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ROMAN AND INVERSE ROMAN DOMINATION IN NETWORK OF TRIANGLES

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Abstract. In graph G (V, E), a function $f : V \to \{0, 1\ 2\}$ is said to be a Roman Dominating Function (RDF). If $\forall u \in V, f(u) = 0$ is adjacent to at least one vertex $v \in V$ such that $f(v) = 2$. The weight of f is given by $w(f) = \sum_{v\in V} f(v)$. The Roman Domination Number (RDN) denoted by $\gamma_R(G)$ is the minimum weight among all RDF in G. If $V - D$ contains a RDF $f^1 : V \to \{0, 1, 2\}$, where D is the set of vertices v, $f(v) > 0$, then f^1 is called Inverse Roman Dominating Function (IRDF) on a graph G with respect to the RDF f. The Inverse Roman Domination Number (IRDN) denoted by $\gamma^1_R(G)$ is the minimum weight among all IRDF in G. In this paper we find RDN and IRDN of few triangulations graphs.

Keywords: Domination Number, Roman Domination Number, Inverse Domination Number.

AMS Subject Classification: 05C69, 94C15, 68R10.

1. Introduction

The concept of domination was introduced by Claude Berge [1] and Ore [2]. The inspiration of domination theory was from the famous problem of covering chessboard with minimum number of chess pieces. Due to its various applications, the idea of domination theory in graphs had received much attention among researchers. It is evident that in a short period of time, there are more than hundred different types of domination defined with thousands of papers published in this concept [3].

Roman domination is one among these hundred different types of domination. Emperor Constantine the great ruled the Roman Empire between 306 CE and 337 CE. During this period Roman Empire was under severe attack due to various conflicts. Due to these conflicts the resources of the empire were severely declined. In order to protect his empire

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the challenge faced by the emperor was to place the limited available legions in the specific location so that the entire empire is secured from the enemy's attack. Hence the emperor came up with an idea that every region was either secured by its own legion or was securable by the neighbor with two legions so that one of which can be sent to the undefended region in case of conflict.

A British mathematician, Ian Stewart in his article titled "Defend the Roman Empire" [4] analyzed Constantine's strategy to defend the empire. C. S. ReVelle and K. E. Rosing studied the deployment of the legions through a form of zero-one integer programming [5]. Inspired by the article written by Ian Stewart, Michael A Henning and Stephen T Hedetniemi formally introduced the concept of Roman domination theory in graphs and also analyzed Constantine's strategy. They believed that the emperor had better chances to protect the empire [6]. The assigning of legions location is not the only problem related with roman domination but also this concept is applicable to many similar problems in the modern world like finding the optimal location of setting up of hospitals, restaurants, fire stations, mobile towers, police stations etc.

In this paper we discuss an important topic of triangulation or triangular tiling. G. K. Francis and J. R. Weeks in 1999 proved that every surface has a triangulation [7]. R.E. Tarjan and C. J. Van Wyk explored the algorithm approach of triangulation [8]. The properties of such triangulation graphs play an important role in the design of cellular network, telecommunication, study heat flux density, surveying and molecular biology. The concept of triangulation and its properties are very important in local area network where the processor or station is considered as nodes and the link or connections are considered as edges [9, 10, 11, 12]. There are various modern networks like World Wide Web, biological network, language network, semantic network, software network, contact network, social network, metabolic network, citations network, human brain network. The availabilities of triangles in these networks and study of these triangular networks plays an important role in understanding its dynamics. Real world networks are massive and complex but identifying sub structures within these networks can provide insight into how the network function and topology affects each other.

Let $G\,(V, E)$ be a graph, a subset $S \subseteq V$ is a domination set of G, if for any vertex $u \in V - S$, then there exist a vertex $v \in S$ such that $uv \in E$. The domination number of G, denoted by $\gamma(G)$ equals the minimum cardinality among all the domination set.

A Roman Dominating Function (RDF) on a graph $G\,(V, E)$ is defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that for every vertex u, $f(u) = 0$ is adjacent to at least one vertex v, such that $f(v) = 2$. For a real valued function $f : V \to R$ the weight of f is $w(f) = \sum\limits_{v \in V} f(v)$. The Roman Domination Number (RDN) denoted by $\gamma_R(G)$ is the minimum weight among all RDF in G.

The positions of legions are kept very secret in order to take advantage over their enemy's strategies. But if the enemies come to know the positions of the legions, then the best chances of attack will be in the places where legions are not placed. So that the time-lag in moving the legions from the adjacent regions could be taken as an advantage. Hence if the emperor comes to know that the enemies know their legions positions. In order to surprise the enemies and create great damage, the question of optimal reorganizing of the legions still defending the Roman Empire need to be answered. Hence inverse roman dominating function was defined.

