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SOME NEW GENERALIZED MODULAR RELATIONS

S. ALI[1*], M. K. MAHMOOD[2], M. S. HAMEED[1], §

Abstract. In view of the Rogers-Ramanujan theta function, several researchers have emphasized the subject of integer partitions and their generating functions for years. The work has not dealt with in an inclusive and closed-form by now. Thus, these generating functions are extremely desirable to be developed in a generalized way. In this paper, a novel methodology is proposed to build many new modular relations. By incorporating these new relations, generalized forms of regular partitions in the spirit of Rogers-Ramanujan and Göllnitz-Gordon functions with four, six, and nine dissections are established. As an application of these generalized generating functions, an infinite family of new congruences modulo 2 has also been developed.

Keywords: Modular relation, Rogers-Ramanujan functions, Göllnitz-Gordon functions, Even-odd method.

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1. Introduction

Without any loss, assume $|q| < 1$ and k is any integer greater or equal to one. The Pochhammer symbol is denoted by $(p;\ q)_k$ and defined by

$$(p;\ q)_k = \prod_{i=1}^{k}(1 - pq^{i-1}),\ (p;\ q)_0 = 1,$$

when $k \longrightarrow \infty$, then Pochhammer symbol becomes q-shifted factorial ($q-$Pochhammer symbol). It is denoted by $(p;\ q)_\infty$ and defined as

$$(p;\ q)_\infty = \prod_{i=1}^{\infty}(1 - pq^{i-1}) \text{ and } (p_1, p_2, \cdots, p_k; q)_\infty = (p_1;\ q)_\infty(p_2;\ q)_\infty \cdots (p_k;\ q)_\infty.$$

[1] Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, 64200, Pakistan.
e-mail: shahbaz.ali@kfueit.edu.pk; ORCID: https://orcid.org/0000-0002-5998-0053.
[*] Corresponding author.
e-mail: shazib.hameed.@kfueit.edu.pk; ORCID: https://orcid.org/0000-0002-1652-6811.
[2] Department of Mathematics, University of the Punjab, Lahore, 54590, Pakistan.
e-mail: khalid.math@pu.edu.pk; ORCID: https://orcid.org/0000-0002-1071-2808.
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The well known Rogers-Ramanujan functions are given below (for detail, see [15, 10]),

$$G(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;\ q)_k}=\frac{1}{(q;\ q^5)_\infty(q^4;\ q^5)_\infty}, \tag{1}$$

$$H(q)=\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;\ q)_k}=\frac{1}{(q^2;\ q^5)_\infty(q^3;\ q^5)_\infty}. \tag{2}$$

From Ramanujan forty identities, Rogers proved ten [11], Darling established one [9], Watson proved eight of which two were proved earlier [7], Bressoud derived fifteen in [6] and Biagioli [2], proved eight identifiers of remaining nine. Berndt et al. [3] proved 35 identities of Ramanujan's forty identities using relations (1) and (2). The rest of five identities have been proved in asymptotic expansions by applying $q \longrightarrow 1_-$ on either side. Göllnitz and Gordon specified two analogous to (1) and (2) in [8, 5]. The following two relations are called Göllnitz-Gordon functions.

$$S(q)=\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;\ q^2)_k}q^{k^2}=\frac{1}{(q;\ q^8)_\infty(q^4;\ q^8)_\infty(q^7;\ q^8)_\infty}, \tag{3}$$

$$T(q)=\sum_{k=0}^{\infty}\frac{(-q;q^2)_k}{(q^2;\ q^2)_k}q^{k(k+1)}=\frac{1}{(q^3;\ q^8)_\infty(q^4;\ q^8)_\infty(q^5;\ q^8)_\infty}. \tag{4}$$

Chen and Huang [13, 14] used the notion of Rogers, Bressoud and Waston [6, 7] established twenty one modular relations in $S(q), T(q)$ and nine in $S(q), T(q), G(q)$ and $H(q)$.
The partitioning problem of an integer is a hard NP-Complete problem. To find the number of partition of any positive integer exactly is an open problem. But, many researchers partially solve this problem. For as in [18]- [23], investigate the several clashes of integers based partitions of integers and then proposed their application in graph labeling.

In this paper, we give some new modular relations in $S(q), T(q), G(q)$ and $H(q)$ by using the notion of even-odd method. By incorporating these modular relations, few generalized forms of regular partition have been established in four, six and nine dissections in the spirit of Rogers-Ramanujan and Göllnitz-Gordon functions.

2. Preliminaries

In order to give some new modular relations, first, we collect some well-known definitions, lemmas, and theorems. The well-known Ramanujan's theta function is defined in [4], as

$$f(a,b)=\sum_{k=-\infty}^{\infty}a^{(k^2+k)/2}b^{(k^2-k)/2},\ |ab|<1. \tag{5}$$

The Jacobi triple product identity can be stated as

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{6}$$

The special case of (5) by replacing $a=-q$ and $b=-q^2$ is

$$f(-q):=f(-q,-q^2)=\sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2-k)/2}=(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty, \tag{7}$$

in whole manuscript, f_k means $f(-q^k)$ for each positive integer k.

Lemma 2.1. *The following 2-dissections holds:*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_{40} f_4^2}{f_8 f_{10}^2} \tag{8}$$

and

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{9}$$

Relations (8) and (9) have been proved in [17].
The following identity given by Ramanujan in [16] and Hirschhorn [12], with a simple proof by means of relation (6).

$$(q;q)_\infty = \frac{(q^{10}, q^{15}, q^{25}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} - q(q^{25}; q^{25})_\infty - q^2\frac{(q^5, q^{20}, q^{25}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty}, \tag{10}$$

Lemma 2.2. *The following two results are true:*

$$f_1 = f_4 S(-q^2) - q f_4 T(-q^2) \tag{11}$$

and

$$\frac{1}{f_1} = \frac{f_4^2}{f_2^3} S(-q^2) + q\frac{f_4^2}{f_2^3} T(-q^2). \tag{12}$$

Xia and Yao proved relations (11) and (12) in [17].

Lemma 2.3. [17]. *If $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c, ac^2d)f(b/d, acd^2). \tag{13}$$

3. Main Results

In this section, we propose some new relations involving Rogers-Ramanujan and Göllnitz-Gordon functions. Later, these relations are incorporated to establish general expression of $\frac{f_t}{f_1}$ for any positive integer t in 4−dissections, 6−dissections and 9−dissections.

