TWMS J. App. and Eng. Math. V.13, N.1, 2023, pp. 341-361

A TECHNIQUE FOR SOLVING SYSTEM OF GENERALIZED EMDEN-FOWLER EQUATION USING LEGENDRE WAVELET

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ABSTRACT. This article is concerned with the development of an efficient numerical algorithm for the solution of a system of generalized nonlinear Emden-Fowler equation. The proposed algorithm is based on the Legendre wavelet operational matrix of integration technique. This method decreases the storage and computational complexity due to its calculation on the subinterval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ of $[0,1]$. The main highlight of this method is to converts the system of the differential equation into an equivalent system of nonlinear algebraic equations, which greatly simplifies approximation. Some numerical example shows that the proposed scheme is very efficient and reliable.

Keywords: Legendre wavelets; Operational matrix of integration; System of Emden-Fowler equation; Initial value problem.

AMS Subject Classification: 65T60, 34A12, 65L10.

1. Introduction

Generalized Emden-Fowler type differential equations [9, 10] are frequently involved in several mathematical models of physical phenomena. It can be encountered in numerous field of mathematical physics and astrophysics problems such as in the study of the shear-free spherically symmetric perfect fluid motion in cosmology [13, 14], pattern formation, population evolution, and chemical reaction [19]. Generalized Emden-Fowler equation can be written as

$$y'' + \frac{\alpha}{t} + g(t,y) = h(t), 0 < t < \infty \tag{1}$$

subject to

$$y(0) = A,\ y'(0) = B, \tag{2}$$

where $'$ denotes the derivative with respect to "t", A and B are constants, and $\alpha > 0$ is known as shape factor. In case $g(t,y) = ct^m y^n, h(t) = 0$ and $B = 0$, initial value problem

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§ Manuscript received: December 12, 2020; accepted: April 18, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.1
This work is supported by SERB, New Delhi. (Grant No: ECR/2017/000560).

(IVPs) (1)-(2) represent Emden-Fowler equation of first kind [1], given by

$$y'' + \frac{\alpha}{t} + ct^m y^n = 0, 0 < t < \infty \tag{3}$$

$$y(0) = A,\ y'(0) = 0. \tag{4}$$

In case $m = 0$ and $c = 1$, IVPs (3)-(4) represents the standard Lane-Emden equation which sculpts many physical phenomena such as theory of stellar structure, thermal behavior of spherical gas cloud and theories of thermionic current [8, 15, 22, 23, 29]. One can find several nonlinear singular boundary value problems based on models of realistic problems in an review article by Verma et. al. [31]. The main difficulty to solve the differential equation (3) is its singular behavior at $t = 0$. There are numerous methods for the numerical solution of Emden-Fowler equation. Some of them which are recently used in literature are perturbation techniques [7], Collocation method and Galerkin method [11], Adomian decomposition method [33], wavelet-Galerkin method [3], operational matrix method associate the first kind shifted Chebyshev polynomial [20, 21, 26] and methods based on Haar wavelet [17, 28, 30, 32].

The Adomian decomposition method has been used on a wide class of problems that provide us a demonstrable, sustainable, and rapidly convergent approximation, but major disadvantage with the method is to construct toughest and very complicated Adomian polynomials. Collocation and Galerkin method discretised the continuous operator into a matrix using the projection method. The major issue with this method is to calculate the inner product numerically. To overcome the difficulties of these methods, several researchers are using wavelet as a basis function, which is known as wavelet Galerkin method. Wavelets as a basis function are most trustworthy and these are great refinement over other basis functions, such as standard polynomial basis and a trigonometric basis.

Recently, the word "wavelet" has fascinated the scientific and engineering community. From last few decades, orthogonal function and wavelets have received great deal of attention and become an important tool for the approximate solution of functional equation. Based on the structure, orthogonal functions may be generally categorized into three families: piecewise constant basis functions such as walsh functions, block pulse functions, etc., orthogonal polynomials (Legendre polynomials, Chebyshev polynomials, etc.) and sine-cosine series (Fourier series) (see [4, 5, 6, 12] and the references therein). The approximation of a continuous function with piecewise constant basis functions results in an estimate which is not continuous. Also, the approximation of a discontinuous function with continuous basis functions results in an estimate which is continuous. None of these two basis functions taken alone can approximate both the continuous and discontinuous model effectively and accurately. So it is essential to use another basis function that can approximate both the continuous and discontinuous models. Wavelets are very useful basis functions to represent both the spatially varying properties due to compact support, orthogonality, and the exact representation of polynomial to a certain degree and ability to represent functions at a different level of resolutions.

Legendre wavelets, Chebyshev wavelets, and Haar wavelets are the members of wavelet families. The numerical method based on Legendre polynomials have better convergence than the method based on Haar wavelet [28, 30, 32] and Chebyshev polynomials [20, 21] due to improved smoothness and good interpolating behavior. The practice of Legendre wavelets on the differential and integral equations deliberated in [25, 34]. Zheng and Yang [35] have used Legendre wavelet operational matrix to solve Lane-Emden equations and some integral equations.

Our aim in this paper is to solve system of generalized Emden-Fowler type differential equations

$$\begin{aligned} u^{''} + \frac{\alpha}{t}u^{'} + f_1(u,v) = f(t),\\ v^{''} + \frac{\beta}{t}v^{'} + f_2(u,v) = g(t), \end{aligned} \tag{5}$$

with the initial conditions

$$u(0) = v(0) = 1, u^{'}(0) = v^{'}(0) = 0. \tag{6}$$

We have used Legendre wavelet operational matrix of integration to deal the system of IVPs (5)-(6). Using Legendre wavelet operational matrix of integration, we have converted IVPs (5)-(6) in a wavelet equations, which can be discretised by Gallerkin or collocation method in subinterval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$. The major advantage of this method is to convert the problem into algebraic equations which facilitate greatly.

We organize the rest of the article in the following order. In section 2, we have discussed Legendre wavelets and computation of operational matrix of integration. Method for the solution of the IVPs (5)-(6) and product of the Legendre wavelet vector function has been discussed in section 3. In section 4, some numerical examples are considered to clarify the applicability and performance of the proposed method. In section 5, we have given conclusion of our work.

2. Introduction of Wavelet

Basic definitions of Legendre wavelet and its detailed properties are found in many papers such as [24, 34, 35]. Here we briefly introduce Legendre wavelets and its operational matrix.