An Inverse Roman Dominating Function (IRDF) is also a roman dominating function. If $V - D$ contains a roman dominating function $f^1 : V \to \{0, 1\ 2\}$, where D is the set of vertices v for which $f(v) > 0$, then f^1 is called Inverse Roman Dominating Function

(IRDF) on a graph G, with respect to roman dominating function f. The inverse roman domination number (IRDN) denoted by $\gamma_R^1(G)$ is the minimum weight among all IRDF in G.

For any undefined terms or notation in this paper, we refer Harary to [13].

2. Preliminary results [14, 15, 16, 17]

Proposition 2.1. *For any given graph $G(n,m)$,*
$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$, $\gamma_R(G) \leq \gamma_R^1(G)$ *and* $\gamma_R(G) \leq \gamma_R^1(G) \leq n$.

Proposition 2.2. *For any Complete graph K_n, with n vertices,*
$\gamma_R(K_n) = \gamma_R^1(K_n) = 2$.

Proposition 2.3. *For the classes of cycle C_n with n vertices,* $\gamma_R(G) = \gamma_R^1(G) = \left\lceil \frac{2n}{3} \right\rceil$.

Proposition 2.4. *For the classes of paths P_n with $n > 2$ vertices,*

$$\gamma_R(P_n) = \left\lceil \frac{2n}{3} \right\rceil \text{ and } \gamma_R^1(G) = \begin{cases} \left\lceil \frac{2n}{3} \right\rceil + 1, & \text{if } n \equiv 0 \,(\text{mod} 3) \\ \left\lceil \frac{2n}{3} \right\rceil, & \text{Otherwise} \end{cases}.$$

Proposition 2.5. *For the classes of Wheel W_n with $n \geq 4$ vertices,*
$\gamma_R(G) = 2$ *and* $\gamma_R^1(G) = \left\lceil \frac{2(n-1)}{3} \right\rceil$.

Proposition 2.6. *For the classes of Star $K_{1,n-1}$ with n vertices,*
$\gamma_R(G) = 2$ *and* $\gamma_R^1(G) = n$.

Proposition 2.7. *For the classes of ladder $G_{2,n}$,*
$\gamma_R(G_{2,n}) = \gamma_R^1(G_{2,n}) = n + 1$.

Proposition 2.8. *For any Graph G of order n with maximum degree Δ and minimum degree δ,* $\left\lceil \frac{2n}{\Delta+1} \right\rceil \leq \gamma_R(G) \leq n - \Delta + 1$, $\left\lceil \frac{2n}{\Delta+1} \right\rceil \leq \gamma_R^1(G) \leq n - \delta + 1$.

Definition 2.1 (Nested Triangle Graph [18])**.** *A graph with n vertices is a planar graph formed from a sequence of $n/3$ triangles, by connecting pairs of corresponding vertices on consecutive triangles in the sequence. It is denoted as N_n.*

Definition 2.2 (TiT Graph [19])**.** *Triangle inside Triangle (TiT) graph or midpoint sequence triangle is obtained by recursively inscribing an equilateral triangle by joining the mid points of the sides of the larger triangle. Inside the smaller equilateral triangle another inscribed equilateral triangle is constructed by joining the midpoints of the sides. It is denoted as G_n.*

Definition 2.3 (Generalized Sierpinski Triangle [20])**.** *Generalized Sierpinski Triangle or Sierpinski gasket is a graph that is constructed from an equilateral triangle by recursively subdividing it into four smaller congruent equilateral triangles and removing the central triangle. It is denoted as $SG(n)$.*

Definition 2.4 (Apollonian Network Graph [21])**.** *A graph formed by a process of recursively subdividing a triangle into three small triangles, i.e., a graph obtained embedding G in the plane by repeatedly selecting a triangular face of the embedding, adding a new vertex 'v' inside the face and connecting the new vertex to each vertex of the face containing it. It is denoted as $A(n)$.*

Definition 2.5 (Triangular Grid Graph [22])**.** *A triangular grid graph is the lattice graph obtained by interpreting the order $(n+1)$ triangular grid as a graph, with the intersection of grid lines being the vertices and the line segments between vertices being the edges.*

Triangular grid graph is also the hexagonal king graph of order n, i.e., the connectivity graph of possible moves of a king chess piece on a hexagonal chess board. It is denoted as $T(n)$.