Theorem 3.1. *The following relations in $G(q)$, $H(q)$, $S(q)$ and $T(q)$ are new.*

$$\frac{G(q)H(q)G(-q)H(-q)}{G^2(q^2)H^2(q^2)} = \frac{f_4f_{10}}{f_2f_{20}}, \tag{14}$$

$$\frac{H^2(-q)G^3(q^2)H(q^4) - G^2(-q)H^3(q^2)G(q^4)}{H(q)G(q)H(q^2)G(q^2)H(q^4)G(q^4)} = 2q\frac{f_2^2f_{20}^2}{f_4f_{10}^4}, \tag{15}$$

$$\frac{G^3(q^2)H(q^4)H^3(-q)G(-q) - H^3(q^2)G(q^4)G^3(-q)H(-q)}{H(q^4)G(q^4)H^3(q^2)G^3(q^2)} = 2q\frac{f_2f_{20}}{f_{10}^3}, \tag{16}$$

$$\frac{G^2(q)H^3(q^2)G(q^4) - H^2(q)G^3(q^2)H(q^4)}{H(-q)G(-q)H(q^2)G(q^2)H(q^4)G(q^4)} = 2q\frac{f_2^2f_{20}^2}{f_4f_{10}^4}, \tag{17}$$

$$\frac{H^3(q^2)G(q^4)G^3(q)H(q) - G^3(q^2)H(q^4)H^3(q)G(q)}{H(q^4)G(q^4)H^3(q^2)G^3(q^2)} = 2q\frac{f_2f_{20}}{f_{10}^3}, \tag{18}$$

$$\frac{G^2(q^{10})}{G^2(q^5)S(-q^2)} = \frac{f_4}{f_5}, \tag{19}$$

$$\frac{H^2(q^5)G^2(q^{10})T(-q^2) - qH^2(q^{10})G^2(q^5)S(-q^2)}{G^2(q^5)S(-q^2)H^2(q^5)} = \frac{f_{25}}{f_5}, \tag{20}$$

$$\frac{G^2(q^{10})}{G^2(-q^5)S(-q^2)} = \frac{f_4f_5f_{20}}{f_{10}^3}, \tag{21}$$

$$\frac{H^2(-q^5)G^2(q^{10})T(-q^2) - qH^2(q^{10})G^2(-q^5)S(-q^2)}{G^2(-q^5)S(-q^2)H^2(-q^5)} = \frac{f_5f_{20}f_{50}^3}{f_{10}^3f_{25}f_{100}}, \tag{22}$$

$$\frac{T^2(q)S^2(q^2) + qS^2(q)T^2(q^2)}{T^2(q^2)S^2(q^2)} = \frac{f_4^3f_8^2}{f_1f_{16}^2}, \tag{23}$$

$$\frac{T^2(-q)S^2(q^2) - qS^2(-q)T^2(q^2)}{T^2(q^2)S^2(q^2)} = \frac{f_1f_4^4f_8^2}{f_2^3f_{16}^2}, \tag{24}$$

$$\frac{G^2(q)G(q^4)H^3(q^2) - H^2(q)H(q^4)G^3(q^2)}{G^2(q^2)H^2(q^2)H(q^4)G(q^4)} = 2q\frac{f_1f_{20}^2}{f_5f_{10}^2}, \tag{25}$$

$$\frac{H^2(-q)H(q^4)G^3(q^2) - G^2(-q)G(q^4)H^3(q^2)}{G^2(q^2)H^2(q^2)H(q^4)G(q^4)} = 2q\frac{f_2^3f_5f_{20}^3}{f_1f_4f_{10}^5}, \tag{26}$$

$$\frac{\Big(G^2(q)H^2(-q) + H^2(q)G^2(-q)\Big)H(q^4)G^3(q^2)H^3(q^2)G(q^4)}{H^2(q^4)G^4(q^2)H^4(q^2)G^2(q^4)} - \frac{G^2(q)G^2(-q)G^2(q^4)H^6(q^2) + H^2(q)H^2(-q)H^2(q^4)G^6(q^2)}{H^2(q^4)G^4(q^2)H^4(q^2)G^2(q^4)} = \frac{4q^2f_2^3f_{20}^4}{f_4f_{10}^7}. \tag{27}$$

$$\frac{f_t}{f_1} = \frac{f_5f_8f_{20}^3f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_2^2f_{10}^3f_{40}f_{4t}f_{10t}^3f_{40t}} + q\frac{f_4^3f_5f_{40}f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_2^3f_8f_{10}^2f_{4t}f_{10t}^3f_{40t}} - q^t\frac{f_5f_8f_{20}^3f_{4t}^2f_{5t}f_{40t}}{f_2^2f_{10}^3f_{40}f_{8t}f_{10t}^2} - q^{t+1}\frac{f_4^3f_5f_{40}f_{4t}^2f_{5t}f_{40t}}{f_2^3f_8f_{10}^2f_{8t}f_{10t}^2}, \tag{28}$$

