $(A \oplus_s B) \oplus_s C = A \oplus_s (B \oplus_s C)$,
(iv) $(A \otimes_s B) \otimes_s C = A \otimes_s (B \otimes_s C)$.

Proof. (i) Let $A \oplus_s B$

$$= \left(\left\langle \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$
$$= \left(\left\langle \sqrt{\zeta^2_{b_{ij}} + \zeta^2_{a_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{a_{ij}}}, \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}}\delta_{a_{ij}} \right\rangle\right)$$
$$= B \oplus_s A.$$

(ii) Let $A \otimes_s B$

$$= \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \right\rangle\right)$$
$$= \left(\left\langle \zeta_{b_{ij}}\zeta_{a_{ij}}, \sqrt{\eta^2_{b_{ij}} + \eta^2_{a_{ij}} - \eta^2_{b_{ij}}\eta^2_{a_{ij}}}, \sqrt{\delta^2_{b_{ij}} + \delta^2_{a_{ij}} - \delta^2_{b_{ij}}\delta^2_{a_{ij}}} \right\rangle\right)$$
$$= B \otimes_s A.$$

(iii) Let $(A \oplus_s B) \oplus_s C$

$$= \left(\left\langle \left(\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right) \oplus_s \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}}\right) \right\rangle\right)$$
$$= \Bigg[\sqrt{\left(\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\right)^2 + \zeta^2_{c_{ij}} - \left(\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\right)^2 \zeta^2_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}\zeta^2_{c_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{c_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{c_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}\zeta^2_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{c_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{c_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}\zeta^2_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$

Let $A \oplus_s (B \oplus_s C)$

$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \left(\sqrt{\zeta^2_{b_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{c_{ij}}}\right)^2 - \zeta^2_{a_{ij}}\left(\sqrt{\zeta^2_{b_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{c_{ij}}}\right)^2},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$
$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} + \zeta^2_{c_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{c_{ij}} - \zeta^2_{b_{ij}}\zeta^2_{c_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}\zeta^2_{c_{ij}}},$$
$$\eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\Bigg]$$

Hence $(A \oplus_s B) \oplus_s C = A \oplus_s (B \oplus_s C)$
Similarly, we can prove that (iv)$(A \otimes_s B) \otimes_s C = A \otimes_s (B \otimes_s C)$. □

Theorem 2.4. *For $A, B \in S_{m\times n}$, then*
(i) $A \oplus_s (A \otimes_s B) \geq A$,
(ii) $A \otimes_s (A \oplus_s B) \leq A$.

Proof. (i) Let $A \oplus_s (A \otimes_s B)$

$$= \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right) \oplus \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \right\rangle\right)$$
$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}[\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}]}, \eta_{a_{ij}}[\sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}],$$
$$\delta_{a_{ij}}[\sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}]\Bigg]$$
$$= \Bigg[\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}[1 - \zeta^2_{a_{ij}}]}, \eta_{a_{ij}}\left(\sqrt{1 - [1 - \eta^2_{a_{ij}}][1 - \eta^2_{b_{ij}}]}\right),$$

$$\delta_{a_{ij}}\left(\sqrt{1-[1-\delta^2_{a_{ij}}][1-\delta^2_{b_{ij}}]}\right)\Big]$$
$\geq A.$
Hence $A\oplus_s(A\otimes_s B)\geq A$.
Similarly, we can prove that $(ii) A\otimes_s(A\oplus_s B)\leq A$. □

The following theorem is obvious.

Theorem 2.5. *For $A,B\in S_{m\times n}$, then*
(i) $A\vee_s B=B\vee_s A$,
(ii) $A\wedge_s B=B\wedge_s A$,

Theorem 2.6. *For $A,B,C\in S_{m\times n}$, then*
(i) $A\oplus_s(B\vee_s C)=(A\oplus_s B)\vee_s(A\oplus_s C)$,
(ii) $A\otimes_s(B\vee_s C)=(A\otimes_s B)\vee_s(A\otimes_s C)$,
(iii) $A\oplus_s(B\wedge_s C)=(A\oplus_s B)\wedge_s(A\oplus_s C)$,
(iv) $A\otimes_s(B\wedge_s C)=(A\otimes_s B)\wedge_s(A\otimes_s C)$.