2.1. **Legendre wavelets.** Legendre wavelet [34] in terms of Legendre polynomial $L_m(t)$ is defined as

$$\Psi_{nm}(t) = \begin{cases} (m+\frac{1}{2})^{1/2} 2^{k/2} L_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \le t \le \frac{\hat{n}+1}{2^k} \\ 0, & otherwise. \end{cases} \tag{7}$$

Here, $\hat{n} = 2n-1$, $n = 1, 2, 3, ..., 2^{(k-1)}$ and $m = 0, 1, 2, 3, ..., M-1$, m is the order of the Legendre Polynomial and M is maximum possible order of the Legendre polynomial. Legendre polynomials are orthogonal with respect to the weight function $w(t) = 1$, hold the following recurrence relations:

$$\begin{aligned} L_0(t) &= 1,\\ L_1(t) &= t,\\ L_{m+1}(t) &= \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t),\ m = 1, 2, 3, ... \end{aligned} \tag{8}$$

Now for $k = 2$ and $M = 3$, there exist six wavelet basis functions which are defined on the interval $[0, 1]$, as follows

$$\begin{aligned} \Psi_{10}(t) &= \sqrt{2},\\ \Psi_{11}(t) &= \sqrt{6}(4t-1),\ 0 \le t \le \frac{1}{2}\\ \Psi_{12}(t) &= \sqrt{10}\left(\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right), \end{aligned} \tag{9}$$

$$\begin{aligned}\Psi_{20}(t) &= \sqrt{2},\\ \Psi_{21}(t) &= \sqrt{6}(4t-3),\ \frac{1}{2}\leq t\leq 1\\ \Psi_{22}(t) &= \sqrt{10}\left(\frac{3}{2}(4t-3)^2-\frac{1}{2}\right).\end{aligned} \tag{10}$$

On the basis of the multi-resolution theory a function $g \in L_2[0,1]$ can be expanded as

$$g(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\Psi_{nm}(t), \tag{11}$$

where

$$c_{nm}(t)=\big\langle(t),\Psi_{nm}(t)\big\rangle. \tag{12}$$

Here, $\langle .,.\rangle$ represents the inner product on a vector space $L_2[0,1]$. Truncating (11), we have

$$g(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\Psi_{nm}(t)=C^T\Psi(t), \tag{13}$$

where $\Psi(t)$ and C are the matrices given by

$$C = [C_{10},C_{11},...,C_{1M-1},C_{20},...,C_{2M-1},...,C_{2^{k-1}0},...,C_{2^{k-1}M-1}]^T, \tag{14}$$

$$\begin{aligned}\Psi(t) = {}& [\Psi_{10}(t),\Psi_{11}(t),...,\Psi_{1M-1}(t),\Psi_{20}(t),...,\Psi_{2M-1}(t),...,\Psi_{2^{k-1}0}(t),...,\\ & \Psi_{2^{k-1}M-1}(t)]^T.\end{aligned} \tag{15}$$

2.2. **Operational matrix of integration of the Legendre wavelet.** In this section, Legendre wavelet operational matrix of integration [24, 35, 27] has been introduced on the interval $[0,1)$. Matrix of integration on the subintervals $[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}})$ of $[0,1]$ is obtained. These matrices are same in each subintervals and hence minimize the computational time. In order to get these matrices, integrate equation (9) from 0 to t for $t\in[0,\frac{1}{2})$ and equation (10) from $\frac{1}{2}$ to t for $t\in[\frac{1}{2},1)$. Now for $t\in[0,\frac{1}{2})$, we have

$$\begin{aligned}\int_0^t\Psi_{10}(t)dt &= \sqrt{2}t=\frac{1}{4}\Psi_{10}(t)+\frac{3^{\frac{1}{2}}}{12}\Psi_{11}(t),\\ \int_0^t\Psi_{11}(t)dt &= \sqrt{6}(2t^2-t)=-\frac{3^{\frac{1}{2}}}{12}\Psi_{10}(t)+\frac{15^{\frac{1}{2}}}{60}\Psi_{12}(t),\\ \int_0^t\Psi_{12}(t)dt &= \sqrt{10}(8t^3-6t^2+t)=-\frac{15^{\frac{1}{2}}}{60}\Psi_{11}(t)+\frac{35^{\frac{1}{2}}}{140}\Psi_{13}(t).\end{aligned}$$

Let $\Psi_1(t)=[\Psi_{10}(t),\Psi_{11}(t),\Psi_{12}(t)]^T$, then we have

$$\int_0^t\Psi_1(t)dt\approx B_{3\times3}\Psi_1(t), \tag{16}$$

where

$$B_{3\times3}=\frac{1}{4}\begin{pmatrix}1 & \frac{3^{\frac{1}{2}}}{3} & 0\\ -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3}\\ 0 & -\frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} & 0\end{pmatrix}. \tag{17}$$

Again for $t \in [1/2, 1)$, we have

$$\begin{aligned}
\int_{\frac{1}{2}}^{t} \Psi_{20}(t)dt &= \sqrt{2}t - \frac{\sqrt{2}}{2} = \frac{1}{4}\Psi_{20}(t) + \frac{3^{\frac{1}{2}}}{12}\Psi_{21}(t), \\
\int_{\frac{1}{2}}^{t} \Psi_{21}(t)dt &= \sqrt{6}(2t^2 - 3t + 1) = -\frac{3^{\frac{1}{2}}}{12}\Psi_{20}(t) + \frac{15^{\frac{1}{2}}}{60}\Psi_{22}(t), \\
\int_{\frac{1}{2}}^{t} \Psi_{22}(t)dt &= \sqrt{10}(8t^3 - 18t^2 + 13t - 3) = -\frac{15^{\frac{1}{2}}}{60}\Psi_{21}(t) + \frac{35^{\frac{1}{2}}}{140}\Psi_{23}(t).
\end{aligned}$$

Let $\Psi_2(t) = [\Psi_{20}(t), \Psi_{21}(t), \Psi_{22}(t)]^T$, then

$$\int_{\frac{1}{2}}^{t} \Psi_2(t)dt \approx B_{3\times3}\Psi_2(t), \tag{18}$$

where

$$B_{3\times3} = \frac{1}{4}\begin{pmatrix} 1 & \frac{3^{\frac{1}{2}}}{3} & 0 \\ -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} \\ 0 & -\frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} & 0 \end{pmatrix}. \tag{19}$$

Razzaghi and Yousefi [25] have constructed the Legendre wavelet operational matrix of integration on the complete interval $[0, 1]$. Hence for $t \in [0, 1]$, we have

$$\int_0^t \Psi_{6\times6}(t)dt = P_{6\times6}(t)\Psi_{6\times6}(t), \tag{20}$$

where

$$\Psi_{6\times6} = [\Psi_{10}(t), \Psi_{11}(t), \Psi_{12}(t), \Psi_{20}(t), \Psi_{21}(t), \Psi_{22}(t)]^T, \tag{21}$$

and

$$P_{6\times6} = \frac{1}{2^2}\begin{pmatrix} 1 & \frac{3^{\frac{1}{2}}}{3} & 0 & 2 & 0 & 0 \\ -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} & 0 & 0 & 0 \\ 0 & -\frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{3^{\frac{1}{2}}}{3} & 0 \\ 0 & 0 & 0 & -\frac{3^{\frac{1}{2}}}{3} & 0 & \frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} \\ 0 & 0 & 0 & 0 & -\frac{3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{5\times3} & 0 \end{pmatrix}. \tag{22}$$