$$\begin{aligned}\frac{f_t}{f_1} &= \frac{f_8 f_{20}^2 f_{2t} f_{8t} f_{20t}^3}{f_2^2 f_5 f_{4t} f_{10t}^3 f_{40t}} + q\frac{f_4 f_{50}^3 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2^3 f_{25} f_{100} f_{4t} f_{10t}^3 f_{40t}} - q^2\frac{f_4^2 f_{10}^{10} f_{25}^2 f_{40} f_{100}^2 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2^4 f_5^3 f_8 f_{20}^6 f_{50}^4 f_{4t} f_{10t}^3 f_{40t}}\\ &\quad -q^t\frac{f_8 f_{20}^2 f_{4t}^2 f_{5t} f_{40t}}{f_2^2 f_5 f_{40} f_{8t} f_{10t}^2} - q^{t+1}\frac{f_4 f_{50}^3 f_{4t}^2 f_{5t} f_{40t}}{f_2^3 f_{25} f_{100} f_{8t} f_{10t}^2}\\ &\quad +q^{t+2}\frac{f_4^2 f_{10}^{10} f_{25}^2 f_{40} f_{100}^2 f_{4t}^2 f_{5t} f_{40t}}{f_2^4 f_5^3 f_8 f_{20}^6 f_{50}^4 f_{8t} f_{10t}^2},\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{f_t}{f_1} &= \frac{f_8 f_{20}^2 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2^2 f_5 f_{40} f_{4t} f_{10t}^3 f_{40t}} + q\frac{f_4 f_{50}^3 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2^3 f_{25} f_{100} f_{4t} f_{10t}^3 f_{40t}} - q^2\frac{f_4^2 f_{10}^{10} f_{25}^2 f_{40} f_{100}^2 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2^4 f_5^3 f_8 f_{20}^6 f_{50}^4 f_{4t} f_{10t}^3 f_{40t}}\\ &\quad -q^t\frac{f_8 f_{20}^2 f_{25t}}{f_2^2 f_5 f_{40}} - q^{t+1}\frac{f_4 f_{50}^3 f_{25t}}{f_2^3 f_{25} f_{100}} + q^{t+2}\frac{f_4^2 f_{10}^{10} f_{25}^2 f_{40} f_{100}^2 f_{25t}}{f_2^4 f_5^3 f_8 f_{20}^6 f_{50}^4}\\ &\quad -q^{2t}\frac{f_8 f_{20}^2 f_{4t} f_{10t}^7 f_{25t}^2 f_{40t} f_{100t}^2}{f_2^2 f_5 f_{40} f_{2t} f_{5t} f_{8t} f_{20t}^5 f_{50t}^4} - q^{2t+1}\frac{f_4 f_{50}^3 f_{4t} f_{10t} f_{5t}^2 f_{40t} f_{50t}^2}{f_2^3 f_{25} f_{100} f_{2t} f_{8t} f_{20t}^3 f_{25t}^2}\\ &\quad +q^{2t+2}\frac{f_4^2 f_{10}^{10} f_{25}^2 f_{40} f_{100}^2 f_{4t} f_{5t}^3 f_{10t} f_{40t} f_{50t}^2}{f_2^4 f_5^3 f_8 f_{20}^6 f_{50}^4 f_{2t} f_{8t} f_{20t}^3 f_{25t}^2},\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{f_1}{f_t} &= \frac{f_2 f_5 f_8 f_{20}^3 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_4 f_{10}^3 f_{40} f_{2t}^2 f_{10t}^3 f_{40t}} - q\frac{f_4^2 f_5 f_{40} f_{5t} f_{8t} f_{20t}^3}{f_8 f_{10}^2 f_{2t}^2 f_{10t}^3 f_{40t}} + q^t\frac{f_2 f_5 f_8 f_{20}^3 f_{4t}^3 f_{5t} f_{40t}}{f_4 f_{10}^3 f_{40} f_{2t}^3 f_{8t} f_{10t}^2}\\ &\quad -q^{t+1}\frac{f_4^2 f_5 f_{40} f_{4t}^3 f_{5t} f_{40t}}{f_8 f_{10}^2 f_{2t}^3 f_{8t} f_{10t}^2},\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{f_1}{f_t} &= \frac{f_2 f_5 f_8 f_{20}^3 f_{8t} f_{20t}^3}{f_4 f_{40} f_{2t}^2 f_{5t} f_{10t}^3 f_{20t} f_{40t}} - q\frac{f_4^2 f_5 f_{10} f_{40} f_{8t} f_{20t}^3}{f_8 f_{5t} f_{10t}^3 f_{20t} f_{40t}} + q^t\frac{f_2 f_5 f_8 f_{20}^3 f_{4t} f_{50t}^3}{f_4 f_{10}^3 f_{40} f_{2t}^3 f_{25t} f_{100t}}\\ &\quad -q^{t+1}\frac{f_4^2 f_5 f_{40} f_{4t} f_{50t}^3}{f_8 f_{10}^2 f_{2t}^3 f_{25t} f_{100t}} - q^{2t}\frac{f_2 f_5 f_8 f_{4t}^2 f_{10t}^{10} f_{25t}^2 f_{40t} f_{100t}^2}{f_4 f_{10}^3 f_{20}^2 f_{40} f_{50}^4 f_{2t}^4 f_{5t}^3 f_{8t}}\\ &\quad +q^{2t+1}\frac{f_4^2 f_5 f_{40} f_{4t}^2 f_{10t}^{10} f_{25t}^2 f_{40t} f_{100t}^2}{f_8 f_{10}^2 f_{2t}^4 f_{5t}^3 f_{2t}^4 f_{8t} f_{20t}^6 f_{50t}^4},\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{f_1}{f_t} &= \frac{f_2 f_5 f_8 f_{20}^3 f_{8t} f_{20t}^2}{f_4 f_{10}^3 f_{40} f_{2t}^2 f_{5t} f_{40t}} - q\frac{f_{25} f_{8t} f_{20t}^2}{f_{2t}^2 f_{5t} f_{40t}} - q^2\frac{f_4 f_5^3 f_{10} f_{40} f_{50}^2 f_{8t} f_{20t}^2}{f_2 f_8 f_{20}^3 f_{25}^2 f_{2t}^2 f_{5t} f_{40t}}\\ &\quad +q^t\frac{f_2 f_5 f_8 f_{20}^3 f_{4t} f_{50t}^3}{f_4 f_{10}^3 f_{40} f_{2t}^3 f_{25t} f_{100t}} - q^{t+1}\frac{f_{25} f_{4t} f_{50t}^3}{f_{2t}^3 f_{25t} f_{100t}} - q^{t+2}\frac{f_4 f_5^3 f_{10} f_{40} f_{50}^2 f_{4t} f_{50t}^3}{f_2 f_8 f_{20}^3 f_{25}^2 f_{2t}^3 f_{25t} f_{100t}}\\ &\quad -q^{2t}\frac{f_2 f_5 f_8 f_{20}^3 f_{4t}^2 f_{10t}^{10} f_{25t}^2 f_{40t} f_{100t}}{f_4 f_{10}^3 f_{40} f_{2t}^4 f_{5t}^3 f_{8t} f_{20t}^6 f_{50t}^4} + q^{2t+1}\frac{f_{25} f_{4t}^2 f_{10t}^{10} f_{25t}^2 f_{40t} f_{100t}^2}{f_{2t}^4 f_{5t}^3 f_{8t} f_{20t}^6 f_{50t}^4}\\ &\quad +q^{2t+2}\frac{f_4 f_5^3 f_{10} f_{40} f_{50}^2 f_{4t}^2 f_{10t}^{10} f_{25t}^2 f_{40t} f_{100t}^2}{f_2 f_8 f_{20}^3 f_{25}^2 f_{2t}^3 f_{5t}^3 f_{8t} f_{20t}^6 f_{50t}^4}.\end{aligned} \tag{33}$$

4. Some Useful Lemmas

To prove defining new modular relations in Theorem 3.1, for this we develop some lemmas in this section.

Lemma 4.1. *For $|q| < 1$, we have*

$$f(q, q^4) = f_5\frac{G(q)}{G(q^2)}, \ \ f(q^2, q^3) = f_5\frac{H(q)}{H(q^2)}. \tag{34}$$

and

$$f(q,q^7)=\frac{f_4f_{16}}{f_8}\frac{S(q)}{S(q^2)},\ f(q^3,q^5)=\frac{f_4f_{16}}{f_8}\frac{T(q)}{T(q^2)}. \tag{35}$$

Proof. By utilizing (1) − (4), we have

$$G(q)=\frac{1}{(q,q^4;\ q^5)_\infty}=\frac{(-q;\ q^5)_\infty(-q^4;\ q^5)_\infty}{(q^2;\ q^{10})_\infty(q^8;\ q^{10})_\infty}=\frac{f(q,q^4)}{f_5(q^2,q^8;\ q^{10})_\infty}=\frac{f(q,q^4)G(q^2)}{f_5},$$

$$H(q)=\frac{1}{(q^2,q^3;\ q^5)_\infty}=\frac{(-q^2;\ q^5)_\infty(-q^3;\ q^5)_\infty}{(q^4;\ q^{10})_\infty(q^6;\ q^{10})_\infty}=\frac{f(q^2,q^3)}{f_5(q^4,q^6;\ q^{10})_\infty}=\frac{f(q^2,q^3)H(q^2)}{f_5},$$