3. Main results

Theorem 3.1. *For any nested triangle graph $G = N_n$, $n \geq 1$,*
$\gamma_R(N_n) = \gamma_R^1(N_n)$ and

$$\gamma_R(N_n) = \begin{cases} 2\left\lceil \frac{n}{2} \right\rceil + \left(n - \left\lceil \frac{n}{2} \right\rceil\right), & n \text{ is Odd} \\ 2\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + \left(n - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)\right), & n \text{ is Even} \end{cases}$$

Proof. The nested triangle graph $G(V, E)$ has $3n$ vertices say $\{u_i, v_i, w_i\}$ for $i = 1, 2, \ldots, n$ with $|E| = 6n - 3$. Vertices $\{u_i, v_i, w_i\}$ form a triangle for $i = 1, 2, \ldots, n$. We also have path $v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n, w_1, w_2, w_3, \ldots, w_n$ with the graph G being a three connected graph. Let $f(V_0, V_1, V_2)$ be a γ_R function, by definition for each $u \in V_0$, $f(u) = 0$ there exist at least one vertex $v \in V_2$, $f(v) = 2$ such that $uv \in E$, $v \in V_1$, $f(v) = 1$. The graph $G = N_n$, $n \geq 1$, has n complete graphs K_3, denoted by T_i, $1 \leq i \leq n$, every K_3 can be dominated by single vertex $v \in T_i$, $1 \leq i \leq n$, $f(v) = 2$, this gives $\gamma_R(N_n) \leq 2n$, but $v \in T_i$ also dominate a vertex in T_{i+1}, $1 \leq i \leq n-1$. Therefore alternate triangle will have the single vertex $v \in T_i$ such that $f(v) = 2$ and $u \in T_{i+1}$, $f(u) = 1$, $1 \leq i \leq n-1$. No two vertices in T_i, $1 \leq i \leq n$ will have $v \in T_i$, $f(v) > 0$. Hence if n is odd, then $\left|\left\lceil \frac{n}{2} \right\rceil\right|$ vertices will have $f(v) = 2$ and remaining vertices $\left|\left(n - \left\lceil \frac{n}{2} \right\rceil\right)\right|$ will have $f(u) = 1$ and if n is even, then $\left|\left\lceil \frac{n}{2} \right\rceil + 1\right|$ vertices will have $f(v) = 2$ and remaining vertices $\left|\left(n - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)\right)\right|$ will have $f(u) = 1$. For γ_R^1 function it has been found that u_i, v_i, w_i, in f, $v \in V$, $f(v) > 0$ are rotated to v_i, w_i, u_i, $1 \leq i \leq n$ respectively, therefore the roman dominating set $D = \{u_i \vee v_j \vee w_k / i, j, k = 1, 2, \ldots, n\}$ and inverse roman dominating set is given as $D^1 = \{v_i \vee w_j \vee u_k / i, j, k = 1, 2, \ldots, n\}$.

Hence $\gamma_R^1(N_n) = \left|D^1\right| = |D| = \gamma_R(N_n)$. □

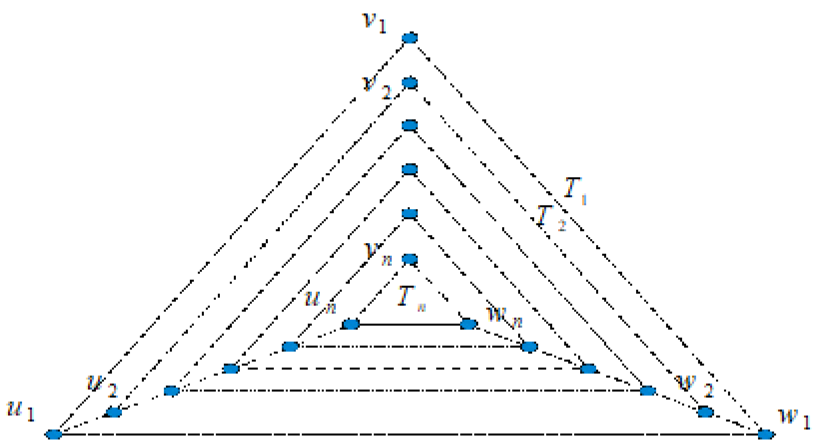

FIGURE 1. Nested triangle graph $G = N_n$.

Theorem 3.2. *For any TiT graph $G = G_n$, $n \geq 1$,*
$\left\lceil \frac{2n}{\Delta+1} \right\rceil \leq \gamma_R(G_n) \leq 2\left\lceil \frac{3n}{4} \right\rceil$, $\gamma_R(G_n) = \gamma_R^1(G_n)$.