Proof. In the following, we shall prove (i), and $(ii)-(iv)$ can be proved analogously.
(i) Let $A\oplus_s(B\vee_s C)$

$$=\Big[\sqrt{\zeta^2_{a_{ij}}+\max\left(\zeta^2_{b_{ij}},\zeta^2_{c_{ij}}\right)-\zeta^2_{a_{ij}}.\max\left(\zeta^2_{b_{ij}},\zeta^2_{c_{ij}}\right)},$$
$$\eta_{a_{ij}}.\max\left(\eta_{b_{ij}},\eta_{c_{ij}}\right),\delta_{a_{ij}}.\max\left(\delta_{b_{ij}},\delta_{c_{ij}}\right)\Big]$$
$$=\Big[\sqrt{\max\left(\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}},\zeta_{a^2_{ij}}+\zeta^2_{c_{ij}}\right)-\max\left(\zeta^2_{a_{ij}}\zeta^2_{b_{ij}},\zeta^2_{a_{ij}}\zeta^2_{c_{ij}}\right)},$$
$$\min\left(\eta_{a_{ij}}\eta_{b_{ij}},\eta_{a_{ij}}\eta_{c_{ij}}\right),\min\left(\delta_{a_{ij}}\delta_{b_{ij}},\delta_{a_{ij}}\delta_{c_{ij}}\right)\Big]$$
$$=\Big[\sqrt{\max\left(\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}},\zeta^2_{a_{ij}}+\zeta^2_{c_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{c_{ij}}\right)},$$
$$\min\left(\eta_{a_{ij}}\eta_{b_{ij}},\eta_{a_{ij}}\eta_{c_{ij}}\right),\min\left(\delta_{a_{ij}}\delta_{b_{ij}},\delta_{a_{ij}}\delta_{c_{ij}}\right)\Big]$$
$$=(A\oplus_s B)\vee_s(A\oplus_s C).$$ □

Theorem 2.7. *For $A,B\in S_{m\times n}$, then*
(i) $(A\wedge_s B)\oplus_s(A\vee_s B)=A\oplus_s B$,
(ii) $(A\wedge_s B)\otimes_s(A\vee_s B)=A\otimes_s B$,
(iii) $(A\oplus_s B)\wedge_s(A\otimes_s B)=A\otimes_s B$,
(iv) $(A\oplus_s B)\vee_s(A\otimes_s B)=A\oplus_s B$.

Proof. In the following, we shall prove (i), and $(ii)-(iv)$ can be proved analogously.
(i) Let $(A\wedge_s B)\oplus_s(A\vee_s B)$

$$=\Bigg[\sqrt{\min\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)+\max\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)-\min\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right).\max\left(\zeta^2_{a_{ij}},\zeta^2_{b_{ij}}\right)},$$
$$\max\left(\eta_{a_{ij}},\eta_{b_{ij}}\right).\min\left(\eta_{a_{ij}},\eta_{b_{ij}}\right),\quad \max\left(\delta_{a_{ij}},\delta_{b_{ij}}\right).\min\left(\delta_{a_{ij}},\delta_{b_{ij}}\right)\Bigg]$$
$$=\left(\left\langle\sqrt{\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}},\eta_{a_{ij}}\eta_{b_{ij}},\delta_{a_{ij}}\delta_{b_{ij}}\right\rangle\right)$$
$$=A\oplus_s B.$$ □

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus,\otimes,\vee_s,\wedge_s$.

Theorem 2.8. *For $A,B\in S_{m\times n}$, then*
(i) $(A\oplus_s B)^C=A^C\otimes_s B^C$,

(ii) $(A \otimes_s B)^C = A^C \oplus_s B^C$,
(iii) $(A \oplus_s B)^C \leq A^C \oplus_s B^C$,
(iv) $(A \otimes_s B)^C \geq A^C \otimes_s B^C$.