The matrix $P_{6\times6}$ can also be expressed as

$$P_{6\times6} = \begin{pmatrix} B_{3\times3} & F_{3\times3} \\ O_{3\times3} & B_{3\times3} \end{pmatrix},$$

where $B_{3\times3}$ is given in (17) and

$$F_{3\times3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

General form of matrix $P_{N\times N}, N = 2^{k-1}M$, can be calculated on the interval $[0,1]$ using similar procedure. It is given by

$$P_{N\times N} = \frac{1}{2^k}\begin{pmatrix} B_{M\times M} & F_{M\times M} & F_{M\times M} & \cdots & F_{M\times M} \\ O_{M\times M} & B_{M\times 3} & F_{M\times M} & \cdots & F_{M\times M} \\ \vdots & O_{M\times M} & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & F_{M\times M} \\ O_{M\times M} & O_{M\times M} & \cdots & O_{M\times M} & B_{M\times M} \end{pmatrix},$$

where

$$B_{M\times M} = \begin{pmatrix} 1 & \frac{1}{3^{\frac{1}{2}}} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{3^{\frac{1}{2}}} & 0 & \frac{1}{3^{\frac{1}{2}}\times 5^{\frac{1}{2}}} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{3^{\frac{1}{2}}\times 5^{\frac{1}{2}}} & 0 & \frac{1}{5^{\frac{1}{2}}\times 7^{\frac{1}{2}}} & \ddots & 0 & 0 \\ 0 & 0 & -\frac{1}{5^{\frac{1}{2}}\times 7^{\frac{1}{2}}} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{(2M-3)^{\frac{1}{2}}\times(2M-1)^{\frac{1}{2}}} \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{(2M-3)^{\frac{1}{2}}\times(2M-1)^{\frac{1}{2}}} & 0 \end{pmatrix},$$

$$F_{M\times M} = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } O_{M\times M} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Here, the matrix $B_{M\times M}$ only depends upon the decomposition level k and order M of Legendre polynomials. The order of matrix $B_{M\times M}$ is M, while the order of $P_{N\times N}$ is $N = 2^{k-1}M$. The matrix $B_{M\times M}$ reduces the computational complexity and storage of the system of algebraic equations for the large value of the decomposition level k. Thus it is a new numerical algorithm based on an operational matrix of integration to deal with a system of differential equations.

3. Methodology

The method is based on converting differential equations into an integral equation through integration and approximation of antiderivative function by Legendre wavelets. The product operation of the Legendre wavelet vector function and an integral power of functions are computed on the subinterval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$. These operations convert the approximation into nonlinear system of algebraic equations. The solution of it gives the solution of the differential equation on the respective subinterval. The solution of the differential equation can be obtained by merging all these solutions on the subintervals. Let us assume that

$$u'' = C_0^T\Psi_1(t) \text{ and } v'' = D_0^T\Psi_1(t). \tag{23}$$

Here, C_0^T and D_0^T are the coefficient matrix of Legendre wavelet vector $\Psi_1(t)$, which is to be determined. Integrating (23) twice in the subinterval $[0, \frac{1}{2^{k-1}})$ and using the initial condition (6), we get

$$u' = C_0^T B\Psi_1(t) \text{ and } v' = D_0^T B\Psi_1(t) \tag{24}$$

and,

$$u = 1 + C_0^T B^2\Psi_1(t) \text{ and } v = 1 + D_0^T B^2\Psi_1(t). \tag{25}$$

The functions $t, tu'', tv'', \alpha u', \beta v', tf_1(u,v), tf_2(u,v), tf(t)$ and $tg(t)$ of (5) can be approximated with the help of product operation of Legendre wavelet vector function, equation (24) and (25) as follows:

$$\begin{aligned} t &= e^T\Psi_1(t), \\ tu'' &= e^T\Psi_1(t)\Psi_1^T C_0 = \Psi_1^T E C_0, \\ tv'' &= e^T\Psi_1(t)\Psi_1^T D_0 = \Psi_1^T E D_0, \\ \alpha u' &= r^T\Psi_1(t)\Psi_1^T B^T C_0 = \Psi_1^T R B^T C_0, \\ \beta v' &= s^T\Psi_1(t)\Psi_1^T B^T D_0 = \Psi_1^T S B^T D_0, \\ tf_1(u,v) &= e^T\Psi_1(t)\Psi_1^T F_1 = \Psi_1^T(t) E F_1, \\ tf_2(u,v) &= e^t\Psi_1(t)\Psi_1^T F_2 = \Psi_1^T(t) E F_2, \\ tf(t) &= e^T\Psi_1(t)\Psi_1^T F = \Psi_1^T(t) E F, \\ tg(t) &= e^T\Psi_1(t)\Psi_1^T G = \Psi_1^T(t) E G. \end{aligned} \tag{26}$$

Substituting (26) in IVPs (5)-(6), we have

$$\begin{aligned} &EC_0 + RB^T C_0 + EF_1 = EF, \\ &ED_0 + SB^T C_0 + EF_2 = EG. \end{aligned} \tag{27}$$

Here, the product of Legendre wavelet vectors [2] are defined as

$$e^T\Psi(t)\Psi^T(t) = \Psi^T(t)E, \tag{28}$$

where $e = [e_1, e_2, e_3]^T$ and E is $M \times M$ matrix. The product has demonstrated for $M = 3$ and $k = 2$ as

$$e^T\Psi(t)\Psi^T(t) = \Psi^T(t)\begin{pmatrix} \sqrt{2}e_1 & \sqrt{2}e_2 & \sqrt{2}e_3 \\ \sqrt{2}e_2 & \sqrt{2}e_1 + \frac{4}{\sqrt{10}}e_3 & \frac{4}{\sqrt{10}}e_2 \\ \sqrt{2}e_3 & \frac{4}{\sqrt{10}}e_2 & \sqrt{2}e_1 + \frac{20}{\sqrt{10}}e_3 \end{pmatrix}. \tag{29}$$

The general procedure for the product operation of the Legendre wavelet vector functions has demonstrated in [35].

Suppose $e^T\Psi(t)\Psi^T(t) = \Psi^T(t)\begin{pmatrix} e_{11} & \cdots & e_{1M} \\ \cdots & \cdots & \cdots \\ e_{M1} & \cdots & e_{MM} \end{pmatrix}$, then

$$e_{ij} = \sum_{p=1}^{p=M} \int_{\frac{m-1}{2^{k-1}}}^{\frac{m}{2^{k-1}}} \Psi(i,1)\Psi(j,1)\Psi(p,1)dt, \; i,j = 1,2,3,... \tag{30}$$

where the matrix e_{ij} is a symmetric matrix. Thus, the system of initial value problem (5)-(6) corresponds to a system of algebraic equations (27). The solution of the algebraic equations for C and D gives the approximate solution on the subinterval $[0, \frac{1}{2^{k-1}})$. Hence approximate solution of the system of IVPs (5)-(6) is given by

$$\begin{aligned} &u_0 = 1 + C_0^T B^2 \Psi_1(t), \\ &v_0 = 1 + D_0^T B^2 \Psi_1(t). \end{aligned} \tag{31}$$

In a similar fashion, we can obtain the solutions u_{n-1} and v_{n-1} on the subintervals $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ with the initial condition $u(\frac{n-1}{2^{k-1}}) \approx u(\frac{n-1}{2^{k-1}} - \Delta t)$, $u'(\frac{n-1}{2^{k-1}}) \approx \frac{u(0.5-\Delta t)-u(0.5-2\Delta t)}{\Delta t}$, $v(\frac{n-1}{2^{k-1}}) \approx v(\frac{n-1}{2^{k-1}} - \Delta t)$ and $v'(\frac{n-1}{2^{k-1}}) \approx \frac{v(0.5-\Delta t)-v(0.5-2\Delta t)}{\Delta t}$, $\Delta t = 1.0 \times 10^{-9}$, for $n =$

$2, 3, ..., 2^{k-1}$. Now combine $u_0, u_1, ..., u_{n-1}$ and $v_0, v_1, ..., v_{n-1}$ to get the approximate solution on the interval $[0, 1]$. We can also decrease Δt and increase the degree M of Legendre polynomial and the decomposition level k for the highly accurate desired solution.