Proof. The TiT graph or midpoint sequence triangle $G_n(V,E)$ has $|V| = 3n$, $|E| = 6n-3$,label the vertices of G_n as given in Fig. 2. G_1 is a complete graph K_3, $V(G_1) = \{v_1, v_2, v_3\}$. We have subdivided the three edges of G_1. The new vertices are denoted as $\{v_4, v_5, v_6\}$ also they are connected to form K_3. $V(G_2) = V(G_1) \cup \{v_4, v_5, v_6\}$.
$E(G_2) = \{v_1v_4, v_2v_4, v_2v_6,\ v_6v_3, v_3v_5, v_1v_5, v_5v_4, v_4v_6, v_6v_5\}$, for $n \geq 2$, the vertex set $V(G_n) = V(G_{n-1}) \cup \{v_{3n-2},\ v_{3n-1}, v_{3n}\}$ and the edge set of the TiT graph is given by

$$E(G_n) = \sum_{k=2}^{n} \Big[(v_{3k-5}, v_{3k-2}), (v_{3k-2}, v_{3k-4}), (v_{3k-4}, v_{3k}), (v_{3k}, v_{3k-3}),$$
$$(v_{3k-3}, v_{3k-1}), (v_{3k-1}, v_{3k-5})\Big] \cup \Big[(v_{3n-2}, v_{3n-1}), (v_{3n-1}, v_{3n}), (v_{3n}, v_{3n-2})\Big].$$

Since Δ is the maximum degree of G, $V_2 \subseteq V$, we must have $|V_0| \leq \Delta\,|V_2|$, $(\Delta+1)\,\gamma_R(G) = (\Delta+1)\,(|V_1| + 2\,|V_2|) \geq 2\,|V_1| + 2\,|V_2| + 2\,|V_0|$. Therefore $(\Delta+1)\,\gamma_R(G) = 2n$. Hence the lower bound.

For the TiT graph $\Delta = 4$, $\deg(v_i) = 4$, $i \neq 1,2,3, \deg(v_i) = 2$, $i = 1,2,3$. for $v \in V$, such that $f(v) = 2$ then $f(u) = 0$, $u \in N(v)$, since $\Delta = 4$, $|u| = 4$, since $|V| = 3n$. Hence the upper bound. The upper bound is very sharp.

Consider G_4, label the vertices as shown in the Fig. 2. By symmetry in the graph, for every $a \in V$, $f(a) > 0$ there exit $b \in V - D$, such that $f^1(b) > 0$, where D is the set of vertices $a \in V$, $f(a) > 0$. $D = \{u_i \vee v_j / i, j = 1, 2, \ldots, n\}$ then $D^1 = \{v_i \vee u_j / i, j = 1, 2, \ldots, n\}$ and $w_i \in V$, $f(w_i) > 0$ is rotated to the adjacent vertex u_i, v_i, $f^1(u_i) > 0$ or $f^1(v_i) > 0$, $1 \leq i \leq n$. This can generalized for G_n.

Hence $\gamma_R^1(G_n) = \left|D^1\right| = |D| = \gamma_R(G_n)$. □

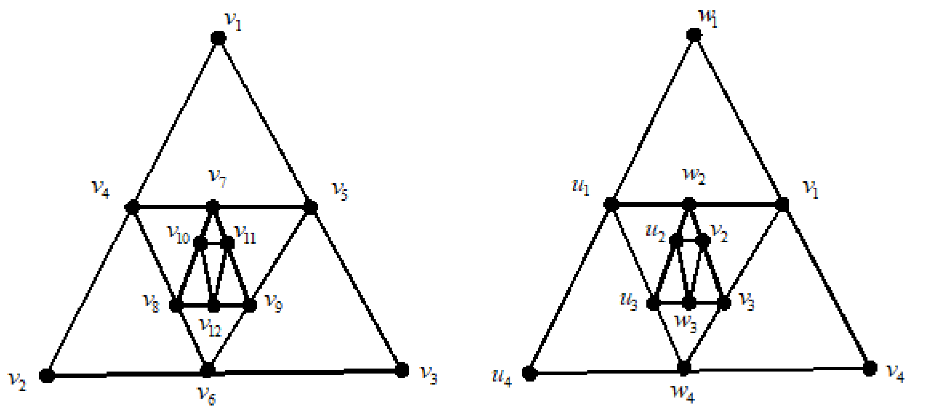

FIGURE 2. TiT Graph G_4 for γ_R and γ_R^1 function.