Proof. We shall prove $(iii), (iv)$, and $(i), (ii)$ are straightfforward.
(iii) Let $(A \oplus_s B)^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}} \right\rangle\right)$.
$A^C \oplus_s B^C = \left(\left\langle \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right)$.
Since $\delta_{a_{ij}}\delta_{b_{ij}} \leq \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}$
$\sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}} \geq \eta_{a_{ij}}\eta_{b_{ij}}$
$\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}} \geq \zeta_{a_{ij}}\zeta_{b_{ij}}$
Hence $(A \oplus_s B)^C \leq A^C \oplus_s B^C$.
(iv) Let $(A \otimes_s B)^C = \left(\left\langle \sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right)$.
$A^C \otimes_s B^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}, \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}} \right\rangle\right)$.
Since $\sqrt{\delta^2_{a_{ij}} + \delta^2_{b_{ij}} - \delta^2_{a_{ij}}\delta^2_{b_{ij}}} \geq \delta_{a_{ij}}\delta_{b_{ij}}$
$\eta_{a_{ij}}\eta_{b_{ij}} \leq \sqrt{\eta^2_{a_{ij}} + \eta^2_{b_{ij}} - \eta^2_{a_{ij}}\eta^2_{b_{ij}}}$
$\zeta_{a_{ij}}\zeta_{b_{ij}} \leq \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}$
Hence $(A \otimes_s B)^C \geq A^C \otimes_s B^C$. □

Theorem 2.9. *For $A, B \in S_{m\times n}$, then*
(i) $(A^C)^C = A$,
(ii) $(A \vee_s B)^C = A^C \wedge_s B^C$,
(iii) $(A \wedge_s B)^C = A^C \vee_s B^C$.

Proof. We shall prove (ii) only, (i) is obvious.
$A \vee_s B = \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\right\rangle\right)$
$(A \vee_s B)^C = \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right)$
$\Rightarrow A^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}}\right\rangle\right)$
$B^C = \left(\left\langle \delta_{b_{ij}}, \eta_{b_{ij}}, \zeta_{b_{ij}}\right\rangle\right)$
$\Rightarrow A^C \wedge_s B^C = \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right)$
Hence $(A \vee_s B)^C = A^C \wedge_s B^C$,
Similarly, we can prove that $(iii)(A \wedge_s B)^C = A^C \vee_s B^C$. □

Based on the Definition 2.2, 2.3 & 2.4., we shall next prove the algebraic properties of Spherical fuzzy matrices under the operations of scalar multiplication and exponentiation.

Theorem 2.10. *For $A, B \in S_{m\times n}$, then $n > 0$,*
(i) $n(A \oplus_s B) = nA \oplus_s nB, n > 0$,
(ii) $n_1A \oplus_s n_2A = (n_1 + n_2)A, n_1, n_2 > 0$,
(iii) $(A \otimes_s B)^n = A^n \otimes_s B^n, n > 0$,
(iv) $A_1^n \otimes_s A_2^n = A^{(n_1+n_2)}, n_1, n_2 > 0$,

Proof. For the two SFMs A and B, and $n, n_1, n_2 > 0$, according to definition,
we can obtain
(i) Let $n(A \oplus_s B)$
$= n\left(\left\langle \sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$

$= \Big(\Big\langle \sqrt{1-[1-\zeta^2_{a_{ij}}]^n[1-\zeta^2_{a_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \Big\rangle\Big)$
$= \Big(\Big\langle \sqrt{1-[1-\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \Big\rangle\Big)$

$nA \oplus_s nB$
$= \Big(\Big\langle \Big(\sqrt{1-[1-\zeta^2_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n\Big) \oplus_s \Big(\sqrt{1-[1-\zeta^2_{b_{ij}}]^n}, [\eta_{b_{ij}}]^n, [\delta_{b_{ij}}]^n\Big)\Big\rangle\Big)$
$= \Big[\sqrt{\Big(1-[1-\zeta^2_{a_{ij}}]^n+1-[1-\zeta^2_{a_{ij}}]^n\Big)-\Big(1-[1-\zeta^2_{a_{ij}}]^n\Big)\Big(1-[1-\zeta^2_{b_{ij}}]^n\Big)},$
$[\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n\Big]$
$= \Big(\Big\langle \sqrt{1-[1-\zeta^2_{a_{ij}}]^n[1-\zeta^2_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \Big\rangle\Big)$
$= \Big(\Big\langle \sqrt{1-[1-\zeta^2_{a_{ij}}+\zeta^2_{b_{ij}}-\zeta^2_{a_{ij}}\zeta^2_{b_{ij}}]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \Big\rangle\Big)$
$= n(A \oplus_s B).$

