HOCHSTADT'S RESULTS FOR INVERSE STURM–LIOUVILLE PROBLEMS WITH FINITE NUMBER OF TRANSMISSION AND PARAMETER DEPENDENT BOUNDARY CONDITIONS

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ABSTRACT. This paper deals with the boundary value problem involving the differential equation
$$-y'' + qy = \lambda y,$$
subject to the parameter dependent boundary conditions with finite number of transmission conditions. The potential function $q \in L^2(0, \pi)$ is real and λ is a spectral parameter. We develop the Hochstadt's results based on the transformation operator for inverse Sturm–Liouville problem when there are finite number of transmission and parameter dependent boundary conditions. Furthermore, we establish a formula for $q(x) - \tilde{q}(x)$ in the finite interval $[0, \pi]$, where $q(x)$ and $\tilde{q}(x)$ are analogous functions.

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1. INTRODUCTION

Sturm–Liouville problems with transmission conditions at interior points arise in a variety of applications in engineering. We refers to [2] for a nice discussion and further information. For the Sturm–Liouville problems, we have three types of problems: direct problems, isospectral problems, and inverse problems. In direct problems, the eigenvalues, eigenfunctions, and some properties of the problem are estimated from the known coefficients [6, 13]. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isospectral Sturm–Liouville problems are studied in [14, 15]. The third type of problems related to the Sturm–Liouville problems are inverse problems. The inverse spectral Sturm–Liouville problem can be regarded as three aspects: existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions.

Inverse problems with the discontinuities conditions inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of

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science and technology. As an example, such problems are related to discontinuous and non-smooth properties of a medium (e.g., see [3, 5, 7] and [10]). We refer to the somewhat complementary surveys in inverse Sturm–Liouville problems in [2, 7, 9, 16, 19] and [21]–[28].

In this paper, we study the inverse problem of Sturm–Liouville equations with finit number of discontinuous and parameter dependent boundary conditions. We discuss the uniqueness of spectral problem by developing the Hochstadt's results for inverse Sturm–Liouville problem using two spectra with finite number of transmission and parameter dependent boundary conditions. Using the above notation, we generalize the Hochstadt's results [8], refining the approach of Levinson [12] to show that precisely how much q has freedom where the μ_n and all but finitely many of the λ_n's are specified. Note that the eigenvalues μ_n is obtained with replacing H_j by $\mathcal{H}_j$ for $j = 1, 2, 3$ in (2). The similar papers for Hochstadt's results in several cases such as discontinuous, left-definite Sturm–Liouville equations with indefinite weight, and singular Sturm–Liouville operators are in [4, 9] and [17]–[19].

2. The Hilbert space formulations and properties of the spectrum

We consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y \tag{1}$$

with the eigenparameter dependent boundary conditions

$$L_1(y) := \lambda\Big(y'(0) + h_1 y(0)\Big) - h_2 y'(0) - h_3 y(0) = 0,$$

$$L_2(y) := \lambda\Big(y'(\pi) + H_1 y(\pi)\Big) - H_2 y'(\pi) - H_3 y(\pi) = 0, \tag{2}$$

and the transmission conditions

$$U_i(y) := y(d_i + 0) - a_i y(d_i - 0) = 0,$$
$$V_i(y) := y'(d_i + 0) - b_i y'(d_i - 0) - c_i y(d_i - 0) = 0, \tag{3}$$

where $q(x)$ is real-valued function in $L^2[0, \pi]$. We also assume that a_i, b_i, c_i d_i, $i = 1, 2, \cdots, m-1$ (with $m \geq 2$) and h_j, H_j, for $j = 1, 2, 3$, are real numbers, satisfying $a_i b_i > 0$, $d_0 = 0 < d_1 < d_2 < \cdots < d_{m-1} < d_m = \pi$ and

$$r_1 := h_3 - h_1 h_2 > 0, \text{ and } r_2 := H_1 H_2 - H_3 > 0.$$

For simplicity, we use the notation $L := L(q(x); h_j; H_j; d_i)$ for the problem (1)–(3). We define the following weight function

$$w(x) = \begin{cases} 1, & 0 \leq x < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < x < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < x \leq \pi. \end{cases} \tag{4}$$

to obtain a self-adjoint operator. The Hilbert space will be $\mathcal{H} := L_2((0, \pi); w) \oplus \mathbb{C}^2$ associated with the weighted inner product

$$\langle F, G \rangle_{\mathcal{H}} := \int_0^\pi f \overline{g} w + \frac{w(0)}{r_1} f_1 \overline{g_1} + \frac{w(\pi)}{r_2} f_2 \overline{g_2}, \quad F := \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \quad G := \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix}. \tag{5}$$