Theorem 3.3. *For a Generalized Sierpinski Triangle $G = SG(n)$, $n \geq 3$, $\gamma_R(SG(n)) = \gamma_R^1(SG((n)) = 6 \cdot 3^{n-2} + 1$.*

Proof. The Generalized Sierpinski Triangle $SG(n)$, $n \geq 0$, has $|V| = \frac{3}{2}\,(3^n + 1)$, $|E| = 3^{n+1}$, $SG(0) = K_3$. Therefore $\gamma_R(SG(0)) = 2$, $\gamma_R(SG(1)) = 3$, for $SG(2)$, γ_R function has $|V_1| = 1$, $|V_2| = 3$ hence $\gamma_R(SG(2)) = 2(3) + 1 = 7$. For $n \geq 3$, $\gamma_R(SG(3)) = 3\gamma_R(SG(2)) = 3^1\gamma_R(SG(2))$, $\gamma_R(SG(4)) = 9\gamma_R(SG(2)) = 3^2\gamma_R(SG(2))$, $\gamma_R(SG(5)) = 27\gamma_R(SG(2)) = 3^3\gamma_R(SG(2))$. Therefore, $\gamma_R(SG(n)) = 3^{n-2}\gamma_R(SG(2))$.

Let $f(V_0, V_1, V_2)$ be a γ_R function, since for all $SG(n)$, $n \geq 3$, $|V_1| = 1$, hence $\gamma_R(SG(n)) = 3^{n-2}SG(2) + 1 = 6 \cdot 3^{n-2} + 1$. Consider $SG(2)$, label the graph as given in Fig. 3. By symmetry in the graph, for every $u \in V, f(u) > 0$ there exit a $v \in V - D$, such that

$f^1(v) > 0$ where D is the set of vertices u, $f(u) > 0$. If $D = \{u_i/i = 2, 4, 6 \ldots\}$ then $D^1 = \{v_i/i = 1, 3, 5 \ldots\}$ with $w_1 \in V$, $f(w_1) = 1$ and $w_2 \in V$, $f^1(w_2) = 1$. The same can be generalized to $SG(n)$.

Hence $\gamma_R^1(G_n) = \left|D^1\right| = |D| = \gamma_R(G_n)$. □

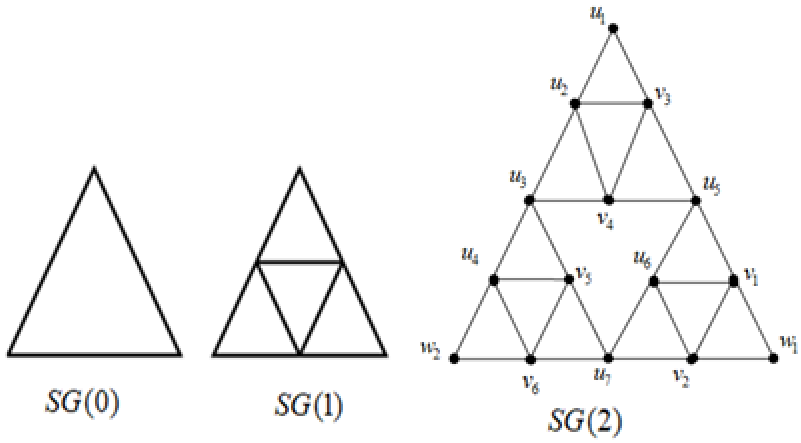

FIGURE 3. First three stages of Generalized Sierpinski Triangle.

Theorem 3.4. *For a Apollonian Network Graph $G = A(n)$, $n \geq 4$, $\gamma_R(A(n)) = 3^{n-3} + 5$ and $\gamma_R^1(A((n)) = 5 \cdot 3^{n-3}$.*

Proof. The apollonian network graph $A(n)$ has $|V| = \frac{1}{2}(3^n + 5)$, $|E| = \frac{3}{2}(3^n + 1)$, $A(0) = K_3$, $\gamma_R(A(0)) = 2$, $\gamma_R(A(1)) = \gamma_R(A(2)) = 2$, $\gamma_R(A(3)) = 5$. For $A(4)$, γ_R function has $|V_1| = 0$, $|V_2| = 4$, hence $\gamma_R(A(4)) = 2(3 + 1) = 8$. $A(4)$ can be embedded in $G = A(n)$, $n \geq 4$, as follows: $A(5) = 3A(4)$, $A(6) = 9A(4)$, $A(7) = 27A(4)$. We get the following relation $\gamma_R(A(5)) = 2(3+1+3) = 14$, $\gamma_R(A(6)) = 2(3 + 1 + 3 + 9) = 32$, generalizing the result we get $\gamma_R(A(n)) = 2\left(3 + \frac{3^{n-3}-1}{2}\right) = 2\left(\frac{6+3^{n-3}-1}{2}\right)$, $\gamma_R(A(n)) = 2\left(\frac{6+3^{n-3}-1}{2}\right) = 3^{n-3} + 5$.