(ii)Let $n_1A \oplus_s n_2B$
$= \Big(\Big\langle \Big(\sqrt{1-[1-\zeta^2_{a_{ij}}]^{n_1}}, [\eta_{a_{ij}}]^{n_1}, [\delta_{a_{ij}}]^{n_1}\Big) \oplus_s \Big(\sqrt{1-[1-\zeta^2_{a_{ij}}]^{n_2}}, [\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_2}\Big)\Big\rangle\Big)$
$= \Big[\sqrt{1-[1-\zeta^2_{a_{ij}}]^{n_1}+1-[1-\zeta^2_{a_{ij}}]^{n_2}-\Big(1-[1-\zeta^2_{a_{ij}}]^{n_1}\Big)\Big(1-[1-\zeta^2_{a_{ij}}]^{n_2}\Big)},$
$[\eta_{a_{ij}}]^{n_1}[\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_1}[\delta_{a_{ij}}]^{n_2}\Big]$
$= \Big(\Big\langle \sqrt{1-[1-\zeta^2_{a_{ij}}]^{n_1+n_2}}, [\eta_{a_{ij}}]^{n_1+n_2}, [\delta_{a_{ij}}]^{n_1+n_2} \Big\rangle\Big)$
$= (n_1+n_2)A.$

(iii) Let $(A \otimes_s B)^n$
$= \Big[\Big(\zeta_{a_{ij}}\zeta_{b_{ij}}\Big)^n, \sqrt{1-[1-\eta^2_{a_{ij}}+\eta^2_{b_{ij}}-\eta^2_{a_{ij}}\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}+\delta^2_{b_{ij}}-\delta^2_{a_{ij}}\delta^2_{b_{ij}}]^n}\Big]$
$= \Big[\Big(\zeta_{a_{ij}}\zeta_{b_{ij}}\Big)^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n[1-\eta^2_{b_{ij}}]^n, 1-[1-\delta^2_{a_{ij}}]^n[1-\delta^2_{b_{ij}}]^n}\Big]$

$A^n \otimes_s B^n$
$= \Big[\Big(\zeta_{a_{ij}}\zeta_{b_{ij}}\Big)^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n+1-[1-\eta^2_{b_{ij}}]^n-\Big(1-[1-\eta^2_{a_{ij}}]^n\Big)\Big(1-[1-\eta^2_{b_{ij}}]^n\Big)},$
$\sqrt{1-[1-\delta^2_{a_{ij}}]^n+1-[1-\delta^2_{b_{ij}}]^n-\Big(1-[1-\delta^2_{a_{ij}}]^n\Big)\Big(1-[1-\delta^2_{b_{ij}}]^n\Big)}\Big]$
$= \Big(\Big\langle \Big(\zeta_{a_{ij}}\zeta_{b_{ij}}\Big)^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n[1-\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}]^n[1-\delta^2_{b_{ij}}]^n} \Big\rangle\Big)$
$= (A \otimes_s B)^n.$

(iv) Let $A^{n_1} \otimes_s A^{n_2}$
$= \Big[\Big(\zeta_{a_{ij}}\Big)^{n_1+n_2}, \sqrt{1-[1-\eta^2_{a_{ij}}]^{n_1}+1-[1-\eta^2_{a_{ij}}]^{n_2}-\Big(1-[1-\eta^2_{a_{ij}}]^{n_1}\Big)\Big(1-[1-\eta^2_{a_{ij}}]^{n-2}\Big)},$
$\sqrt{1-[1-\delta^2_{a_{ij}}]^{n_1}+1-[1-\delta^2_{a_{ij}}]^{n-2}-\Big(1-[1-\delta^2_{a_{ij}}]^{n_1}\Big)\Big(1-[1-\delta^2_{a_{ij}}]^{n_2}\Big)}\Big]$
$= \Big(\Big\langle \Big(\zeta_{a_{ij}}\Big)^{n_1+n_2}, \sqrt{1-[1-\eta^2_{a_{ij}}]^{n_1+n_2}}, \sqrt{1-[1-\delta^2_{a_{ij}}]^{n_1+n_2}} \Big\rangle\Big)$
$= A^{(n_1+n_2)}.$