The corresponding norm will be denoted by $\|F\|_{\mathcal{H}} = \langle F, F \rangle_{\mathcal{H}}^{1/2}$. Next, we introduce

$$R_1(y) := y'(0) + h_1 y(0), \qquad R_1'(y) := h_2 y'(0) + h_3 y(0),$$

$$R_2(y) := y'(\pi) + H_1 y(\pi), \qquad R_2'(y) := H_2 y'(\pi) + H_3 y(\pi).$$

In this Hilbert space, we construct the operator

$$A : \mathcal{H} \to \mathcal{H}$$

with domain

$$\mathrm{dom}\,(A) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \middle| \begin{array}{l} f, f' \in AC\left(\cup_0^{m-1} (d_i, d_{i+1}) \right),\ \ell f \in L^2(0, \pi) \\ U_i(f) = V_i(f) = 0,\ f_1 = R_1(f),\ f_2 = R_2(f) \end{array} \right\}$$

by

$$AF = \begin{pmatrix} \ell f \\ R_1'(f) \\ R_2'(f) \end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in \mathrm{dom}\,(A).$$

By construction, the eigenvalue problem for A,

$$AY = \lambda Y, \qquad Y = \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in \mathrm{dom}\,(A),$$

is equivalent to the eigenvalue problem (1)–(3) for L.

Lemma 2.1. [20] *The operator A is self-adjoint.*

In particular, the eigenvalues of A, and hence of L, are real and simple.

Suppose that the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ are solutions of (1) under the initial conditions

$$\varphi(0, \lambda) = \lambda - h_2, \qquad \varphi'(0, \lambda) = h_3 - \lambda h_1, \tag{6}$$

and

$$\psi(\pi, \lambda) = H_2 - \lambda, \qquad \psi'(\pi, \lambda) = \lambda H_1 - H_3 \tag{7}$$

as well as the jump conditions (3), respectively. It is easy to see that the equation (1) under the initial conditions (6) or (7) has a unique solution $\varphi_1(x, \lambda)$ or $\psi_m(x, \lambda)$, which is an entire function of $\lambda \in \mathbb{C}$ for each fixed point $x \in [0, d_1)$ or $x \in (d_{m-1}, \pi]$. From the linear differential equations, we obtain that the modified Wronskian

$$W(u, v) = w(x)\big(u(x)v'(x) - u'(x)v(x) \big)$$

is constant on $x \in [0, d_1) \cup_1^{m-2} (d_i, d_i + 1) \cup (d_{m-1}, \pi]$ for two solutions $\ell u = \lambda u$, $\ell v = \lambda v$ satisfying the transmission conditions (3). Moreover, we set

$$\Delta(\lambda) := W(\varphi(\lambda), \psi(\lambda)) = L_1(\psi(\lambda)) = -w(\pi)L_2(\varphi(\lambda)).$$

Then $\Delta(\lambda)$ is an entire function whose roots λ_n coincide with the eigenvalues of L. In this section, we obtain the asymptotic form of solutions and characteristic function.

Theorem 2.1. [20] *Let $\lambda = \rho^2$ and $\tau := \mathrm{Im}\rho$. For equation (1) with boundary conditions (2) and jump conditions (3) as $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$\varphi(x,\lambda) = \begin{cases} \rho^2 \cos\rho x + O(\rho\exp(|\tau|x)), & 0 \le x < d_1, \\ \rho^2[\alpha_1\cos\rho x + \alpha_1'\cos\rho(x-2d_1)] + O(\rho\exp(|\tau|x)), & d_1 < x < d_2, \\ \rho^2[\alpha_1\alpha_2\cos\rho x + \alpha_1'\alpha_2\cos\rho(x-2d_1) + \alpha_1\alpha_2'\cos\rho(x-2d_2) \\ \quad + \alpha_1'\alpha_2'\cos\rho(x+2d_1-2d_2)] + O(\rho\exp(|\tau|x)), & d_2 < x < d_3, \\ \vdots \\ \rho^2[\alpha_1\alpha_2\ldots\alpha_{m-1}\cos\rho x + \\ \quad + \alpha_1'\alpha_2\ldots\alpha_{m-1}\cos\rho(x-2d_1) + \cdots \\ \quad + \alpha_1\alpha_2\ldots\alpha_{m-1}'\cos\rho(x-2d_{m-1}) + \\ \quad + \alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\cos\rho(x+2d_1-2d_2) + \cdots \\ \quad + \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\cos\rho(x+2d_i-2d_j) \\ \quad + \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\cos\rho(x-2d_i+2d_j-2d_k) + \cdots \\ \quad + \alpha_1'\alpha_2'\ldots\alpha_{m-1}'\cos\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})] \\ \quad + O(\rho\exp(|\tau|x)), & d_{m-1} < x \le \pi, \end{cases}$$