We have analyzed the convergence of the Legendre wavelet operational matrix of integration using the following theorem.

Theorem 3.1. *Assume that the $u(t), v(t) \in L_2[0,1]$ with the bounded second order derivative i.e. $\exists\ K \in R$ such that $\max\{|u''(t)|, |v''(t)|\} \leq K$ then the error norm satisfy the following inequalities*

$$\begin{aligned} ||e_1(M,k)(t)|| &= \sup_{t\in[0,1]} |u(t) - u(M,k)(t)| \\ &\leq \frac{A}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}, \\ ||e_2(M,k)(t)|| &= \sup_{t\in[0,1]} |v(t) - v(M,k)(t)| \\ &\leq \frac{A}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}, \end{aligned}$$

where $A = \frac{\sqrt{5}K}{\sqrt{24}}$.

Proof. The error for the variable u is defined as

$$\begin{aligned} |e_1(M,k)(t)| &= |u(t) - u(M,k)(t)| \\ &= \left| \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}\Psi_{nm}(t) \right|, \end{aligned}$$

where

$$u(M,k)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\Psi_{nm}(t).$$

Hence

$$\begin{aligned} \|e_1(M,k)\|^2 &= \int_{-1}^{1} \left\langle \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}\Psi_{nm}(t), \sum_{p=2^{k-1}+1}^{\infty} \sum_{q=M}^{\infty} c_{pq}\Psi_{pq}(t) \right\rangle dt, \\ &= \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \sum_{p=2^{K-1}+1}^{\infty} \sum_{q=M}^{\infty} c_{nm}c_{pq} \int_{-1}^{1} \Psi_{nm}(t)\Psi_{pq}(t)dt, \end{aligned} \tag{32}$$

$$\|e_1(M,k)\|^2 \leq \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2, \tag{33}$$

Here,

$$\begin{aligned} c_{nm} &= \int_0^1 u(t)\Psi_{nm}(t)dt, \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u(t)\sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}L_m(2^k t-\hat{n})dt, \\ &= \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} u(t)L_m(2^k t-\hat{n})dt. \end{aligned}$$

Substituting $x = 2^k t - \hat{n}$, we get

$$\begin{aligned} c_{nm} = & \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}\int_{-1}^{1} u\left(\frac{x+\hat{n}}{2^k}\right)L_m(x)\frac{dx}{2^k}, \\ = & \frac{\sqrt{m+\frac{1}{2}}}{2^{\frac{k}{2}}}\int_{-1}^{1} u\left(\frac{x+\hat{n}}{2^k}\right)L_m(x)dx. \end{aligned} \tag{34}$$

Since,

$$(2m+1)L_m(x) = \left(L'_{m+1}(x) - L'_{m-1}(x)\right). \tag{35}$$

So, we have

$$c_{nm} = \frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}}\int_{-1}^{1} u\left(\frac{x+\hat{n}}{2^k}\right)d(L_{m+1}(x) - L_{m-1}(x)). \tag{36}$$

Consequently, integrating equation (36) by part, we have

$$\begin{aligned} c_{nm} &= \frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(u\left(\frac{x+\hat{n}}{2^k}\right)(L_{m+1}(x) - L_{m-1}(x))\right)_{-1}^{1}, \\ & -\frac{1}{2^{\frac{k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} u'\left(\frac{x+\hat{n}}{2^k}\right)\frac{1}{2^k}(L_{m+1}(x) - L_{m-1}(x))dx\right), \\ &= -\frac{1}{2^{\frac{3k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} u'\left(\frac{x+\hat{n}}{2^k}\right)(L_{m+1}(x) - L_{m-1}(x))dx\right). \end{aligned}$$

Now using equation (35), we have

$$c_{nm} = -\frac{1}{2^{\frac{3k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} u'\left(\frac{x+\hat{n}}{2^k}\right)d\left(\frac{L_{m+2}(x) - L_m(x)}{2m+3} - \frac{L_m(x) - L_{m-2}(x)}{2m-1}\right)\right).$$

Proceeding to a similar manner, we have

$$\begin{aligned} c_{nm} &= \frac{1}{2^{\frac{5k}{2}+1}\sqrt{m+\frac{1}{2}}}\left(\int_{-1}^{1} u''\left(\frac{x+\hat{n}}{2^k}\right)\left(\frac{L_{m+2}(x) - L_m(x)}{2m+3} - \frac{L_m(x) - L_{m-2}(x)}{2m-1}\right)dx\right), \\ &= \frac{1}{2^{\frac{5k}{2}+1}\sqrt{m+\frac{1}{2}}(2m+3)(2m-1)}\left(\int_{-1}^{1} u''\left(\frac{x+\hat{n}}{2^k}\right)T_m(x)dx\right), \end{aligned}$$

where $T_m(x) = (2m-1)L_{m+2}(x) - 2(2m+1)L_m(x) + (2m+3)L_{m-2}(x)$ and hence,

$$\begin{aligned} |c_{nm}| &\leq \frac{1}{2^{\frac{5k}{2}+1}\sqrt{m+\frac{1}{2}}(2m+3)(2m-1)} \int_{-1}^{1} \left| u''\left(\frac{x+\hat{n}}{2^k}\right)\right| |T_m(x)|dx \\ &\leq \frac{\sqrt{3}K}{2^{\frac{5k}{2}}(2m-3)^2}, \end{aligned} \tag{37}$$

where

$$|T_m(x)| \leq \sqrt{24}\frac{(2m+3)}{\sqrt{(2m-3)}}.$$

It is shown in [18]. Since $n \leq 2^k$, then the inequality (37) can be expressed as

$$|c_{nm}| \leq \frac{\sqrt{3}K}{n^{\frac{5}{2}}(2m-3)^2}. \tag{38}$$

Hence

$$\begin{aligned} \|e_1(M,k)\|^2 &\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} \frac{3K^2}{n^5(2m-3)^4} \\ &= 3K^2 \sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(2m-3)^4} \qquad (39) \\ &\leq \frac{5K^2}{24(2^{k-1}+1)^4(M-\frac{3}{2})^3}. \qquad (40) \end{aligned}$$

$$\|e_1(M,k)\| \leq \frac{A}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}, \tag{41}$$

where $A = \frac{\sqrt{5}K}{\sqrt{24}}$ is a real constant. Similarly, we can show

$$\|e_2(M,k)\| \leq \frac{A}{(2^{k-1}+1)^2(M-\frac{3}{2})^{\frac{3}{2}}}. \tag{42}$$

□

4. Numerical Experiment

In this section, we present four comparative examples to review the applicability and reliability of the method.