Hence proved. □

Theorem 2.11. *For $A, B \in S_{m\times n}$, then $n > 0$,*
(i) $nA \leq nB$,
(ii) $A^n \leq B^n$.

Proof. (*i*) Let $A \leq B$
$\Rightarrow \zeta_{a_{ij}} \leq \zeta_{b_{ij}}$ and $\eta_{a_{ij}} \geq \eta_{b_{ij}}$ and $\delta_{a_{ij}} \geq \delta_{b_{ij}}$ for all i, j.

$\Rightarrow \sqrt{1-[1-\zeta^2_{a_{ij}}]^n} \leq \sqrt{1-[1-\zeta^2_{b_{ij}}]^n}$,
$[\eta_{a_{ij}}]^n \geq [\eta_{b_{ij}}]^n$ and
$[\delta_{a_{ij}}]^n \geq [\delta_{b_{ij}}]^n$. for all i, j.
(ii) Also, $[\zeta_{a_{ij}}]^n \geq [\zeta_{b_{ij}}]^n$,
$\sqrt{1-[1-\eta^2_{a_{ij}}]^n} \leq \sqrt{1-[1-\eta^2_{b_{ij}}]^n}$,
$\sqrt{1-[1-\delta^2_{a_{ij}}]^n} \leq \sqrt{1-[1-\delta^2_{b_{ij}}]^n}$, for all i, j. □

Theorem 2.12. *For $A, B \in S_{m\times n}$, then $n > 0$,*
(i) $n(A \wedge_s B) = nA \wedge_s nB$,
(ii) $n(A \vee_s B) = nA \vee_s nB$.

Proof. (i) Let $n(A \wedge_s B)$

$$= \left[\sqrt{1-[1-\min\left(\zeta^2_{a_{ij}}, \zeta^2_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= \left[\sqrt{1-[\max\left(1-\zeta^2_{a_{ij}}, 1-\zeta^2_{b_{ij}}\right)]^n}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= \left[\sqrt{1-\left(\max\left([1-\zeta^2_{a_{ij}}]^n, [1-\zeta^2_{b_{ij}}]^n\right)\right)}, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= \left[\max\left(\sqrt{1-[1-\zeta^2_{a_{ij}}]^n}, \sqrt{1-[1-\zeta^2_{b_{ij}}]^n}\right), \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)\right]$$

$= nA \wedge_s nB$. Hence $n(A \wedge_s B) = nA \wedge_s nB$,
Similarly, we can prove that $(ii) n(A \vee_s B) = nA \vee_s nB$. □

Theorem 2.13. *For $A, B \in S_{m\times n}$, then $n > 0$,*
(i) $(A \wedge_s B)^n = A^n \wedge_s B^n$,
(ii) $(A \vee_s B)^n = A^n \vee_s B^n$.

Proof. (i) Let $(A \wedge_s B)^n$

$$= \left[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt{1-[\max\left(1-\eta^2_{a_{ij}}, 1-\eta^2_{b_{ij}}\right)]^n}, \sqrt{1-[\max\left(1-\delta^2_{a_{ij}}, 1-\delta^2_{b_{ij}}\right)]^n}\right]$$
$$= \left[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \sqrt{1-\left(\min\left([1-\eta^2_{a_{ij}}]^n, [1-\eta^2_{b_{ij}}]^n\right)\right)}, \sqrt{1-\left(\min\left([1-\delta^2_{a_{ij}}]^n, [1-\delta^2_{b_{ij}}]^n\right)\right)}\right]$$
$$= \left[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}\right), \max\left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\right]$$

$A^n \wedge_s B^n$

$$= \left[\left([\zeta_{a_{ij}}]^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}]^n}\right) \bigwedge \left([\zeta_{b_{ij}}]^n, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\right]$$
$$= \left[\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}\right), \max\left(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n}\right)\right]$$
$$= (A \wedge_s B)^n.$$

Hence $(A \wedge_s B)^n = A^n \wedge_s B^n$,
Similarly, we can prove that $(ii)(A \vee_s B)^n = A^n \vee_s B^n$. □

Theorem 2.14. *For $A, B \in S_{m\times n}$, then $n > 0$,*
$(A \oplus_s B)^n \neq A^n \oplus_s B^n$.