Hence for $n \geq 4$, $\gamma_R(A(n)) = 3^{n-3} + 5$.

Inverse roman domination number is given as $\gamma_R^1(A(0)) = \gamma_R^1(A(1)) = 2$, $\gamma_R^1(A(2)) = 3$, $\gamma_R^1(A(3)) = 6$, $\gamma_R^1(A(4)) = (2 + 3)3 = 15$, where $|V_1| = 9$ and $|V_2| = 3$. $A(4)$ can be embedded in $A(n)$, $n \geq 4$, we have the following $\gamma_R^1(A(5)) = 3^1\gamma_R^1(A(4)) = (2 + 3)3 \cdot 3$, $\gamma_R^1(A(6)) = 3^2\gamma_R^1(A(4)) = (2 + 3)3 \cdot 3 \cdot 3$, $\gamma_R^1(A(n)) = 3^{n-3} \cdot 5$.

Hence for $n \geq 4$, $\gamma_R^1(A(n)) = 5 \cdot 3^{n-3}$. □

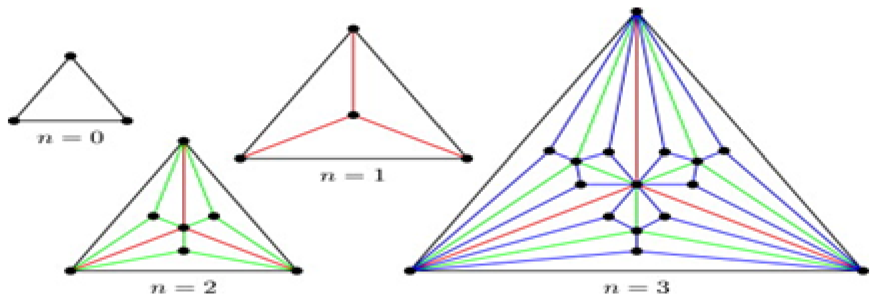

Figure 4. First four stages of Apollonian Network Graph.

Theorem 3.5. *For a Triangular Grid Graph $G = T(n)$, $n \geq 1$, if $|V| = 15k$, then $\gamma_R(T(n)) \leq 6k$, $n\ k \in N$.*

Proof. A triangular grid graph $T(n)$, $|V| = \frac{(n+1)(n+2)}{2}$, $|E| = \frac{3}{2}\left(n^2+n\right)$, for $T(4)$, $|V| = 15$, $|E| = 30$. It can also be seen that $T(4)$ can be embedded in $T(n)$ with minimum number of common vertex being dominated. If $\gamma_R\left(T(4)\right) = 6$, then $|V| = \frac{(n+1)(n+2)}{2} = 15k$, $\left(n+\frac{3}{2}\right)^2 - \frac{1}{4} = 30k$, $n = \sqrt{30k+\frac{1}{4}} - \frac{3}{2}$. For $n, k \in N$, we have $k = 1$, $n = 4$, $|V| = 15$, $T(4) = 6$, $k = 3$, $n = 8$, $|V| = 45$, $T(8) = 18$. Hence if $|V| = 15k$, then $\gamma_R(T(n)) \leq 6k$. Similar inequalities can also be obtained for $|V| = 3k$, $|V| = 6k$, $|V| = 10k$ etc. But the bounds are not close because more number of common vertices with $a \in V$, $f(a) = 0$ is dominated by $b \in V$, $f(b) = 2$. □

Theorem 3.6. *For a Triangular Grid Graph $G = T(n)$, $n \geq 1$, $\gamma_R(G) = \gamma_R^1(G)$.*

Proof. A triangular grid graph $T(n)(V,E)$ has $|V| = \frac{(n+1)(n+2)}{2}$, $|E| = \frac{3}{2}\left(n^2+n\right)$, Consider $T(4)$, label the vertices as shown in the Fig. 5, $\{u_{i,j}, v_{i,j}, w_{i,j}\}$, $0 \leq i, j,\ w \leq 4$ (i is the level and j being the ascending order of the vertices) are the vertices and L_i, $0 \leq i \leq 4$ are levels of $T(4)$. Only for L_i, i being even will have a central vertex labeled as $w_{i,j}$, $i = 0, 2, 4, \ldots$, $0 \leq j \leq n-1$. By symmetry in the graph, for every $a \in V$, $f(a) > 0$ there exit $b \in V - D$, such that $f^1(b) > 0$, where D is the set of vertices $a \in V$, $f(a) > 0$. If roman dominating set is given by $D = \{u_{i,j} \vee v_{i,j}/0 \leq i, j \leq n\}$ then the inverse roman dominating set $D^1 = \{v_{i,j} \vee u_{i,j}/0 \leq i, j \leq n\}$ or the other case is that, if $D = \{u_{i,j} \vee v_{i,j}/0 \leq i, j \leq n\}$ then $D^1 = \{u_{i\pm1,j\pm1} \vee v_{i\pm1,j\pm1}/1 \leq i, j \leq n-1\}$ and also $w_i \in V$, $f(w_{i,j}) > 0$ is rotated to the adjacent vertex $u_{i,j}$ or $v_{i,j}$, $f^1(u_{i,j}) > 0$ or $f^1(v_{i,j}) > 0$, $0 \leq i, j \leq n$. This can generalized for $T(n)$.