Example 4.1. Let us consider the system of linear Emden-Fowler equation

$$\begin{aligned} u'' + \frac{3}{t}u' - 4(u+v) = 0, \\ v'' + \frac{2}{t}v' + 3(u+v) = 0, \end{aligned} \tag{43}$$

with initial conditions

$$\begin{aligned} u(0) = 1,\ u'(0) = 0, \\ v(0) = 1,\ v'(0) = 0, \end{aligned} \tag{44}$$

which approximate a redder and very cool i.e. late type star[16, 20].
The exact solutions are given by $u(t) = 1+t^2$, and $v(t) = 1-t^2$. Now by using the technique

developed in section 3, the approximation of IVPs (43)-(44) in terms of Legendre wavelet is given by

$$\begin{aligned} &e\Psi(t)\Psi^T(t)C + 3\Psi^T(t)B^TC - 4e\Psi(t)\Psi^T(t)(F+G) = 0, \\ &e\Psi(t)\Psi^T(t)D + 2\Psi^T(t)B^TD + 3e\Psi(t)\Psi^T(t)(F+G) = 0. \end{aligned} \tag{45}$$

This can be approximate into a system of linear algebraic equations using product operations of Legendre wavelet vectors, which is given by

$$\begin{aligned} EC + 3B^TC - 4E(F+G) = 0, \\ ED + 2B^TD + 3E(F+G) = 0. \end{aligned} \tag{46}$$

On solving the system of linear algebraic equations (46), we have the approximate solution on the subinterval $[0, \frac{1}{2})$ for $M = 3$ and $k = 2$

$$\begin{aligned} u_0(t) &= 1 + C_0^T B^2 \Psi(t), \\ &= 1.000000000002 - 1.53541074794518 \times 10^{-10} * t + 0.99999999991011 * t^2, \\ &\cong 1 + t^2, \\ v_0(t) &= 1 + D_0^T B^2 \Psi(t) \\ &= 1.000000000001059 - 4.03923688374473 \times 10^{-10} * t - 1.0000000001442 * t^2, \\ &\cong 1 - t^2, \end{aligned}$$

where

$$\begin{aligned} C_0 &= [1.414213562, 2.717959249 \times 10^{-10}, -5.500145852 \times 10^{-10}]^T, \\ D_0 &= [-1.414213563, 1.038541152 \times 10^{-9}, -9.45736677 \times 10^{-10}]^T. \end{aligned}$$

In a similar way, on solving differential equations (43) on the interval $[\frac{1}{2}, 1)$, the numerical solutions $u_1(t)$ and $v_1(t)$ can be obtained using the initial conditions given by

$$\begin{aligned} u\left(\frac{1}{2}\right) = u\left(\frac{1}{2} - \Delta t\right),\ u'\left(\frac{1}{2}\right) = \frac{u(0.5-\Delta t) - u(0.5 - 2\Delta t)}{\Delta t}, \\ v\left(\frac{1}{2}\right) = v\left(\frac{1}{2} - \Delta t\right),\ v'\left(\frac{1}{2}\right) = \frac{v(0.5-\Delta t) - v(0.5 - 2\Delta t)}{\Delta t}, \end{aligned} \tag{47}$$

where $\Delta t = 1.0 \times 10^{-9}$. The approximate solution on the interval $[\frac{1}{2}, 1)$ is given by

$$\begin{aligned} u_1(t) &= u\left(\frac{1}{2}\right) + \left(t - \frac{1}{2}\right) u'\left(\frac{1}{2}\right) + C_1^T B^2 \Psi_2(t), \\ &= 1.0000000002 + 4.00000033096148 \times 10^{-10} * t + 1.00000000159999 * t^2, \\ &\cong 1 + t^2, \\ v_1(t) &= v\left(\frac{1}{2}\right) + \left(t - \frac{1}{2}\right) v'\left(\frac{1}{2}\right) + D_1^T B^2 \Psi_2(t), \\ &= 1.00000000005071 + 1.643224227879 \times 10^{-09} * t - 1.00000000037423 * t^2, \\ &\cong 1 - t^2, \end{aligned}$$

where

$$\begin{aligned} C_1 &= [1.414213564, -1.330183920 \times 10^{-9}, 5.933262504 \times 10^{-10}]^T, \\ D_1 &= [-1.414213563, 6.056012001 \times 10^{-10}, -2.183635061 \times 10^{-10}]^T. \end{aligned}$$

Hence, the approximate solutions are the same as the exact solutions.

Example 4.2. We consider the system of linear Emden-Fowler equation

$$\begin{aligned} u'' + \frac{3}{t}u' - 12t^2(u+v) = 0, \\ v'' + \frac{3}{t}v' + 12t^2(u+v) = 0, \end{aligned} \tag{48}$$

with initial conditions

$$\begin{aligned} u(0) = 1,\ u'(0) = 0, \\ v(0) = 1,\ v'(0) = 0. \end{aligned} \tag{49}$$

The exact solutions are $u(t) = 1 + t^4$, and $v(t) = 1 - t^4$. Now by using the technique developed in section 3, the approximation of IVPs (48)-(49) in terms of Legendre wavelet is given by

$$\begin{aligned} e^T\Psi_1(t)\Psi_1^T(t)C^T + g_1^T\Psi_1(t)\Psi_1^T(t)B^TC^T - e_1\Psi(t)\Psi^T(t)(F+G) = 0, \\ e^T\Psi_1(t)\Psi_1^T(t)D^T + g_1^T\Psi_1(t)\Psi_1^T(t)B^TD^T + e_1\Psi(t)\Psi^T(t)(F+G) = 0. \end{aligned} \tag{50}$$

Approximate (50) into a system of linear algebraic equations using product operations of Legendre wavelet vectors, we get

$$\begin{aligned} EC^T + G_1B^TC^T - E_1(F+G) = 0, \\ ED^T + G_1B^TD^T + E_1(F+G) = 0. \end{aligned} \tag{51}$$

On solving system of linear algebraic equation (51) for C and D, we have the approximate solution on the subinterval $[0, \frac{1}{2})$ for $M = 4$ and $k = 2$, given by

$$\begin{aligned} u_0(t) = 1 + C_0^TB^2\Psi(t), \\ v_0(t) = 1 + D_0^TB^2\Psi(t), \end{aligned} \tag{52}$$

where Legendre wavelet coefficients are given by

$$\begin{aligned} C_0 = & [0.70710678, 0.61237243, 0.15811388, -5.3075714 \times 10^{-10}]^T, \\ D_0 = & [-.70710678, -.61237243, -.15811388, -2.8124285 \times 10^{-10}]^T. \end{aligned} \tag{53}$$

In similar way the approximate solutions $u_1(t)$ and $v_1(t)$ on the subinterval $[\frac{1}{2}, 1)$ is obtained from the solution of system of differential equation (48) with boundary conditions

$$u\left(\frac{1}{2}\right) = u\left(\frac{1}{2} - \Delta t\right),\ u'\left(\frac{1}{2}\right) = \frac{u(0.5-\Delta t) - u(0.5 - 2\Delta t)}{\Delta t}, \tag{54}$$

$$v\left(\frac{1}{2}\right) = v\left(\frac{1}{2} - \Delta t\right),\ v'\left(\frac{1}{2}\right) = \frac{v(0.5-\Delta t) - v(0.5 - 2\Delta t)}{\Delta t}, \tag{55}$$

where $\Delta t = 1.0 \times 10^{-9}$. The Legendre wavelet coefficients corresponding to the solutions $u_1(t)$ and $v_1(t)$ are given by

$$\begin{aligned} C_1 = & [4.9939416, 1.8043116, 0.17223119, -0.47725221 \times 10^{-2}]^T, \\ D_1 = & [-4.9939416, -1.8043116, -0.17223119, 0.47725230 \times 10^{-2}]^T. \end{aligned} \tag{56}$$

Then, combining the solutions u_0, u_1 and v_0, v_1 on $[0, 1]$ yields the approximate solution of the IVPs (48)-(49). We have compared the approximate solution ($Leg(M, k)$) with the solution obtained from truncated shifted Chebyshev series method [21, 20] ($Cheb(N)$), Haar wavelet collocation method [30] ($HWCM(J)$) and exact solution ($Exact$) in the Tables 1-4. Graphs of ($Leg(M, k)$), ($Cheb(N)$), and exact solution are given in Figures 1-4. To ensure the accuracy and reliability of the method, the absolute errors $e_1 =$

$|Exact - Leg(M,k)|$, $e_2 = |Exact - Cheb(N)|$ and $e_3 = |Exact - HWCM(J)|$ are also shown in the Tables.

