TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 166-174

A GENERALIZATION OF RELATION-THEORETIC CONTRACTION PRINCIPLE

D. KHANTWAL[1*], S. ANEJA[2], G. PRASAD[3], U. C. GAIROLA[3], §

Abstract. In the present paper, we generalize relation-theoretic contraction principle using weaker class of contraction mappings which is assumed to be hold on the elements of a particular subset of the whole space, whose elements are relaxed under the underlying relation. We also relaxed the assumption of continuity from the main result of Alam and Imdad by introducing the notion of $(\mathcal{R}, k)$-continuity. Moreover, our results do not require the underlying binary relation to be T-closed for existence of fixed points in relational metric spaces.

Keywords: Fixed point, Contraction mapping, Binary relation, K-continuous mapping.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The Banach contraction principle [3] is very popular tool for guaranteeing the existence and uniqueness of solution of considerable problem arising in several branches of Mathematics. Several extensions of this core result are available in the existing literature of metric fixed point theory (see [1, 5, 6, 8, 10] and references therein) . In 2015, Alam and Imdad [1] extended the classical Banach contraction principle using an arbitrary binary relation. In doing so, the authors introduced relation-theoretic contraction condition which is assumed to be hold on those elements which are related under the underlying binary relation rather than the whole space. It is easy to see that under universal relation the result of Alam and Imdad [1] reduces to the Banach contraction principle [3].

In this paper, we introduce notion of $(\mathcal{R}, k)$-continuous mappings which is relatively weaker notion of the class of continuous mappings as compare to the class of k-continuous mappings and $\mathcal{R}$-continuous mappings. Using this concept, we extend the relation-theoretic

[1] Department of Mathematics, Graphic Era Hill University, Dehradun, Uttarakhand, India.
e-mail: deepakkhantwal15@gmail.com; ORCID: hhttps://orcid.org/0000-0002-1081-226X.
[*] Corresponding author.
[2] Government Inter College, Purwala Dogi, Tehri-Garhwal, Uttarakhand, India.
e-mail: surbhianejajgd@gmail.com; ORCID: https://orcid.org/0000-0002-3099-6246.
[3] Department of Mathematics, H.N.B Garhwal University, Srinagar (Garhwal), Uttarakhand, India.
e-mail: gopiprasad127@gmail.com; ORCID: hhttps://orcid.org/0000-0002-8724-3309.
e-mail: ucgairola@rediffmail.com; ORCID: hhttps://orcid.org/0000-0002-3981-1033.
§ Manuscript received: October 19, 2020; accepted: December 31, 2020.

contraction principle due to Alam and Imdad [1] for a weaker class of contraction mappings. Our results guarantee the existence of fixed points in such cases wherein all the classical fixed point theorems to an arbitrary binary relation can not be applied. Our results also show that the assumption of T-closedness of underlying binary relation in whole space is not necessary condition for the existence of fixed points in relational metric spaces.

2. Preliminaries

We start our consideration by giving a brief review of the definitions and basic properties of binary relations. Throughout this paper, we assume that $\mathbb{N}$ and $\mathbb{N}_0$ stand for the set of positive integers and the set of non-negative integers respectively.

Definition 2.1. [1]. *A binary relation $\mathcal{R}$ on a non-empty set X is a subset of $X \times X$. We say that x relates to y under $\mathcal{R}$ if and only if $(x, y) \in \mathcal{R}$.*

Definition 2.2. [1]. *Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. We say x and y are $\mathcal{R}$-comparable if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.*

Definition 2.3. [1]. *Let $\mathcal{R}$ be a binary relation on a non-empty set X.*

1. *The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}$.*
2. *The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e. $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.*

Definition 2.4. [1]. *A binary relation $\mathcal{R}$ defined on a non-empty set X is called*

(a) *reflexive if $(x, x) \in \mathcal{R}$ for all $x \in X$,*
(b) *irreflexive if $(x, x) \notin \mathcal{R}$ for all $x \in X$,*
(c) *symmetric if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,*
(d) *antisymmetric if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$,*
(e) *transitive if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$,*
(f) *complete, connected or dichotomous if $[x, y] \in \mathcal{R}$ for all $x, y \in X$,*
(f) *weekly complete, weakly connected or trichotomous if $[x, y] \in \mathcal{R}$ or $x = y$ for all $x, y \in X$.*

Definition 2.5. [1]. *Let X be a non-empty set and $\mathcal{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if*

$$(x_n, x_{n+1}) \in \mathcal{R}, \qquad \text{for all } n \in \mathbb{N}_0.$$

Definition 2.6. [1]. *Let (X, d) be a metric space. A binary relation $\mathcal{R}$ defined on X is called d-*self-closed *if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence on X and*

$$x_n \xrightarrow{d} x$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x_n] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

Definition 2.7. [1, 9]. *Let $\mathcal{R}$ be a binary relation on a non-empty set X and T be a self-mapping on X. If for $x, y \in X$ with*

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}$$

then the binary relation $\mathcal{R}$ is called T-closed and mapping T is called comparative mapping on X under binary relation $\mathcal{R}$.