Hence $\gamma_R^1(G_n) = \left|D^1\right| = |D| = \gamma_R(G_n)$. □

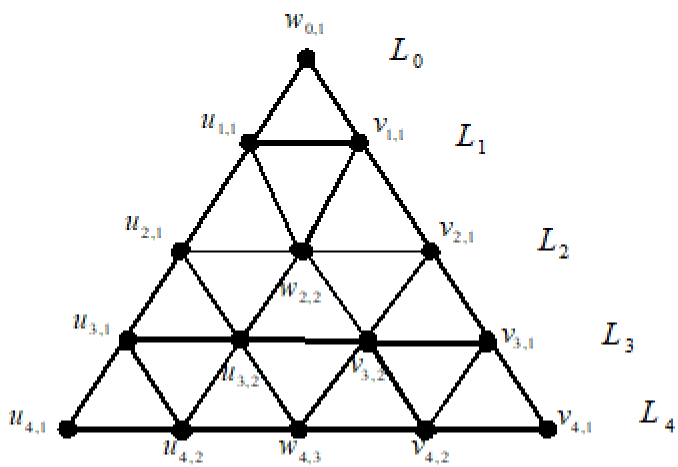

Figure 5. Triangular Grid Graph $T(4)$.

There are two kinds of γ_R function one being triangular (semi hexagonal) and other being hexagonal. We show that both the bounds are close but triangular γ_R function has closer bounds when compared to the hexagonal γ_R function. In triangular function there is no pattern but whereas in hexagonal function we get a pattern for labeling the vertex hence easy to find the RDF in hexagonal function. The difference between the triangular and hexagonal γ_R function is that for triangular γ_R function most of the time $\Delta = 4$ is considered for $f(v) = 2$, $v \in V$. Whereas for the hexagonal γ_R function $\Delta = 6$ is considered for $f(v) = 2$, $v \in V$. In Theorems 3.7 and 3.8, $\gamma_R(T_n)$ is proved with equality sign with respect to triangular and hexagonal γ_R function, but in general $\gamma_R(T_n)$, $n \geq 1$, the symbol will be less than or equal to.

Theorem 3.7. *In a Triangular Grid Graph $G = T(n)$, $n \geq 1$. If γ_R is a triangular function, then* $\gamma_R(T(n)) = 6 + \sum\limits_{m=1}^{k} 2\left\lceil \frac{2m+5}{3} \right\rceil$, $n = 2k+4$, $k \in N$ *and*

$$\gamma_R(T(n)) = 6 + \left[\sum_{m=1}^{k} 2\left\lceil \frac{2m+5}{3} \right\rceil\right] + 2\left\lceil \frac{2k+6-2\left\lceil\frac{2k+5}{3}\right\rceil}{3} \right\rceil,$$
$$n > 5, n = 2k+5, k \in N$$

Proof. **Case 1:** For n is even, $n = 2k+4$.
Let $G = T(n)$ has $n+1$ levels L_p, $0 \leq p \leq n$ given in Fig. 6. At each level m there are $m+1$ vertices, $0 \leq m \leq n$. Vertices are labeled as $v_{i,j}$, i is the level and j being the ascending order of the vertices. $f(v_{1,1}) = f(v_{3,4}) = f(v_{4,2}) = 2$. For each vertex $v \in L_{2m+4}$, $1 \leq m \leq \frac{n-4}{2}$, there are $2m+5$ vertices, for $v \in L_{2m+4}$ such that $f(v) = 2$ then $f(u) = 0$, $u \in N[v]$, $|N[v]| = 3$. Hence $\left\lceil \frac{2m+5}{3} \right\rceil$ vertices will have $f(v) = 2$ and hence the result.
Case 2: For n is odd, $n = 2k+5$.
Vertices $v \in L_{2k+3}$, $1 \leq k \leq \frac{n-3}{2}$, are dominated by vertices in level $v \in L_{2k+4}$, $1 \leq k \leq \frac{n-4}{2}$. $f(v_{1,1}) = f(v_{3,4}) = f(v_{4,2}) = f(v_{5,5}) = 2$, $f(v_{5,1}) = 1$, hence $\gamma_R(T_5) = 9$. $n > 5$, for n is odd, we already have the case for n is even, only one more level of vertices needs to be covered by γ_R function. If there are $2k+5$ vertices, for $v \in L_{2k+4}$, then $2k+6$ vertex will be in the next level. For each vertex $v \in L_{2k+4}$, $1 \leq k \leq \frac{n-4}{2}$, $2\left\lceil \frac{2k+5}{3} \right\rceil$ vertices are dominated in level $v \in L_{2k+5}$. Therefore remaining vertices need to dominated in level $v \in L_{2k+5}$.