It is nothing, but just(34). Now

$$\begin{aligned}S(q)=\frac{1}{(q,q^4,q^7;q^8)_\infty}&=\frac{(-q;\ q^8)_\infty(-q^4;\ q^8)_\infty(-q^7;\ q^8)_\infty}{(q^2;\ q^{16})_\infty(q^8;\ q^{16})_\infty(q^{14};\ q^{16})_\infty}\\&=\frac{f(q,q^7)f_8}{f_4f_{16}(q^2,q^8,q^{14};\ q^{16})_\infty}=\frac{f(q,q^7)S(q^2)f_8}{f_4f_{16}},\\T(q)=\frac{1}{(q,q^4,q^5;q^8)_\infty}&=\frac{(-q^3;\ q^8)_\infty(-q^4;\ q^8)_\infty(-q^5;\ q^8)_\infty}{(q^6;\ q^{16})_\infty(q^8;\ q^{16})_\infty(q^{10};\ q^{16})_\infty}\\&=\frac{f(q^3,q^5)f_8}{f_4f_{16}(q^6,q^8,q^{10};\ q^{16})_\infty}=\frac{f(q^3,q^5)T(q^2)f_8}{f_4f_{16}}.\end{aligned}$$

□

Lemma 4.2. *For $|q|<1$, we have*

$$f(q,q^9)\ =\ \frac{1}{2q^2}\Big(\frac{f_5^2f_{10}H^2(q)}{f_{20}^2H^2(q^2)}-\frac{f_{10}^7H(q^2)}{f_5^2f_{20}^4H(q^4)}\Big) \tag{36}$$

and

$$f(q^3,q^7)\ =\ \frac{1}{2q}\Big(\frac{f_5^2f_{10}G^2(q)}{f_{20}^2G^2(q^2)}-\frac{f_{10}^7G(q^2)}{f_5^2f_{20}^4G(q^4)}\Big). \tag{37}$$

Proof. Putting, $a=c=q^2$ and $b=d=q^3$ in (13),

$$\begin{aligned}f^2(q^2,q^3)&=f(q^4,q^6)f(q^5,q^5)+q^2f(q,q^9)f(1,q^{10})\\f_5^2\frac{H^2(q)}{H^2(q^2)}&=f_{10}^2\frac{H(q^2)}{H(q^4)}(-q^5;\ q^{10})_\infty^2+2q^2f(q,q^9)\frac{f_{20}}{(q^{10};\ q^{20})_\infty}\\&=f_{10}^6\frac{H(q^2)}{f_5^2f_{20}^2H(q^4)}+2q^2f(q,q^9)\frac{f_{20}^2}{f_{10}},\end{aligned}$$

after some simplification, we have (36). Next take, $a=c=q$ and $b=d=q^4$ in (13),

$$\begin{aligned}f^2(q,q^4)&=f(q^2,q^8)f(q^5,q^5)+qf(q^3,q^7)f(1,q^{10})\\f_5^2\frac{G^2(q)}{G^2(q^2)}&=f_{10}^2\frac{G(q^2)}{G(q^4)}(-q^5;\ q^{10})_\infty^2+2qf(q^3,q^7)\frac{f_{20}}{(q^{10};\ q^{20})_\infty}\\&=f_{10}^6\frac{G(q^2)}{f_5^2f_{20}^2G(q^4)}+2qf(q^3,q^7)\frac{f_{20}^2}{f_{10}}.\end{aligned}$$

□

Lemma 4.3. *We have following two relations,*

$$\frac{f_1}{f_5} = \frac{f_{10}^2}{2qf_{20}^2}\Big(\frac{H(q^2)G^2(q)}{H(q^4)G^2(q^2)} - \frac{H^2(q)G(q^2)}{H^2(q^2)G(q^4)}\Big) \tag{38}$$

and

$$\frac{f_5}{f_1} = \frac{f_4 f_{10}^5}{2qf_2^3 f_{20}^3}\Big(\frac{H^2(-q)G(q^2)}{H^2(q^2)G(q^4)} - \frac{H(q^2)G^2(-q)}{H(q^4)G^2(q^2)}\Big). \tag{39}$$

Proof. We setting, $a = -q,\ b = -q^4,\ c = -q^2$ and $d = -q^3$ in (13),

$$f(-q,-q^4)f(-q^2,-q^3) = f(q^3,q^7)f(q^4,q^6) - qf(q^2,q^8)f(q,q^9)$$

by using (34), (35), (36) and (37), we have

$$f_1f_5 = f_{10}\frac{H(q^2)}{2qH(q^4)}\Big(\frac{f_5^2 f_{10}G^2(q)}{f_{20}^2G^2(q^2)} - \frac{f_{10}^7G(q^2)}{f_5^2f_{20}^4G(q^4)}\Big)$$
$$-qf_{10}\frac{G(q^2)}{2q^2G(q^4)}\Big(\frac{f_5^2 f_{10}H^2(q)}{f_{20}^2H^2(q^2)} - \frac{f_{10}^7H(q^2)}{f_5^2f_{20}^4H(q^4)}\Big).$$

$$f_1f_5 = \frac{f_{10}}{2q}\Big(\frac{f_5^2f_{10}}{f_{20}^2G^2}\frac{H(q^2)G^2(q)}{H(q^4)G^2(q^2)} - \frac{f_{10}^7}{f_5^2f_{20}^4}\frac{H(q^2)G(q^2)}{H(q^4)G(q^4)}$$
$$-\frac{f_5^2f_{10}}{f_{20}^2G^2}\frac{G(q^2)H^2(q)}{G(q^4)H^2(q^2)} + \frac{f_{10}^7}{f_5^2f_{20}^4}\frac{G(q^2)H(q^2)}{G(q^4)H(q^4)}\Big).$$

$$\frac{f_1}{f_5} = \frac{f_{10}^2}{2qf_{20}^2}\Big(\frac{H(q^2)G^2(q)}{H(q^4)G^2(q^2)} - \frac{G(q^2)H^2(q)}{G(q^4)H^2(q^2)}\Big).$$

by replacing q by $-q$ in (38) and using the relation,

$$\prod_{m=1}^{\infty}(1-(-q)^m) = \frac{f_2^3}{f_1f_4}, \tag{40}$$

in above equation we have (39). □

5. Proof of Theorem 3.1

Proof. By utilizing (34), we have

$$f(q,q^4)f(q^2,q^3) = f_5^2\frac{G(q)}{G(q^2)}\frac{H(q)}{H(q^2)},$$
$$\frac{f_2f_5^3}{f_1f_{10}} = f_5^2\frac{G(q)}{G(q^2)}\frac{H(q)}{H(q^2)},$$
$$\Rightarrow \frac{f_5}{f_1} = \frac{f_{10}G(q)H(q)}{f_2G(q^2)H(q^2)}, \tag{41}$$

in (41) replacing q by $-q$ and using the relation (40), we obtain

$$\frac{f_1}{f_5} = \frac{f_2^2f_{20}G(-q)H(-q)}{f_4f_{10}^2G(q^2)H(q^2)}. \tag{42}$$