Definition 2.8. [2]. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and $x \in X$. A self-mapping T on X is called* $\mathcal{R}$-continuous mapping *at point x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called* $\mathcal{R}$-continuous *if it is $\mathcal{R}$-continuous at each point of X.*

Clearly every continuous mapping is $\mathcal{R}$-continuous, for any binary relation $\mathcal{R}$ and under universal relation the notion of $\mathcal{R}$-continuity coincides with the definition of usual continuity.

Definition 2.9. [7]. *A self-mapping T of a metric space (X, d) is called k- continuous, $k = 1, 2, 3 \ldots$, at the point $x \in X$ if $T^k(x_n) \to Tx$, whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}(x_n) \to x$ in X. Moreover, T is called k-continuous if it is k-continuous at each point of X.*

It is obvious by definition of k-continuity that every continuous mapping T of a metric space (X, d) is k-continuous mapping and continuity coincides with the notion of 1-continuity. However k-continuity of a function (for $k \geq 2$) does not implies the continuity of function (see Example 1.2 in [7]).

Definition 2.10. [2] *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in X.*

Every complete metric space is $\mathcal{R}$-complete, for any binary relation $\mathcal{R}$ on X and both the definition coincides under the universal relation.

Definition 2.11. *Let $\mathcal{R}$ be a binary relation on a non-empty set X. For $x, y \in X$, a path of lengh $k \in \mathbb{N}$ in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, \ldots, z_k\} \subseteq X$ satisfying the following conditions:*

(i) *$z_0 = x$ and $z_k = y$;*
(ii) *$(z_i, z_{i+1}) \in \mathcal{R}$, for all $i \in \{0, 1, 2, \ldots, k-1\}$.*

We denote by $\gamma(x, y, \mathcal{R})$, the family of all paths in $\mathcal{R}$ from x to y and by $X(T; \mathcal{R})$, the set of all points $x \in X$ satisfying $(x, Tx) \in \mathcal{R}$. The following result is analoge of the Banach contraction principle in relational metric space and proved in [1].

Theorem 2.1. *Let (X, d) be a complete metric space, $\mathcal{R}$ is binary relation on X and $T : X \to X$ be a self mapping on X. Suppose that the following conditions hold:*

(a) *$X(T; \mathcal{R})$ is non-empty,*
(b) *$\mathcal{R}$ is T-closed,*
(c) *either T is continuous or $\mathcal{R}$ is d-self-closed,*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y), \text{ for all } x, y \in X \text{ with } (x, y) \in \mathcal{R}. \tag{1}$$

Then T has a fixed point. Moreover, if

(e) *$\gamma(x, y, \mathcal{R}^s)$ is non-empty, for each $x, y \in X$,*

then T has a unique fixed point.

Proposition 2.1. *Let $\mathcal{R}$ be a binary relation on metric space X endowed with metric d and T be a self-mapping on X. For non-empty subset A of $X(T; \mathcal{R})$ and $\alpha \in [0, 1)$, the the following conditions are equivalent:*

(I) *$d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in A$ with $(x, y) \in \mathcal{R}$,*
(II) *$d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in A$ with $[x, y] \in \mathcal{R}$.*

Proof. The proof of the Preposition 2.1 is similar to the proof of Preposition 2.3 in [1]. □

3. Main Results

Firstly, we introduce the notion of $(\mathcal{R}, k)$-continuous mappings, which is weaker than the class of continuous mapping, k-continuous mappings and $\mathcal{R}$-continuous mappings:

Definition 3.1. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$. A self mapping T on X is said to be $(\mathcal{R}, k)$-continuous at a point $x \in X$ if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ in X such that $T^{k-1}(x_n) \xrightarrow{d} x$, we have $T^k(x_n) \xrightarrow{d} Tx$. Moreover T is called $(\mathcal{R}, k)$-continuous if it is $(\mathcal{R}, k)$-continuous at each point of X.*

By definition of $(\mathcal{R}, k)$-continuity, it is clear that every $\mathcal{R}$-continuous mapping is $(\mathcal{R}, k)$-continuous mapping and both the definitions coincide for $k = 1$. Also every k-continous mapping is $(\mathcal{R}, k)$-continuous mapping and under universal relation the notion of $(\mathcal{R}, k)$-continuity coincides with the definition of k-continuity introduced by Pant and Pant in [7].