TABLE 1. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M = 4, k = 2$, $N = 3$ and $J = 2$ for u of Example 4.2.

t	$Exact$	$Leg(4,2)$	$Cheb(3)$	$HWCM(2)$	e_1	e_2	e_3
0.0	1.0000	0.9991	1.0000	1.0000	8.0×10^{-04}	2.1×10^{-10}	0.0000
0.2	1.0016	1.0013	0.9623	1.0015	2.0×10^{-04}	3.9×10^{-02}	7.6×10^{-05}
0.4	1.0256	1.0259	0.9228	1.0248	3.0×10^{-04}	1.0×10^{-01}	7.0×10^{-04}
0.6	1.1296	1.1263	0.9914	1.1284	3.2×10^{-03}	1.3×10^{-01}	1.1×10^{-03}
0.8	1.4096	1.4030	1.2782	1.4069	6.5×10^{-03}	1.3×10^{-01}	2.6×10^{-03}
1.0	2.0000	1.9914	1.8933	1.9960	8.5×10^{-03}	1.1×10^{-01}	3.9×10^{-03}

TABLE 2. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M = 4, k = 2$, $N = 3$ and $J = 2$ for v of Example 4.2.

t	$Exact$	$Leg(4,2)$	$Cheb(3)$	$HWCM(2)$	e_1	e_2	e_3
0.0	1.0000	1.0008	1.0000	1.0000	8.0×10^{-04}	0.0000	0.0000
0.2	0.9984	0.9986	1.0376	0.9984	2.0×10^{-04}	3.9×10^{-02}	7.6×10^{-05}
0.4	0.9744	0.9740	1.0771	0.9751	3.0×10^{-04}	1.0×10^{-01}	7.0×10^{-04}
0.6	0.8704	0.8735	1.0085	0.8715	3.2×10^{-03}	1.3×10^{-01}	1.1×10^{-03}
0.8	0.5904	0.5969	0.7217	0.5930	6.5×10^{-03}	1.3×10^{-01}	2.6×10^{-03}
1.0	0.0000	0.0085	0.1066	0.0039	8.5×10^{-03}	1.1×10^{-01}	3.9×10^{-03}

TABLE 3. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M = 5, k = 2$, $N = 4$ and $J = 3$ for u of Example 4.2.

t	$Exact$	$Leg(5,2)$	$Cheb(4)$	$HWCM(3)$	e_1	e_2	e_3
0.0	1.0000	1.0000	1.0000	1.0000	1.8×10^{-12}	9.9×10^{-11}	0.0000
0.2	1.0016	1.0016	1.0016	1.0015	2.6×10^{-11}	5.6×10^{-11}	4.7×10^{-05}
0.4	1.0256	1.0256	1.0256	1.0254	3.9×10^{-11}	3.2×10^{-10}	1.0×10^{-04}
0.6	1.1296	1.1295	1.1295	1.1292	7.1×10^{-10}	5.5×10^{-10}	3.0×10^{-04}
0.8	1.4096	1.4095	1.4095	1.4090	1.0×10^{-09}	8.2×10^{-10}	5.0×10^{-04}
1.0	2.0000	1.9999	1.9999	1.9990	1.1×10^{-09}	1.4×10^{-09}	9.0×10^{-04}

TABLE 4. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M = 5, k = 2$, $N = 4$ and $J = 3$ for v of Example 4.2.

t	$Exact$	$Leg(5,2)$	$Cheb(4)$	$HWCM(3)$	e_1	e_2	e_3
0.0	1.0000	1.0000	1.0000	1.0000	1.8×10^{-12}	1.0×10^{-10}	0.0000
0.2	0.9984	0.9984	0.9984	0.9984	1.7×10^{-11}	1.5×10^{-10}	4.7×10^{-05}
0.4	0.9744	0.9744	0.9744	0.9745	1.8×10^{-11}	1.7×10^{-10}	1.0×10^{-04}
0.6	0.8704	0.8704	0.8707	0.8707	8.4×10^{-10}	5.6×10^{-11}	3.0×10^{-04}
0.8	0.5904	0.5904	0.5904	0.5909	1.4×10^{-09}	1.5×10^{-10}	5.0×10^{-04}
1.0	0.0000	1.5×10^{-09}	-2.4×10^{-10}	0.0009	1.5×10^{-09}	2.4×10^{-10}	9.0×10^{-04}

Example 4.3. We consider the non-linear Emden-Fowler equation

$$\begin{aligned} u'' + \frac{2}{t}u' + v^2 &= t^6 - 2t^3 + 12t + 1, \\ v'' + \frac{3}{t}v' + u^2 &= t^6 + 2t^3 - 15t + 1, \end{aligned} \tag{57}$$

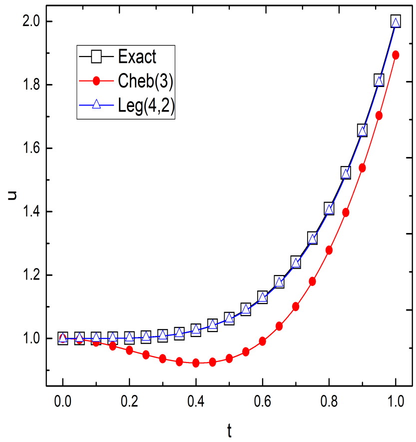

FIGURE 1. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 4, k = 2$ and $N = 3$ for u of Example 4.2.

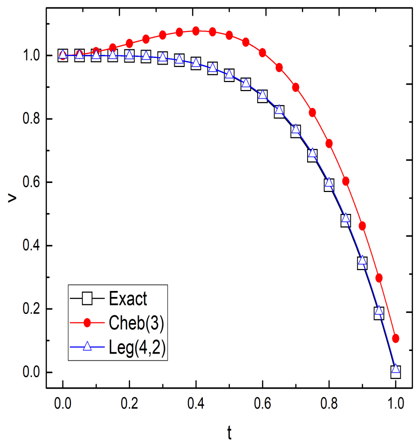

FIGURE 2. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 4, k = 2$ and $N = 3$ for v of Example 4.2.

with initial conditions

$$\begin{aligned} u(0) &= 1,\ u'(0) = 0, \\ v(0) &= 1,\ v'(0) = 0. \end{aligned} \tag{58}$$

The exact solutions of the IVPs (57)-(58) is $u(t) = 1 + t^3$, and $v(t) = 1 - t^3$. We have compared the approximate solution $Leg(M,k)$ with the solution obtained from truncated shifted chebyshev series method $Cheb(N)$, Haar wavelet collocation method $(HWCM(J))$ and exact solution $(Exact)$ in Tables 5-8. Graphs of $(Leg(M,k)), (Cheb(N))$, and exact solution are given in Figures 5-8. Tables and Figures show that approximate solution approaches exact solution with increase in the values of M and k.