By the product of (41) and (42),

$$1 = \frac{f_1}{f_5}\frac{f_5}{f_1} = \frac{f_2f_{20}G(-q)H(-q)G(q)H(q)}{f_4f_{10}G(q^2)H(q^2)G(q^2)H(q^2)}.$$

This is (14). By using (39) and (41),

$$\frac{f_{10}G(q)H(q)}{f_2G(q^2)H(q^2)} = \frac{f_4f_{10}^5}{2qf_2^3f_{20}^3}\Big(\frac{H^2(-q)G(q^2)}{H^2(q^2)G(q^4)} - \frac{H(q^2)G^2(-q)}{H(q^4)G^2(q^2)}\Big),$$

$$\frac{2qf_2^2f_{20}^3}{f_4f_{10}^4} = \Big(\frac{H^2(-q)G(q^2)}{H^2(q^2)G(q^4)} - \frac{H(q^2)G^2(-q)}{H(q^4)G^2(q^2)}\Big)\frac{G(q^2)H(q^2)}{G(q)H(q)},$$

$$\frac{H^2(-q)G^2(q^2)}{H(q^2)G(q^4)G(q)H(q)} - \frac{H^2(q^2)G^2(-q)}{H(q^4)G(q^2)G(q)H(q)} = \frac{2qf_2^2f_{20}^3}{f_4f_{10}^4},$$

after some simplification, we have (15). By the reciprocal of (42),

$$\frac{f_5}{f_1} = \frac{f_4f_{10}^2G(q^2)H(q^2)}{f_2^2f_{20}G(q)H(q)}, \tag{43}$$

by utilizing (39) and (43),

$$\frac{f_4f_{10}^2G(q^2)H(q^2)}{f_2^2f_{20}G(q)H(q)} = \frac{f_4f_{10}^5}{2qf_2^3f_{20}^2}\Big(\frac{H^2(-q)G(q^2)}{H^2(q^2)G(q^4)} - \frac{H(q^2)G^2(-q)}{H(q^4)G^2(q^2)}\Big),$$

$$\frac{H^2(-q)G(q^2)G(-q)H(-q)}{H^2(q^2)G(q^4)G(q^2)H(q^2)} - \frac{H(q^2)G^2(-q)G(-q)H(-q)}{H(q^4)G^2(q^2)G(q^2)H(q^2)} = \frac{2qf_2f_{20}}{f_{10}^3},$$

after some simplification, we get (16). Replacing by q by $-q$ in (15) and (16) and using the relation (40), we get (17) and (18) respectively. By (10),

$$f_1 = \frac{(q^{10},q^{15},q^{25};\ q^{25})_\infty}{(q^5,q^{20};\ q^{25})_\infty} - qf_{25} - q^2\frac{(q^5,q^{20},q^{25};\ q^{25})_\infty}{(q^{10},q^{15};\ q^{25})_\infty},$$

$$f_1 = \frac{f_5f_{25}^2}{f^2(q^5,\ q^{20})} - qf_{25} - q^2\frac{f_5f_{25}^2}{f^2(q^{10},\ q^{15})},$$

by using(34),

$$f_1 = \frac{G^2(q^{10})f_5}{G^2(q^5)} - qf_{25} - q^2\frac{H^2(q^{10})f_5}{H^2(q^5)}, \tag{44}$$

by (11) and (44),

$$\begin{aligned} f_4S(-q^2) - qf_4T(-q^2) &= \frac{G^2(q^{10})f_5}{G^2(q^5)} - qf_{25} - q^2\frac{H^2(q^{10})f_5}{H^2(q^5)}, \\ &= \frac{G^2(q^{10})f_5}{G^2(q^5)} - q\Big(f_{25} + q\frac{H^2(q^{10})f_5}{H^2(q^5)}\Big), \end{aligned} \tag{45}$$

equating even and odd parts of (45),

$$f_4S(-q^2) = \frac{G^2(q^{10})f_5}{G^2(q^5)},\ \Rightarrow\ \frac{G^2(q^{10})}{G^2(q^5)S(-q^2)} = \frac{f_4}{f_5}$$

and

$$f_{25} + q\frac{H^2(q^{10})f_5}{H^2(q^5)} = f_4T(-q^2),\ \Rightarrow\ \frac{f_{25}}{f_5} = \frac{f_4T(-q^2)}{f_5} - q\frac{H^2(q^{10})}{H^2(q^5)}. \tag{46}$$

by using (19) and (46), we have (20). Setting q by $-q$ in (19), (20) and using relation (40), we have (21), (22) respectively. We set, $a = q,\ b = q^3,\ c = d = q^2$ in (13)

$$f(q,q^3)f(q^2,q^2) = f^2(q^3,q^5) + qf^2(q,q^7), \tag{47}$$

using (35) in (47),

$$\begin{aligned} \frac{f_4^5}{f_1} &= \frac{f_4^2 f_{16}}{f_8^2}\frac{T^2(q)}{T^2(q^2)} + q\frac{f_4^2 f_{16}}{f_8^2}\frac{S^2(q)}{S^2(q^2)}, \\ \Rightarrow \frac{f_4^3 f_8^2}{f_1 f_{16}^2} &= \frac{T^2(q)}{T^2(q^2)} + q\frac{S^2(q)}{S^2(q^2)}. \end{aligned} \tag{48}$$

This is (23), after replacing q by $-q$ and utilizing relation (40) in (48), then we have (24). By setting $a = -q,\ b = -q^4,\ c = -q^2,\ d = -q^3$ in (13), we have

$$f(-q,-q^4)f(-q^2,-q^3) = f(q^3,q^7)f(q^4,q^6) - qf(q^2,q^8)f(q,q^9), \tag{49}$$

utilizing (34), (36), (37) in (49),

$$2qf_1f_5 = \frac{f_5^2f_{10}^2G^2(q)H(q^2)}{f_{20}^2G^2(q^2)H(q^4)} - \frac{f_{10}^8G(q^2)H(q^2)}{f_5^2f_{20}^4G(q^4)H(q^4)} - \frac{f_5^2f_{10}^2H^2(q)G(q^2)}{f_{20}^2H^2(q^2)G(q^4)} + \frac{f_{10}^8G(q^2)H(q^2)}{f_5^2f_{20}^4G(q^4)H(q^4)}, \tag{50}$$

after some simplification, we have (25) and then replacing q by $-q$ in (25) and using (40), we get (26). By the product of (38) and (39),

$$1 = \frac{f_1}{f_5}\frac{f_5}{f_1} = \frac{f_4f_{10}^7}{4q^2f_2^3f_{20}^4}\Big(\frac{H(q^2)G^2(q)}{H(q^4)G^2(q^2)} - \frac{H^2(q)G(q^2)}{H^2(q^2)G(q^4)}\Big)\Big(\frac{H^2(-q)G(q^2)}{H^2(q^2)G(q^4)} - \frac{H(q^2)G^2(-q)}{H(q^4)G^2(q^2)}\Big),$$

this yields,

$$\begin{aligned} &\frac{G^2(q)H^2(-q)}{H(q^4)G(q^2)H(q^2)G(q^4)} - \frac{H^2(q^2)G^2(q)G^2(-q)}{H^2(q^4)G^4(q^2)} \\ &-\frac{H^2(q)H^2(-q)G^2(q^2)}{H^4(q^2)G^2(q^4)} + \frac{H^2(q)G^2(-q)}{H(q^4)G(q^2)H(q^2)G(q^4)} = \frac{4q^2f_2^3f_{20}^4}{f_4f_{10}^7}. \end{aligned}$$