Remark 3.1. *Every continuous, k-continuous and $\mathcal{R}$-continuous mapping is a $(\mathcal{R}, k)$-continuous mapping but converse is not true. The following example illustrates that $(\mathcal{R}, k)$-continuity does not imply $\mathcal{R}$-continuity and k-continuity as well.*

Example 3.1. *Let $X = [-1, 2]$ be a metric space equipped with usual metric $d(x, y) = |x - y|$. Define a binary relation $\mathcal{R} = \{(\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N}\}$ on X and the mapping $T : X \to X$ defined by*

$$T(x) = \begin{cases} 1/3, & \text{if } x \in [-1, 0], \\ 1/2, & \text{if } x \in (0, 1] \\ x, & \text{if } x \in (1, 2]. \end{cases}$$

Clearly T is not a continuous mapping in X and $\{x_n\} = \{\frac{1}{2^n}\}, n \in \mathbb{N}$ is $\mathcal{R}$-preserving sequence in X as $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$. Since $\{x_n\} \to 0$ as $n \to \infty$ then $Tx_n \to 1/2 \neq T0$. Hence T is not $\mathcal{R}$-continuous mapping in X. Now, for each $k = 1, 2, 3, \ldots$,

$$T^k(x) = \begin{cases} 1/2, & \text{if } x \in [-1, 1], \\ x, & \text{if } x \in (1, 2]. \end{cases}$$

Since $T^k(x)$ is continuous everywhere in X, except at $x = 1$. Also there does not exist any $\mathcal{R}$-preserving sequence $\{x_n\}$ in X such that $T^{k-1}(x_n) \to 1$ as $n \to \infty$. So T is obviously $(\mathcal{R}, k)$-continuous mapping in X. However, for $\{x_n\} = \{1 + \frac{1}{n}\}, n \in \mathbb{N}$, $T^{k-1}(x_n) \to 1$ and $T^k(x_n) \to 1 \neq T1$ yields T is not k-continuous mapping in X. Hence, the mapping T is $(\mathcal{R}, k)$-continuous mapping in X, but T is neither continuous nor k-continuous and $\mathcal{R}$-continuous mapping in X.

Now we state our main results.

Theorem 3.1. *Let (X, d) be a complete metric space endowed with a binary relation $\mathcal{R}$ on X and $T : X \to X$ be a mapping. Suppose that A be any non-empty subset of $X(T; \mathcal{R})$ and the following conditions are hold:*

(a) *$T(A) \subseteq A$;*
(b) *T is $(\mathcal{R}, k)$-continuous mapping or $\mathcal{R}$ is d-self-closed;*
(c) *For all $x, y \in A$ with $(x, y) \in \mathcal{R}$, there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y); \tag{2}$$

Then T has a fixed point in X.

Proof. Let A be any non-empty subset of $X(T;\mathcal{R})$ and $x_0 \in A$. Then we have $(x_0, Tx_0) \in \mathcal{R}$. If $x_0 = Tx_0$, then the proof is completed. So in view of condition (a), there exists a point say x_1 in A such that $x_1 = Tx_0$. Again since $x_1 \in A$, so $(x_1, Tx_1) \in \mathcal{R}$. If $x_1 = Tx_1$ then x_1 is a fixed point of T and proof is completed. Therefore $x_1 \neq Tx_1$ and in view of condition (a), there exists a point say $x_2 \in A$ such that $(x_2, Tx_2) \in \mathcal{R}$. Continuing this process again and again, we get a sequence of points $\{x_n\}$ in A such that

$$x_{n+1} = Tx_n \text{ and } (x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Thus the sequence $\{x_n\}$ is $\mathcal{R}$-preserving . Applying condition (c) to (3) for all $n \in \mathbb{N}_0$, we deduce that