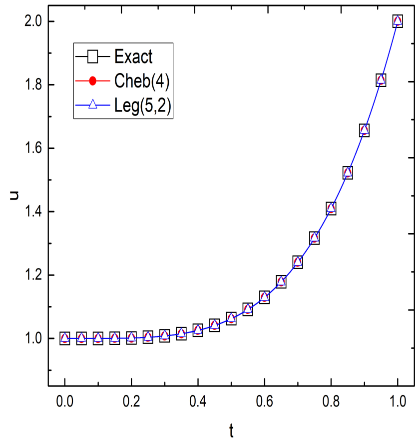

FIGURE 3. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 5, k = 2$ and $N = 4$ for u of Example 4.2.

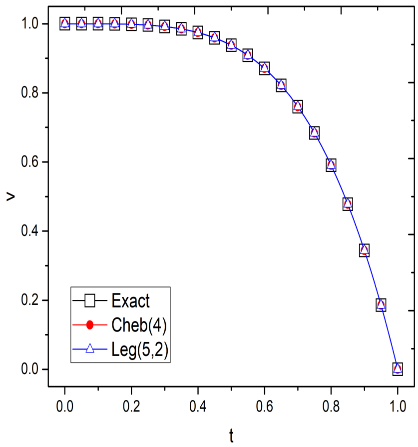

FIGURE 4. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 5, k = 2$ and $N = 4$ for v of Example 4.2.

Example 4.4. We consider the non linear Emden-Fowler equation

$$\begin{aligned} u'' + \frac{5}{t}u' + 8(e^{u} + 2e^{-\frac{v}{2}}) = 0, \\ v'' + \frac{3}{t}v' - 8(e^{-v} + e^{\frac{u}{2}}) = 0, \end{aligned} \tag{59}$$

with initial conditions

$$\begin{aligned} u(0) = 0,\ u'(0) = 0, \\ v(0) = 0,\ v'(0) = 0. \end{aligned} \tag{60}$$

Table 5. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M=3, k=2, N=2$ and $J=1$ for u of Example 4.3.

t	$Exact$	$Leg(3,2)$	$Cheb(2)$	$HWCM(1)$	e_1	e_2	e_3
0.0	1.0000	1.0062	1.0000	1.0000	6.2×10^{-03}	8.9×10^{-09}	0.0000
0.2	1.0080	1.0062	1.0117	1.0100	1.7×10^{-03}	3.7×10^{-03}	2.0×10^{-03}
0.4	1.0640	1.0662	1.0470	1.0602	2.2×10^{-03}	1.7×10^{-03}	3.7×10^{-03}
0.6	1.2160	1.1952	1.1057	1.2072	2.1×10^{-02}	1.1×10^{-01}	8.7×10^{-03}
0.8	1.5120	1.4782	1.1880	1.4988	3.3×10^{-02}	3.2×10^{-01}	1.3×10^{-02}
1.0	2.0000	1.9486	1.2938	1.9835	5.1×10^{-02}	7.1×10^{-01}	1.6×10^{-02}

Table 6. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M=3, k=2, N=2$ and $J=1$ for v of Example 4.3.

t	$Exact$	$Leg(3,2)$	$Cheb(2)$	$HWCM(1)$	e_1	e_2	e_3
0.0	1.0000	0.9937	1.0000	1.0000	6.2×10^{-03}	1.8×10^{-10}	0.0000
0.2	0.9920	0.9937	0.9989	0.9906	1.7×10^{-03}	3.0×10^{-03}	1.3×10^{-03}
0.4	0.9360	0.9337	0.9559	0.9413	2.2×10^{-03}	1.9×10^{-02}	5.3×10^{-03}
0.6	0.7840	0.8035	0.9008	0.7939	1.9×10^{-02}	1.2×10^{-01}	9.9×10^{-03}
0.8	0.4880	0.5173	0.8237	0.5020	2.9×10^{-02}	3.4×10^{-01}	1.4×10^{-02}
1.0	0.0000	0.0442	0.7246	0.0179	4.4×10^{-02}	7.2×10^{-01}	1.8×10^{-02}

Table 7. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M=5, k=3, N=4$ and $J=3$ for u of Example 4.3.

t	$Exact$	$Leg(5,3)$	$Cheb(4)$	$HWCM(3)$	e_1	e_2	e_3
0.0	1.0000	1.0000	1.0000	1.0000	1.1×10^{-10}	8.6×10^{-11}	0.0000
0.2	1.0080	1.0080	1.0080	1.0078	1.9×10^{-11}	7.3×10^{-11}	1.0×10^{-04}
0.4	1.0640	1.0639	1.0640	1.0636	1.4×10^{-09}	3.6×10^{-11}	3.0×10^{-04}
0.6	1.2160	1.2159	1.2160	1.2153	4.3×10^{-09}	2.7×10^{-11}	6.0×10^{-04}
0.8	1.5120	0.5119	0.5120	1.5111	8.8×10^{-09}	8.4×10^{-11}	8.0×10^{-04}
1.0	2.0000	1.9999	2.0000	1.9989	1.3×10^{-08}	7.4×10^{-11}	1.0×10^{-03}

Table 8. Comparison of $Leg(M,k)$, $Cheb(N)$ and $HWCM(J)$ with exact solution at $M=5, k=3, N=4$ and $J=3$ for v of Example 4.3.

t	$Exact$	$Leg(5,3)$	$Cheb(4)$	$HWCM(3)$	e_1	e_2	e_3
0.0	1.0000	1.0000	1.0000	1.0000	4.1×10^{-11}	7.9×10^{-12}	0.0000
0.2	0.9920	0.9920	0.9920	0.9922	8.4×10^{-11}	7.6×10^{-12}	2.0×10^{-04}
0.4	0.9360	0.9359	0.9360	0.9364	1.1×10^{-09}	7.0×10^{-12}	4.0×10^{-04}
0.6	0.7840	0.7839	0.7840	0.7846	1.6×10^{-09}	4.2×10^{-12}	6.0×10^{-04}
0.8	0.4880	0.4879	0.4880	0.4888	9.4×10^{-10}	1.1×10^{-11}	8.0×10^{-04}
1.0	0.0000	-2.2×10^{-09}	1.2×10^{-11}	0.0011	2.2×10^{-09}	1.2×10^{-11}	1.1×10^{-03}

The exact solution of the nonlinear IVPs (57)-(58) is $u=-2ln(1+t^2)$, and $v=2ln(1+t^2)$. We have compared the approximate solution $(Leg(M,k))$ with exact solution $(Exact)$ in the Tables 9-10 and Figures 9-10.