This is (27) after simplification. We replacing q by q^t in (11),

$$f_t = f_{4t}S(-q^{2t}) - q^tf_{4t}T(-q^{2t}), \tag{51}$$

taking product of (12) and (51),

$$\begin{aligned} \frac{f_t}{f_1} &= \frac{f_4^2f_{4t}}{f_2^3}S(-q^{2t})S(-q^2) + q\frac{f_4^2f_{4t}}{f_2^3}S(-q^{2t})T(-q^2) \\ &-q^t\frac{f_4^2f_{4t}}{f_2^3}S(-q^2)T(-q^{2t}) - q^{t+1}\frac{f_4^2f_{4t}}{f_2^3}T(-q^{2t})T(-q^2), \end{aligned} \tag{52}$$

by (44), we have

$$\frac{f_1}{f_5} = \frac{G^2(q^{10})}{G^2(q^5)} - q\ \frac{f_{25}}{f_5} - q^2\ \frac{H^2(q^{10})}{H^2(q^5)}, \tag{53}$$

by using (38) and (53), we have

$$\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}} - q\frac{f_{40}f_4^2}{f_8f_{10}^2} = \frac{G^2(q^{10})}{G^2(q^5)} - q\Big(\frac{f_{25}}{f_5} + q\ \frac{H^2(q^{10})}{H^2(q^5)}\Big), \tag{54}$$

equating the even and odd parts of (54), we obtain

$$\frac{G^2(q^{10})}{G^2(q^5)} = \frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}} \tag{55}$$

and

$$\frac{f_{25}}{f_5} + q\frac{H^2(q^{10})}{H^2(q^5)} = \frac{f_{40}f_4^2}{f_8f_{10}^2}, \tag{56}$$

from (19) and (20),

$$\frac{f_{25}}{f_5}=\frac{f_4}{f_5}T(-q^2)-q\frac{H^2(q^{10})}{H^2(q^5)} \tag{57}$$

by using (57) in (56), we have

$$\frac{f_4}{f_5}T(-q^2)=\frac{f_{40}f_4^2}{f_8f_{10}^2}\ \Rightarrow\ T(-q^2)=\frac{f_4f_5f_{40}}{f_8f_{10}^2}, \tag{58}$$

replacing q by q^t in (58), we get

$$T(-q^{2t})=\frac{f_{4t}f_{5t}f_{40t}}{f_{8t}f_{10t}^2}, \tag{59}$$

by (19) and (55),

$$S(-q^2)=\frac{f_2f_5f_8f_{20}^3}{f_4^2f_{10}^3f_{40}}, \tag{60}$$

(60) becomes, after replacing q by q^t

$$S(-q^{2t})=\frac{f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_{4t}^2f_{10t}^3f_{40t}}, \tag{61}$$

by using (58)-(61), then we have following relations

$$S(-q^2)S(-q^{2t})=\frac{f_2f_5f_8f_{20}^3f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_4^2f_{10}^3f_{40}f_{4t}^2f_{10t}^3f_{40t}},\ T(-q^2)T(-q^{2t})=\frac{f_4f_5f_{40}f_{4t}f_{5t}f_{40t}}{f_8f_{10}^2f_{8t}f_{10t}^2} \tag{62}$$

$$S(-q^{2t})T(-q^2)=\frac{f_4f_5f_{40}f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_8f_{10}^2f_{4t}^2f_{10t}^3f_{40t}},\ S(-q^2)T(-q^{2t})=\frac{f_2f_5f_8f_{20}^3f_{4t}f_{5t}f_{40t}}{f_4^2f_{10}^3f_{40}f_{8t}f_{10t}^2} \tag{63}$$

by using (62) and (63) in (52),

$$\begin{aligned}\frac{f_t}{f_1}&=\frac{f_4^2f_{4t}}{f_2^3}\Big(\frac{f_2f_5f_8f_{20}^3f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_4^2f_{10}^3f_{40}f_{4t}^2f_{10t}^3f_{40t}}\Big)+q\frac{f_4^2f_{4t}}{f_2^3}\Big(\frac{f_4f_5f_{40}f_{2t}f_{5t}f_{8t}f_{20t}^3}{f_8f_{10}^2f_{4t}^2f_{10t}^3f_{40t}}\Big)\\&-q^t\frac{f_4^2f_{4t}}{f_2^3}\Big(\frac{f_2f_5f_8f_{20}^3f_{4t}f_{5t}f_{40t}}{f_4^2f_{10}^3f_{40}f_{8t}f_{10t}^2}\Big)-q^{t+1}\frac{f_4^2f_{4t}}{f_2^3}\Big(\frac{f_4f_5f_{40}f_{4t}f_{5t}f_{40t}}{f_8f_{10}^2f_{8t}f_{10t}^2}\Big),\end{aligned}$$

this is nothing but just (28). By replacing q by $-q$ in (44) and using the relation (40), we have

$$\frac{1}{f_1}=\frac{G^2(q^{10})f_4f_{10}^3}{G^2(-q^5)f_2^3f_5f_{20}}+q\frac{f_4f_{50}^3}{f_2^3f_{25}f_{100}}-q^2\frac{H^2(q^{10})f_4f_{10}^3}{H^2(-q^5)f_2^3f_5f_{20}}, \tag{64}$$

by the product of (51) and (64), we have

$$\begin{aligned}\frac{f_t}{f_1}\ &=\ \frac{f_4f_{10}^3f_{4t}G^2(q^{10})S(-q^{2t})}{f_2^3f_5f_{20}G^2(-q^5)}+q\frac{f_4f_{50}^3f_{4t}S(-q^{2t})}{f_2^3f_{25}f_{100}}-q^2\frac{f_4f_{10}^3f_{4t}H^2(q^{10})S(-q^{2t})}{f_2^3f_5f_{20}H^2(-q^5)}\\&-q^t\frac{f_4f_{10}^3f_{4t}G^2(q^{10})T(-q^{2t})}{f_2^3f_5f_{20}G^2(-q^5)}-q^{t+1}\frac{f_4f_{50}^3f_{4t}T(-q^{2t})}{f_2^3f_{25}f_{100}}\\&+q^{t+2}\frac{f_4f_{10}^3f_{4t}H^2(q^{10})T(-q^{2t})}{f_2^3f_5f_{20}H^2(-q^5)},\end{aligned} \tag{65}$$

from (41),

$$\frac{H(q^2)}{H(q)}=\frac{f_1f_{10}}{f_2f_5}\frac{G(q)}{G(q^2)}, \tag{66}$$

by replacing q by q^5 in (66), then replacing q by $-q$ and using the relation (40),