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}),$$

which by induction yields that

$$d(x_{n+1}, x_{n+2}) \leq \alpha^{n+1} d(x_0, x_1), \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Using triangular inequality and (4), for all $m > n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) d(x_0, Tx_0) \\ &= \alpha^n d(x_0, Tx_0) \sum_{j=1}^{m-n-1} \alpha^j \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $\{x_n\}$ is $\mathcal{R}$-preserving Cauchy sequence in A. Since $A \subseteq X$ and (X, d) is a complete metric space, so there exists $x^* \in X$ such that

$$x_n \xrightarrow{d} x^*.$$

Now in view of assumption (b), we suppose that T is $(\mathcal{R}, k)$-continuous. Then for $\mathcal{R}$-preserving sequence $\{x_n\} = \{T^{k-1}(x_{n-k+1})\}$ and

$$T^{k-1}(x_{n-k+1}) \xrightarrow{d} x^*.$$

We have

$$x_{n+1} = T^k(x_{n-k+1}) \xrightarrow{d} T(x^*)$$

and uniqueness of the limit of sequence $\{x_n\}$ implies $T(x^*) = x^*$, that is x^* is a fixed point of T.

Alternately, let us assume that $\mathcal{R}$ is d-self-closed. As $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and

$$x_n \xrightarrow{d} x^*,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$. In view of Preposition 2.1, we obtain

$$d(x_{n_k+1}, Tx^*) = d(Tx_{n_k}, Tx^*) \leq \alpha d(x_{n_k}, x^*) \to 0 \text{ as } k \to \infty,$$

so $x_{n_k+1} \xrightarrow{d} T(x^*)$. Owing to the uniqueness of limit, we obtain $T(x^*) = x^*$. □

Example 3.2. *Consider $X = [-1, 1]$ equipped with usual metric $d(x, y) = |x - y|$ so that (X, d) is a complete metric space. Define a binary relation $\mathcal{R} = \{(0, 0), (0, -1), (-1, 0), (1/2, 1), (1, 0)\} \cup \{(\frac{1}{n}, \frac{1}{n+1}) : n \in \mathbb{N}\}$ on X and the mapping $T : X \to X$ by*

$$T(x) = \begin{cases} 0, & \text{if } x \in [-1, 0], \\ 1, & \text{if } x \in (0, 1]. \end{cases}$$

Then $\{x_n\} = \{\frac{1}{n}\}$ *is* $\mathcal{R}$*-preserving sequence in* X *as* $(x_n, x_{n+1}) \in \mathcal{R}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to 0$ *and* $Tx_n \to 1 \neq T0$*. Hence* T *is not* $\mathcal{R}$*-continuous mapping in* X*. However* T *is* $(\mathcal{R}, k)$*-continuous mapping for* $k = 2$*, as* $T^2(x) = 0$ *for all* $x \in X$ *is a continuous mapping. Clearly* $X(T; \mathcal{R}) = \{-1, 0, 1/2\}$ *as for each* $x \in X(T; \mathcal{R}), (x, Tx) \in \mathcal{R}$*. Also for* $A = \{-1, 0\} \subset X(T; \mathbb{R})$*,* $T(A) \subset A$ *and* T *satisfied condition (2) with* $\alpha = 0$*. Hence all the assumptions of Theorem 3.1 are satisfied and* T *has a fixed point at* $x = 0$*.*

Even though the binary relation $\mathcal{R}$ *used in the Example 3.2 is not* T*-closed still* T *has a fixed point in* X*, which shows that the assumption of* T*-closedness of the binary relation is not a necessary condition for existence of fixed point in relational metric spaces. Also for* $(x, y) = (1, 0) \in \mathcal{R}$*,* T *does not satisfies relation-theoretic contraction (1) of Alam and Imbad* [1].

Remark 3.2. *It is remarkable here that, if* (X, d) *be a* $\mathcal{R}$*-complete metric space, and* T *be a self-mapping on* X *such that* T *has a fixed point* x^* *in* X*, then* T *always satisfies our Theorem 3.1. Clearly for* $A = \{x^*\}$*,* T *satisfies all the conditions of Theorem 3.1.*

In view of Remark 3.1, we obtain the following corollary as a direct consequences of Theorem 3.1.