5. Conclusion

Wavelet analysis is an emerging advancement in the field of applied science. In this article, the applicability of the Legendre wavelet operational matrix of integration has been illustrated to obtain the solution of the system of generaized Emden-Fowler equations. The numerical results are compared with the exact solution, solution obtained from the truncated shifted Chebyshev series with the operation matrix method and Haar wavelet

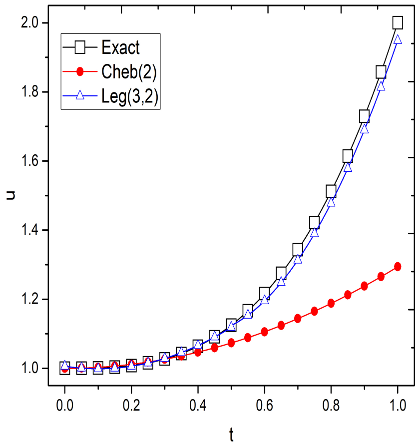

FIGURE 5. Comparison of $Leg(M, k)$, $Cheb(N)$ with exact solution at $M = 3, k = 2$ and $N = 2$ for u of Example 4.3.

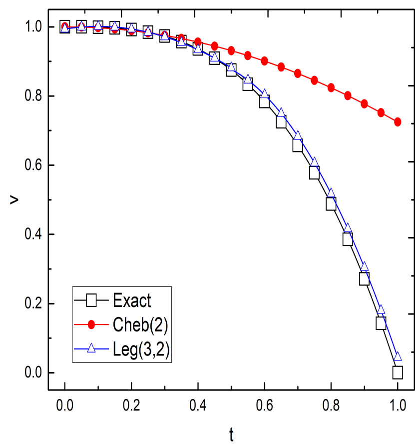

FIGURE 6. Comparison of $Leg(M, k)$, $Cheb(N)$ with exact solution at $M = 3, k = 2$ and $N = 2$ for v of Example 4.3.

TABLE 9. Comparison of $Leg(M, k)$ with exact solution of u in Example 4.4.

t	$Exact$	$Leg(3,2)$	e_1	$Leg(4,2)$	e_1	$Leg(6,2)$	e_1
0.0	0.0000	0.0046	4.6×10^{-03}	-0.0004	4.0×10^{-04}	3.81×10^{-05}	3.8×10^{-05}
0.2	-0.0784	-0.0799	1.5×10^{-03}	-0.0786	2.0×10^{-04}	-0.0784	3.1×10^{-05}
0.4	-0.2968	-0.2948	1.9×10^{-03}	-0.2966	1.0×10^{-04}	-0.2971	2.0×10^{-04}
0.6	-0.6149	-0.6287	1.3×10^{-02}	-0.6159	9.0×10^{-04}	-0.6143	6.0×10^{-04}
0.8	-0.9893	-1.0023	1.3×10^{-02}	-0.9895	1.0×10^{-04}	-0.9874	1.9×10^{-03}
1.0	-1.3862	-1.3928	6.5×10^{-03}	-1.3792	7.0×10^{-03}	-1.3776	8.6×10^{-03}

collocation method to show the efficiency and accuracy of the method. It is pointed out that the proposed method converges to the exact solution more speedily as compared to

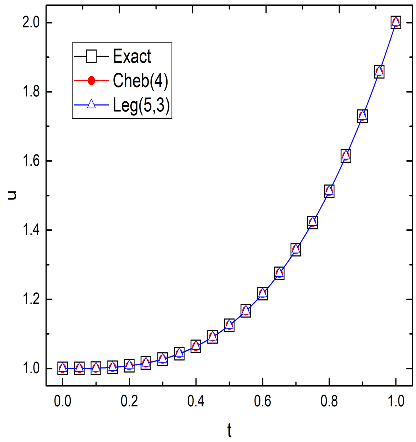

FIGURE 7. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 5, k = 3$ and $N = 4$ for u of Example 4.3.

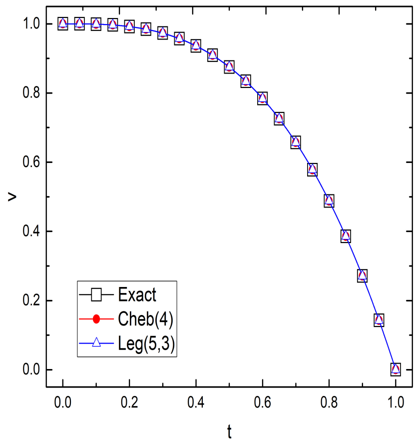

FIGURE 8. Comparison of $Leg(M,k)$, $Cheb(N)$ with exact solution at $M = 5, k = 3$ and $N = 4$ for v of Example 4.3.

TABLE 10. Comparison of $Leg(M,k)$ with exact solution of v in Example 4.4.

t	$Exact$	$Leg(3,2)$	e_1	$Leg(4,2)$	e_1	$Leg(6,2)$	e_1
0.0	0.0000	-0.0048	4.8×10^{-03}	0.0004	4.0×10^{-04}	-2.4×10^{-05}	2.4×10^{-05}
0.2	0.0784	0.0800	1.6×10^{-03}	0.0786	2.0×10^{-04}	0.0784	2.6×10^{-05}
0.4	0.2968	0.2943	2.4×10^{-03}	0.2963	4.0×10^{-04}	0.2968	8.4×10^{-05}
0.6	0.6149	0.6283	1.3×10^{-02}	0.6132	1.7×10^{-03}	0.6113	3.6×10^{-03}
0.8	0.9893	0.9923	2.9×10^{-03}	0.9747	1.5×10^{-02}	0.9715	1.7×10^{-02}
1.0	1.3862	1.3510	3.5×10^{-02}	1.3273	5.9×10^{-02}	1.3242	6.2×10^{-02}

other methods [20, 30] and reduces the computational work. It is also noticed that the proposed method has a prominent ability to deal with singular initial value problems due

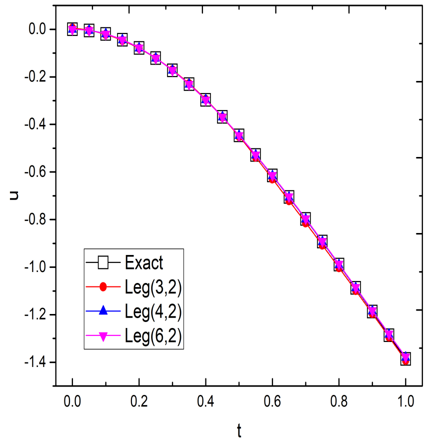

FIGURE 9. Comparison of $Leg(M,k)$ with exact solution of u in Example 4.4.

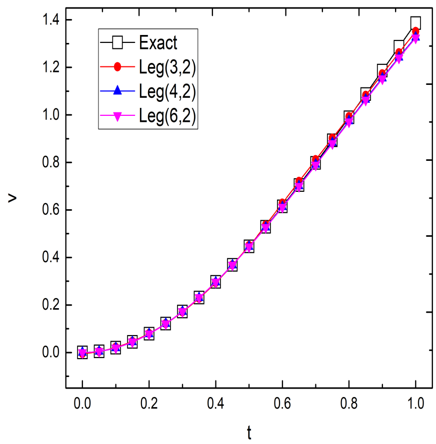

FIGURE 10. Comparison of $Leg(M,k)$ with exact solution of v in Example 4.4.

to its straightforward execution of the method. Hence, the present method is a more efficient, reliable and accurate numerical algorithm to solve the system of generalized Emden-Fowler equation with initial boundary conditions.

Acknowledgement. This research was supported by SERB, New Delhi (Grant No. ECR/2017/000560). The authors would like to extend their gratitude to anonymous referees for their valuable suggestions for the improvement of the presentation and content of this paper.

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