Proof. Let $(A \oplus_s B)^n$

$$= \Big[\Big(\sqrt{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}} - \zeta^2_{a_{ij}}\zeta^2_{b_{ij}}}\Big)^n, \sqrt{1-[1-\eta^2_{a_{ij}}\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}\delta^2_{b_{ij}}]^n} \Big]$$

$$A^n = \Big(\Big\langle [\zeta_{a_{ij}}]^n, \sqrt{1-[1-\eta^2_{a_{ij}}]^n}, \sqrt{1-[1-\delta^2_{a_{ij}}]^n} \Big\rangle\Big)$$

$$B^n = \Big(\Big\langle [\zeta_{b_{ij}}]^n, \sqrt{1-[1-\eta^2_{b_{ij}}]^n}, \sqrt{1-[1-\delta^2_{b_{ij}}]^n} \Big\rangle\Big)$$

$$A^n \oplus_s B^n = \Big[\sqrt{[\zeta^n_{a_{ij}}]^2 + [\zeta^n_{b_{ij}}]^2 - [\zeta^n_{a_{ij}}]^2[\zeta^n_{b_{ij}}]^2}, \Big(\sqrt{1-[1-\eta^2_{a_{ij}}]^n}\Big)^n . \Big(\sqrt{1-[1-\eta^2_{b_{ij}}]^n}\Big)^n, \Big(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}\Big)^n . \Big(\sqrt{1-[1-\delta^2_{a_{ij}}]^n}\Big)^n \Big]$$

Hence $(A \oplus_s B)^n \neq A^n \oplus_s B^n$. □

3. New operation (@) on Spherical fuzzy matrices

In this section, we define a new operation(@) on Spherical fuzzy matrices and proved their algebraic properties. Further, we discuss the Disstributivity laws in the case where the operations of $\oplus, \otimes, \vee_s$ and $\wedge_s$ combined each other.

Definition 3.1. *A Spherical fuzzy matrices A and B of the form, $A = (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ and $B = (\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle)$. Then*

$$A@B = \left(\left\langle \sqrt{\frac{\zeta^2_{a_{ij}} + \zeta^2_{b_{ij}}}{2}}, \sqrt{\frac{\eta^2_{a_{ij}} + \eta^2_{b_{ij}}}{2}}, \sqrt{\frac{\delta^2_{a_{ij}} + \delta^2_{b_{ij}}}{2}} \right\rangle\right).$$

Remark 3.1. *Obviously, for every two Spherical fuzzy matrices A and B, then $A@B$ is a Spherical fuzzy matrix.*

Simple illustration given: For $A@B$,

$$0 \leq \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2} + \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2}$$
$$\leq \frac{\zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}} + \eta_{b_{ij}} + \delta_{b_{ij}}}{2} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Theorem 3.1. *For any Spherical fuzzy matrix A, then $A@A = A$.*

Proof. Let $A@A = \left(\left\langle \sqrt{\frac{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}}{2}}, \sqrt{\frac{\eta^2_{a_{ij}} + \eta^2_{a_{ij}}}{2}}, \sqrt{\frac{\delta^2_{a_{ij}} + \delta^2_{a_{ij}}}{2}} \right\rangle\right)$

$$= \left(\left\langle \left(\sqrt{\frac{\zeta^2_{a_{ij}} + \zeta^2_{a_{ij}}}{2}}\right)^2, \left(\sqrt{\frac{\eta^2_{a_{ij}} + \eta^2_{a_{ij}}}{2}}\right)^2, \left(\sqrt{\frac{\delta^2_{a_{ij}} + \delta^2_{a_{ij}}}{2}}\right)^2 \right\rangle\right)$$

$$= \left(\left\langle \frac{2\zeta^2_{a_{ij}}}{2}, \frac{2\eta^2_{a_{ij}}}{2}, \frac{2\delta^2_{a_{ij}}}{2} \right\rangle\right)$$

$= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle)$. Since $\zeta^2_{a_{ij}} \leq \zeta_{a_{ij}}, \eta^2_{a_{ij}} \leq \eta_{a_{ij}}, \delta^2_{a_{ij}} \leq \delta_{a_{ij}}$

$= A$. □

Remark 3.2. *For $a, b \in [0,1]$,then $ab \leq \frac{a+b}{2}, \frac{a+b}{2} \leq a + b - ab$.*

Theorem 3.2. *For $A, B \in S_{m\times n}$, then*
(i) $(A \oplus_s B) \vee_s (A@B) = A \oplus_s B$,
(ii) $(A \otimes_s B) \wedge_s (A@B) = A \otimes_s B$,

(iii) $(A \oplus_s B) \wedge_s (A@B) = A@B$,
(iv) $(A \otimes_s B) \vee_s (A@B) = A@B$.

Proof. we shall prove (i) and (iii), (ii) and (iv) can be proved analogously.
(i) Let $(A \oplus_s B) \vee_s (A@B)$

$$= \left[\max\left(\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \sqrt{\frac{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2}{2}} \right), \min\left(\eta_{a_{ij}} \eta_{b_{ij}}, \sqrt{\frac{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2}{2}} \right), \right.$$

$$\left. \min\left(\delta_{a_{ij}} \delta_{b_{ij}}, \sqrt{\frac{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2}{2}} \right) \right]$$

$$= \left(\left\langle \sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \eta_{a_{ij}} \eta_{b_{ij}}, \delta_{a_{ij}} \delta_{b_{ij}} \right\rangle \right)$$

$= A \oplus_s B$.

$(iii)(A \oplus_s B) \wedge_s (A@B)$

$$= \left[\min\left(\sqrt{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2 - \zeta_{a_{ij}}^2 \zeta_{b_{ij}}^2}, \sqrt{\frac{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2}{2}} \right), \max\left(\eta_{a_{ij}} \eta_{b_{ij}}, \sqrt{\frac{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2}{2}} \right), \right.$$

$$\left. \max\left(\delta_{a_{ij}} \delta_{b_{ij}}, \sqrt{\frac{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2}{2}} \right) \right]$$

$$= \left(\left\langle \sqrt{\frac{\zeta_{a_{ij}}^2 + \zeta_{b_{ij}}^2}{2}}, \sqrt{\frac{\eta_{a_{ij}}^2 + \eta_{b_{ij}}^2}{2}}, \sqrt{\frac{\delta_{a_{ij}}^2 + \delta_{b_{ij}}^2}{2}} \right\rangle \right)$$

$= A@B$,
Hence proved. □

Remark 3.3. *The Spherical fuzzy matrix forms a semilattice, associativity, commutativity, idempotency under the Spherical fuzzy matrix operation of algebraic sum and algebraic product. The distributive law also holds for* $\oplus_s, \otimes_s$ *and* $\wedge_s, \vee_s, @$ *are combined each other.*

4. Applications

The formation of Spherical fuzzy semilattice structure, Spherical fuzzy matrix and algebraic structure on this matrix, the results are applicable.

5. Conclusion

In this paper, spherical fuzzy matrices and its algebraic operations are defined. Then some properties, such as idempotency, commutativity, associativity, absorption law, distributivity, De Morgan's laws over complement are proved. Finally, we have defined a new operation(@) on Spherical fuzzy matrices and discussed distributive laws in the case where the operations of $\oplus_s, \otimes_s, \wedge_s$ and $\vee_s$ are combined each other. This result can be applied further application of Spherical fuzzy matrix theory. For the development of Spherical fuzzy semilattice and its algebraic property the results of this paper would be helpful. In the future, the application of the proposed aggregating operators of SFMs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

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