Corollary 3.1. *Theorem 3.1 remain true if we replace condition* (b) *by one of the following conditions (besides retaining the rest of the hypothesis):*

(b1) T *is continuous*
(b2) T *is* k*-continuous*
(b3) T *is* $\mathcal{R}$*-continuous.*

Theorem 3.2. *Let* (X, d) *be a metric space endowed with a binary relation* $\mathcal{R}$ *and* T *be a self-mapping on* X*. Suppose that* $X(T; \mathcal{R})$ *be a non-empty subset of* X *and the following conditions are satisfied:*

(a) $X(T; \mathcal{R})$ *is invariant under* T*;*
(b) T *is* $(\mathcal{R}, k)$*-continuous mapping or* $\mathcal{R}$ *is* d*-self-closed;*
(c) *for all* $x, y \in X(T; \mathcal{R})$ *with* $(x, y) \in \mathcal{R}$*, there exists* $\alpha \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq \alpha d(x, y); \tag{5}$$

(d) $X(T; \mathcal{R})$ *is* $\mathcal{R}$*-complete.*

Then T *has at least one fixed point in* X*.*

Proof. Let $x_0 \in X(T; \mathcal{R})$, then $(x_0, Tx_0) \in \mathcal{R}$. If $x_0 = Tx_0$, the proof is completed. So there exists a point say x_1 in $X(T; \mathcal{R})$ such that $x_1 = Tx_0$. Again since $x_1 \in X(T; \mathcal{R})$ so $(x_1, Tx_1) \in \mathcal{R}$. If $x_1 = Tx_1$ again the proof is completed. Otherwise, continuing in the process, we get a $\mathcal{R}$-preserving sequence $\{x_n\}$ in $X(T; \mathcal{R})$ such that

$$x_{n+1} = Tx_n \text{ and } (x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Proceeding as in the proof of Theorem 3.1, we get $\{x_n\}$ is a $\mathcal{R}$-preserving Cauchy sequence in $X(T; \mathcal{R})$. Since $X(T; \mathcal{R})$ is $\mathcal{R}$-complete so there exists $x^* \in X(T; \mathcal{R})$ such that

$$x_n \xrightarrow{d} x^*.$$

Now in view of assumption (b), we suppose that T is $(\mathcal{R}, k)$-continuous mapping on X. Then for every $\mathcal{R}$-preserving sequence $\{x_n\} = \{T^{k-1}(x_{n-k+1})\}$ in X such that

$$T^{k-1}(x_{n-k+1}) \xrightarrow{d} x^*,$$

we have

$$x_{n+1} = T^k(x_{n-k+1}) \xrightarrow{d} T(x^*).$$

Owing to the uniqueness of limit, we obtain $T(x^*) = x^*$, that is x^* is a fixed point of T.

Alternately, let us assume that $\mathcal{R}$ is d-self-closed. As $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and

$$x_n \xrightarrow{d} x^*,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$. In view of Preposition 2.1, we obtain

$$d(x_{n_k+1}, Tx^*) = d(Tx_{n_k}, Tx^*) \leq \alpha d(x_{n_k}, x^*) \to 0 \text{ as } k \to \infty,$$

so $x_{n_k+1} \xrightarrow{d} T(x^*)$. Again, owing to the uniqueness of limit, we obtain $T(x^*) = x^*$. □

Theorem 3.3. *In addition to the hypothesis of Theorem 3.2, suppose that $\mathcal{R}$ is a transitive relation on X and $\gamma(x, y, \mathcal{R})$ is non-empty, for all $x, y \in X(T; \mathcal{R})$. Then T has a unique fixed point in $X(T; \mathcal{R})$.*

Proof. Let x^* and y^* be two distinct fixed points of T in $X(T; \mathcal{R})$ then $x^* = Tx^*, y^* = Ty^*$. Since $\gamma(x, y, \mathcal{R})$ is non-empty, there is a path (say $\{z_1, \ldots, z_2\}$) of some finite length k in $\mathcal{R}$ from x^* to y^*, so that

$$z_0 = x^*, \quad z_k = y^*, \quad (z_i, z_{i+1}) \in \mathcal{R} \quad \text{for each } i = 0, 1, 2, \ldots, k-1.$$

By transitivity of $\mathcal{R}$, we get

$$(x^*, z_1) \in \mathcal{R}, \ \ (z_1, z_2) \in \mathcal{R}, \ldots, (z_{k-1}, y^*) \in \mathcal{R} \implies (x^*, y^*) \in \mathcal{R}.$$

The condition (5) implies that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \alpha d(x^*, y^*)$$

which is not possible. Thus T has a unique fixed point in $X(T; \mathcal{R})$. □

Now we consider some special cases wherein our results deduce and generalize several well-known fixed point theorems of the existing literature.