Hence the proof. □

Theorem 3.8. *In a Triangular Grid Graph $G = T(n)$, $n \geq 1$. If γ_R is a Hexagonal function, then*

$$\gamma_R(T_n) = 1 + \sum_{i=2}^{n} L_i,\ L_n = \begin{cases} 2k, & n = 3k-1 \\ k+1, & n = 3k \\ k, & n = 3k+1 \end{cases},\ k \in N.$$

Proof. Let $G = T(n)$ has $n+1$ levels L_p, $0 \leq p \leq n$ given in Fig. 6. At each level m there are $m+1$ vertices, $0 \leq m \leq n$. Vertices are labeled as $v_{i,j}$, i is the level and j being the ascending order of the vertices.

In hexagonal γ_R function $f(v_{2,2}) = 2$ then $f(v_{0,1}) = 1$, $f(v_{1,1}) = f(v_{1,2}) = 0$. Weight of the function f is given by $w(f) = \sum\limits_{v \in V} f(v)$. For each vertex, $v \in L_{3x-1}$, $1 \leq x \leq n$,

$w(f) = \sum\limits_{v \in L_{3x-1}} f(v) = 2x$, $v \in L_{3y}$, $1 \le y \le n$, $w(f) = \sum\limits_{v \in L_{3y}} f(v) = 1(y+1) = y+1$, $v \in L_{3z+1}$, $1 \le z \le n$, $w(f) = \sum\limits_{v \in L_{3z+1}} f(v) = 1$, $(z) = z$.

Hence the proof. □

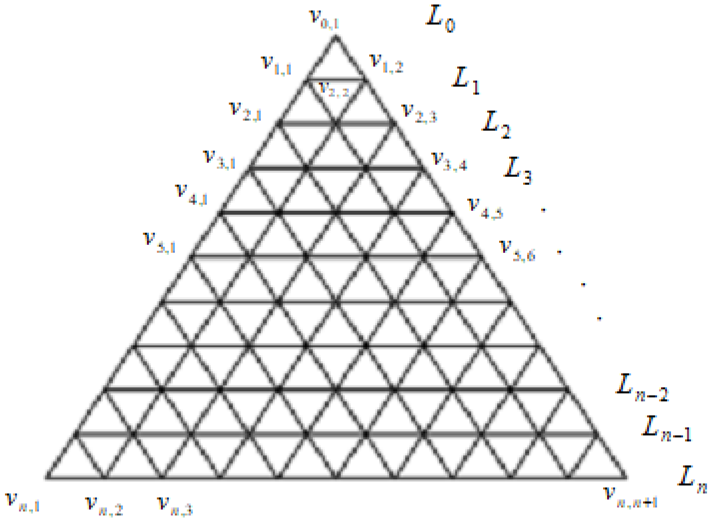

Figure 6. Triangular Grid Graph $T(n)$.

Theorem 3.9. *For any Triangular Grid Graph $G = T(n)$, $n > 3$,*
$2\left\lceil \frac{|V|}{6} \right\rceil \le \gamma_R(T(n)) \le 2\left\lceil \frac{|V|}{5} \right\rceil$.

Proof. Let $G = T(n)$, $|V| = \frac{(n+1)(n+2)}{2}$, $|E| = \frac{3}{2}\left(n^2+n\right)$, for any level L_p, $0 \le p \le n$, $\Delta = 6$, for $v \in V$ such that $f(v) = 2$. Then $f(u) = 0$, $u \in N(v)$, $|N(v)| = 6$, hence the lower bound. Given $f(v) > 0$, $v \in L_p$, $0 \le p \le n$ then at any level L_{p+2}, $v \in L_{p+2}$ in the next level such that $f(v) = 2$. Then $f(u) = 0$, $u \in N(v)$, $|N(v)| = 5$, hence the upper bound. □

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