$$\frac{H(q^{10})}{H(q^5)} = \frac{f_5 f_{50}}{f_{10} f_{25}} \frac{G(q^5)}{G(q^{10})} \Rightarrow \frac{H(q^{10})}{H(-q^5)} = \frac{f_{10}^2 f_{25} f_{100}}{f_5 f_{20} f_{50^2}} \frac{G(-q^5)}{G(q^{10})}, \tag{67}$$

by taking square of (67), then using the reciprocal of (55) to replacing q by $-q$,

$$\frac{H^2(q^{10})}{H^2(-q^5)} = \frac{f_4 f_{10}^7 f_{25}^2 f_{40} f_{100}^2}{f_2 f_5^2 f_8 f_{20}^5 f_{50}^4}, \tag{68}$$

by replacing q by $-q$ in (55),

$$\frac{G^2(q^{10})}{G^2(-q^5)} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}}, \tag{69}$$

utilizing (59), (61), (68) and (69) we have the following relations,

$$\frac{G^2(q^{10})}{G^2(-q^5)} S(-q^{2t}) = \frac{f_2 f_8 f_{20}^3 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_4 f_{10}^3 f_{40} f_{4t}^2 f_{10t}^3 f_{40t}}, \tag{70}$$

$$\frac{G^2(q^{10})}{G^2(-q^5)} T(-q^{2t}) = \frac{f_2 f_8 f_{20}^3 f_{4t} f_{5t} f_{40t}}{f_4 f_{10}^3 f_{40} f_{8t} f_{10t}^2}, \tag{71}$$

$$\frac{H^2(q^{10})}{H^2(-q^5)} T(-q^{2t}) = \frac{f_4 f_{10}^7 f_{25}^2 f_{40} f_{100}^2 f_{4t} f_{5t} f_{40t}}{f_2 f_5^2 f_8 f_{20}^5 f_{50}^4 f_{8t} f_{10t}^2}, \tag{72}$$

$$\frac{H^2(q^{10})}{H^2(-q^5)} S(-q^{2t}) = \frac{f_4 f_{10}^7 f_{25}^2 f_{40} f_{100}^2 f_{2t} f_{5t} f_{8t} f_{20t}^3}{f_2 f_5^2 f_8 f_{20}^5 f_{50}^4 f_{4t}^2 f_{10t}^3 f_{40t}}. \tag{73}$$

Using (59), (61) and (70)– (73) in (65), then after a little bit simplification, we have (29). In (44), replace q by q^t,

$$f_t = \frac{G^2(q^{10t}) f_{5t}}{G^2(q^{5t})} - q f_{25t} - q^{2t} \frac{H^2(q^{10t}) f_{5t}}{H^2(q^{5t})}. \tag{74}$$

By the product of (64) and (74),

$$\begin{aligned} \frac{f_t}{f_1} = {} & \frac{G^2(q^{10t}) G^2(q^{10}) f_4 f_{5t} f_{10}^3}{G^2(q^{5t}) G^2(-q^5) f_2^3 f_5 f_{20}} + q \frac{G^2(q^{10t}) f_4 f_{50}^3 f_{5t}}{G^2(q^{5t}) f_2^3 f_{25} f_{100}} - q^2 \frac{G^2(q^{10t}) H^2(q^{10}) f_4 f_{10}^3 f_{5t}}{G^2(q^{5t}) H^2(-q^5) f_2^3 f_5 f_{20}}, \\ & -q^t \frac{G^2(q^{10}) f_4 f_{10}^3 f_{5t}}{G^2(-q^{5t}) f_2^3 f_5 f_{20}} - q^{t+1} \frac{f_4 f_{50}^3 f_{25t}}{f_2^3 f_{25} f_{100}} + q^{t+2} \frac{H^2(q^{10}) f_4 f_{10}^3 f_{5t}}{H^2(-q^{5t}) f_2^3 f_5 f_{20}}, \\ & -q^{2t} \frac{G^2(q^{10}) H^2(q^{10t}) f_4 f_{10}^3 f_{5t}}{G^2(q^{-5}) H^2(q^{5t}) f_2^3 f_5 f_{20}} - q^{2t+1} \frac{H^2(q^{10t}) f_4 f_{50}^3 f_{5t}}{H^2(q^{5t}) f_2^3 f_{25} f_{100}}, \\ & +q^{2t+1} \frac{H^2(q^{10t}) H^2(q^{10}) f_4 f_{5t} f_{10}^3}{H^2(q^{5t}) H^2(-q^5) f_2^3 f_5 f_{20}}, \end{aligned} \tag{75}$$

by replacing q by $-q$ in (68) and using the relation (40),

$$\frac{H^2(q^{10})}{H^2(q^5)} = \frac{f_4 f_5^2 f_{10} f_{40} f_{50}^2}{f_2 f_8 f_{20}^3 f_{25}^2}. \tag{76}$$

Replace q by q^t in (55), (68), (69) and(76), we have

$$\frac{G^2(q^{10t})}{G^2(q^{5t})} = \frac{G^2(q^{10t})}{G^2(-q^{5t})} = \frac{f_{2t} f_8 f_{20t}^3}{f_{4t} f_{10}^{3t} f_{40t}}, \tag{77}$$

$$\frac{H^2(q^{10t})}{H^2(q^{5t})} = \frac{f_{4t} f_{10t} f_{5t}^2 f_{40t} f_{50t}^2}{f_{2t} f_{8t} f_{20t}^3 f_{25t}^2}, \quad \frac{H^2(q^{10t})}{H^2(-q^{5t})} = \frac{f_{4t} f_{10}^{7t} f_{25t}^2 f_{40t} f_{100t}^2}{f_{2t} f_{5t}^2 f_{8t} f_{20t}^5 f_{50t}^4}. \tag{78}$$

By using (68), (69), (77) and (78), we have following relations.

$$\frac{G^2(q^{10t})G^2(q^{10})}{G^2(q^{5t})G^2(-q^5)} = \frac{f_2f_8f_{20}^3f_{2t}f_{8t}f_{20t}^3}{f_4f_{10}^3f_{40}f_{4t}f_{10t}^3f_{40t}}, \tag{79}$$

$$\frac{G^2(q^{10t})H^2(q^{10})}{G^2(q^{5t})H^2(-q^5)} = \frac{f_4f_{10}^7f_{25}^2f_{40}f_{100}^2f_{2t}f_{8t}f_{20t}^3}{f_2f_5^2f_8f_{20}^5f_{50}^4f_{4t}f_{10t}^3f_{40t}}, \tag{80}$$

$$\frac{G^2(q^{10})H^2(q^{10t})}{G^2(-q^5)H^2(-q^{5t})} = \frac{f_2f_8f_{20}^3f_{4t}f_{10t}^3f_{25t}^2f_{40t}f_{100t}^2}{f_4f_{10}^3f_{40}f_{2t}f_{5t}^2f_{8t}f_{20t}^5f_{50t}^4}, \tag{81}$$

$$\frac{H^2(q^{10t})H^2(q^{10})}{H^2(q^{5t})H^2(-q^5)} = \frac{f_4f_{10}^7f_{25}^2f_{40}f_{100}^2f_{4t}f_{10t}f_{5t}^2f_{40t}f_{50t}^2}{f_2f_5^2f_8f_{20}^5f_{50}^4f_{2t}f_{8t}^3f_{20t}^3f_{25t}^2}. \tag{82}$$