- The Theorem 3.1 and Theorem 3.2 are proper generalizations of Theorem 2.1 of Alam and Imdad [1]. Clearly as, assumption of $(\mathcal{R}, k)$-continuity is weaker than the assumption of $\mathcal{R}$-continuity and contraction conditions (2) and (5) are weaker than the condition (1).
- Under universal relation ($\mathcal{R} = X^2$ and $X(T; \mathcal{R}) = X$), Theorem 3.2 reduces to the classical Banach contraction principle and condition $(a), (b)$ and (c) trivially hold.
- By Choosing $\mathcal{R} :=\preceq$, the partial order in Theorem 3.1, we get a generalized version of Nieto and Rodrìguez-Lòpez [6, see Theorem 2.4,Theorem 2.5] and, Ran and Reurings theorem [8, see Theorem 2.1].
- Taking $\mathcal{R} :=\prec\succ$, the tolerance relation associate with a partial order $\preceq$ in Theorem 3.1, we obtained generalized version of the result of Turinici [11]

Now we furnish an illustrative example in support of Theorem 3.2, which does not satisfy the hypothesis of all the classical results in relational metric spaces.

Example 3.3. *Let $X = [0, 1]$ and d be the standard metric $d(x, y) = |x-y|$ so that (X, d) is a complete metric space. Define a binary relation $\mathcal{R} = \{(0, 0), (1/4, 0), (1/8, 3/4), (3/4, 1/8)\}$ on X and the mapping $T : X \to X$ by*

$$T(x) = \begin{cases} 0, & \text{if } x \in [0, 1/2], \\ 1/4, & \text{if } x \in (1/2, 1]. \end{cases}$$

Then it is easy to verify that mapping T is not continuous but is 2-continuous mapping as $T^2(x) = 0$ for all $x \in X$ and $X(T; \mathcal{R}) = \{0, 1/4\}$. We also observe that $X(T; \mathcal{R})$ is *invariant under the mapping T and T satisfies the condition (3) for $\alpha = 0$. Hence all the conditions of our Theorem 3.2 are satisfied and T has a fixed point at $x = 0$.*

Notice that the binary relation $\mathcal{R}$ in Example 3.3 is neither T-closed nor one of the earlier known standard binary relation such as reflexive, symmetric, transitive, anti-symmetric, complete or weakly complete. Therefore theorems contained in [1, 2, 3, 4, 6, 8, 9, 11, 12, 13] can not be applied in the above example. Thus our results extend all the classical results to an arbitrary binary relation.

Conclusion

In this paper, we introduced a new class of continuous mappings, (namely $(\mathcal{R}, k)$-continuous mappings) which is weaker than the class of continuous mappings and $\mathcal{R}$-continuous mappings. Further, we generalized relation-theoretic contraction principle due to Alam and Imdad [1] for a weaker class of contraction mappings by replacing the assumption of continuity to the notion of $(\mathcal{R}, k)$-continuity.

In order to ensure the existence of fixed points for linear and nonlinear contraction mappings T, the underlying binary relation is required to be T-closed in the relational metric spaces. In our results, the underlying binary relation does not require to be T-closed in the whole space. Moreover, our results also have guaranteed the existence of fixed points in such cases wherein all the classical fixed point results in relational metric spaces remain silent.

Conflict of Interest. The authors declare that they have no conflict of interest.

Acknowledgement. The authors are very much indebted to the referee(s) for their valuable suggestions for the improvement of this manuscript.

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Deepak Khantwal is currently working as an assistant professor in the Department of Mathematics of Graphic Era Hill University, Dehradun, Uttarakhand, India. He received his M.Sc and Ph.D. degree form HNB Garhwal University, Srinagar (Garhwal), Uttarakhand, India. His area of interest includes Nonlinear Analysis and Fixed point theory.

Surbhi Aneja received her M.Sc and Ph.D. degree form HNB Garhwal University, Srinagar (Garhwal) Uttarakhand, India. She is currently working as lecturer (Maths) in Government Inter College, Purwala Dogi, Uttarakhand, India. Her area of interest includes Inventory theory and Fixed point theory.

Gopi Prasad is currently working as an assistant professor in the Department of Mathematics of HNB Garhwal University, Srinagar (Garhwal), Uttarakhand, India. He received his M.Sc and Ph.D. degree form HNB Garhwal University, Srinagar (Garhwal) Uttarakhand, India. His area of interest includes Functional Analysis and fixed point theory.

Dr. U. C. Gairola is Professor & Head in the Department of Mathematics, HNB Garhwal University, BGR Campus, Pauri Garhwal, Uttarakhand, India. He received his Ph.D. Degree from Gurukul Kangri Vishwavidhyalya, Haridwar, Uttarakhand, India. His major research interests include Fixed point theory in metric spaces, Banach spaces and Product spaces.