Utilizing (68), (69) and (79) (82), we have (30). Utilizing the same fashion for the proofs of (31), (32) and (33) □

Theorem 5.1. *If* $t = 2m,\ m > 0$, *then*

$$\sum_{n=0}^{\infty} S_{2m}(2n)q^n \equiv \frac{f_2}{f_5}\Big(\frac{f_{2m}^3}{f_{10m}} - q^m f_{10m}^2\Big)(mod\ 2), \tag{83}$$

$$\sum_{n=0}^{\infty} S_{2m}(2n+1)q^n \equiv \frac{f_{10}}{f_1}\Big(\frac{f_{2m}^3}{f_{10m}} - q^m f_{10m}^2\Big)(mod\ 2) \tag{84}$$

and if $t = 2m+1,\ m \geq 0$, *then*

$$\sum_{n=0}^{\infty} S_{2m+1}(2n)q^n \equiv \frac{f_2f_{2m+1}^3}{f_5f_{10m+5}} - q^{m+1}\frac{f_{10}f_{20m+10}}{f_1}(mod\ 2), \tag{85}$$

$$\sum_{n=0}^{\infty} S_{2m+1}(2n+1)q^n \equiv \frac{f_{10}f_{2m+1}^3}{f_1f_{10m+5}} - q^m\frac{f_2f_{20m+10}}{f_5}(mod\ 2), \tag{86}$$

Where,

$$\sum_{n=0}^{\infty} S_t(n)q^n = \frac{f_t}{f_1f_5f_{5t}}.$$

Proof. By utilizing (28),

$$\sum_{n=0}^{\infty} S_t(n)q^n = \frac{f_8f_{20}^3f_{2t}f_{8t}f_{20t}^3}{f_2^2f_{10}^3f_{40}f_{4t}f_{10t}^3f_{40t}} + q\frac{f_4^3f_{40}f_{2t}f_{8t}f_{20t}^3}{f_2^3f_8f_{10}^2f_{4t}f_{10t}^3f_{40t}} - q^t\frac{f_8f_{20}^3f_{4t}^2f_{40t}}{f_2^2f_{10}^3f_{40}f_{8t}f_{10t}^2} - q^{t+1}\frac{f_4^3f_{40}f_{4t}^2f_{40t}}{f_2^3f_8f_{10}^2f_{8t}f_{10t}^2}. \tag{87}$$

Taking $t = 2m,\ m > 0$ in (87),

$$\sum_{n=0}^{\infty} S_{2m}(n)q^n = \frac{f_8f_{20}^3f_{4m}f_{16m}f_{40m}^3}{f_2^2f_{10}^3f_{40}f_{8m}f_{20m}^3f_{80m}} + q\frac{f_4^3f_{40}f_{4m}f_{16m}f_{40m}^3}{f_2^3f_8f_{10}^2f_{8m}f_{20m}^3f_{80m}} - q^{2m}\frac{f_8f_{20}^3f_{8m}^2f_{80m}}{f_2^2f_{10}^3f_{40}f_{16m}f_{20m}^2} - q^{2m+1}\frac{f_4^3f_{40}f_{8m}^2f_{80m}}{f_2^3f_8f_{10}^2f_{16m}f_{20m}^2}. \tag{88}$$

By evaluating the even and odd parts of (88),

$$\sum_{n=0}^{\infty} S_{2m}(2n)q^n = \frac{f_4 f_{10}^3 f_{2m} f_{8m} f_{20m}^3}{f_1^2 f_5^3 f_{20} f_{4m} f_{10m}^3 f_{40m}} - q^m \frac{f_4 f_{10}^3 f_{4m}^2 f_{40m}}{f_1^2 f_5^3 f_{20} f_{8m} f_{10m}^2} \tag{89}$$

$$\sum_{n=0}^{\infty} S_{2m}(2n+1)q^n = \frac{f_2^3 f_{20} f_{2m} f_{8m} f_{20m}^3}{f_1^3 f_4 f_5^2 f_{4m} f_{10m}^3 f_{40m}} - q^m \frac{f_2^3 f_{20} f_{4m}^2 f_{40m}}{f_1^3 f_4 f_5^2 f_{8m} f_{10m}^2} \tag{90}$$

Employing the relation $f_m^2 \equiv f_{2m}(mod\ 2)$ on (89) and (90), then we have (83) and (84). For (85) and (86) taking $t = 2m+1,\ m \geq 0$ in (87) and then using same fashion. □

If we take $m = 0$ in (86), then we have the following result.

Corollary 5.1.

$$S_1(2n+1) \equiv 0(mod\ 2),\ where \sum_{n=0}^{\infty} S_1(n)q^n = \frac{1}{f_5^2} \tag{91}$$

Conclusion

In this paper, we investigated some generalized form of $\frac{f_t}{f_1}$ in four, six, and nine dissections by using the notion of Rogers-Ramanujan and Göllnitz-Gordon functions. Moreover, we developed some new families of congruences by using the proposed generalized forms of $\frac{f_t}{f_1}$. In future work, one can use these proposed generating relations in different kinds of generating functions for partitions of an integer. Furthermore, one can generate the sequence of generating functions of restricted partitions by means of proposed generating relations.

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Competing Interests

All authors confirm that there are no competing interests between them.

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Shahbaz Ali received his degrees of M.Sc and M.Phil from the University of Punjab, Pakistan. Presently, he is working as a lecturer of Mathematics at Khawaja Fareed University of Engineering and Information Technology (KFUEIT) Rahim Yar Khan and also pursuing his PhD degree in the Department of Mathematics, University of Punjab under the supervision of Dr. Muhammad Khalid Mahmood. His area of research is Number Theory, Combinatorics and Fuzzy Theory.

Muhammad Khalid Mahmood received his degrees of M.Sc., M,Phil., and PhD from University of the Punjab, Pakistan. He has been working in Mathematics Department, University of the Punjab, Pakistan since 2001. His educational background has a silver medal, a gold medal and national talent scholarship and President UGC scholarship. He has a vast teaching and research experience. His research areas are Number Theory, Mathematical Modeling, Combinatorics and Fuzzy Theory.

Muhammad Shazib Hameed received his M.Sc. and M.Phil. degrees from Bahauddin Zakariya University, Multan Punjab, Pakistan. He is currently working on his Ph.D. under the supervision of Dr. Zaheer Ahmad from Khwaja Fareed University of Engineering and Information Technology (KFUEIT), Rahim Yar Khan. He is currently working as a Lecturer of mathematics at the same university. His research interests include numerical analysis, fuzzy algebra, and fuzzy theory.