(8)

and

$$\varphi'(x,\lambda) = \begin{cases} \rho^3[-\sin\rho x] + O(\rho^2\exp(|\tau|x)), & 0 \le x < d_1, \\ \rho^3[-\alpha_1\sin\rho x - \alpha_1'\sin\rho(x-2d_1)] + O(\rho^2\exp(|\tau|x)), & d_1 < x < d_2, \\ \rho^3[-\alpha_1\alpha_2\sin\rho x - \alpha_1'\alpha_2\sin\rho(x-2d_1) - \\ \quad - \alpha_1\alpha_2'\sin\rho(x-2d_2) - \alpha_1'\alpha_2'\sin\rho(x+2d_1-2d_2)] \\ \quad + O(\rho^2\exp(|\tau|x)), & d_2 < x < d_3, \\ \vdots \\ \rho^3[-\alpha_1\alpha_2\ldots\alpha_{m-1}\sin\rho x - \alpha_1'\alpha_2\ldots\alpha_{m-1}\sin\rho(x-2d_1) - \cdots - \alpha_1\alpha_2\ldots\alpha_{m-1}' \\ \quad \sin\rho(x-2d_{m-1}) - \alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\sin\rho(x+2d_1-2d_2) - \cdots \\ \quad - \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\sin\rho(x+2d_i-2d_j) \\ \quad - \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\sin\rho(x-2d_i+2d_j-2d_k) + \cdots \\ \quad - \alpha_1'\alpha_2'\ldots\alpha_{m-1}'\sin\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})] \\ \quad + O(\rho^2\exp(|\tau|x)), & d_{m-1} < x \le \pi, \end{cases}$$

where

$$\alpha_i = \frac{a_i+b_i}{2} \quad and \quad \alpha_i' = \frac{a_i-b_i}{2},$$

for $i = 1, 2, \cdots, m-1$. The characteristic function satisfies

$$\begin{aligned} \Delta(\lambda) = {} & \rho^5 w(\pi)\big[\alpha_1\alpha_2\ldots\alpha_{m-1}\sin\rho\pi + \alpha_1'\alpha_2\ldots\alpha_{m-1}\sin\rho(\pi-2d_1) + \cdots + \alpha_1\alpha_2\ldots\alpha_{m-1}' \\ & \sin\rho(\pi-2d_{m-1}) + \alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\sin\rho(\pi+2d_1-2d_2) + \cdots \\ & + \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\sin\rho(\pi+2d_i-2d_j) \\ & + \alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\sin\rho(\pi-2d_i+2d_j-2d_k) + \cdots \\ & + \alpha_1'\alpha_2'\ldots\alpha_{m-1}'\sin\rho(\pi+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})\big] \\ & + O(\rho^4\exp(|\tau|\pi)). \end{aligned}$$

The eigenvalues $\lambda_n = \rho_n^2$ of the boundary value problem L satisfy

$$\rho_n = n + o(n) \quad as \quad n \to \infty.$$

From the above theorem, we obtain

$$|\varphi^{(\nu)}(x, \lambda)| = O\left(|\rho|^{\nu+2} \exp(|\tau|x)\right), \qquad 0 \le x \le \pi, \ \nu = 0, 1.$$

Substituting x to $\pi - x$ in the Eq. (1) and using a simple calculation, we get the asymptotic form of $\psi(x, \lambda)$ and $\psi'(x, \lambda)$ and particularly

$$|\psi^{(\nu)}(x, \lambda)| = O(|\rho|^{\nu+2} \exp(|\tau|(\pi - x))), \qquad 0 \le x \le \pi, \ \nu = 0, 1.$$

Moreover, the eigenfunctions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ associated with a certain eigenvalue λ_n, satisfy the relation

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n). \tag{9}$$

Using (6) with a simple calculation, we obtain

$$\beta_n = \frac{\psi'(0, \lambda_n) + h_1 \psi(0, \lambda_n)}{r_1}.$$

We also define the norming constant by

$$\gamma_n := \|\Phi_n(x)\|_{\mathcal{H}}^2, \tag{10}$$

where

$$\Phi_n(x) := \Phi(x, \lambda_n) = \begin{pmatrix} \varphi_n(x) = \varphi(x, \lambda_n) \\ R_1(\varphi_n) \\ R_2(\varphi_n) \end{pmatrix}. \tag{11}$$

Then it is straightforward to verify:

Lemma 2.2. *All zeros λ_n of $\Delta(\lambda)$ are simple and the derivative is given by*

$$\dot{\Delta}(\lambda_n) = -\gamma_n \beta_n.$$

Lemma 2.3. *If $\varphi(x, \lambda_n)$ is the eigenfunction corresponding to eigenvalues λ_n, then*

$$\gamma_n = \mu(\rho_n; d_i; a_i; b_i) \left[1 + O\left(\frac{1}{n}\right)\right], \tag{12}$$

where $\mu(\rho_n; d_i; a_i; b_i) =$

$$
\begin{cases}
\rho_n^4 \frac{\pi}{2}, \qquad \text{for } m = 1, \\[2mm]
\rho_n^4 \left[\frac{d_1}{2} + \frac{\pi - d_1}{2a_1 b_1}\left(\alpha_1^2 + \alpha_1'^2 + \alpha_1 \alpha_1' \cos \rho_n(2d_1)\right) \right], \qquad \text{for } m = 2, \\[2mm]
\rho_n^4 \left[\frac{d_1}{2} + \frac{d_2 - d_1}{2a_1 b_1}\left(\alpha_1^2 + \alpha_1'^2 + \alpha_1 \alpha_1' \cos \rho_n(2d_1)\right) \right. \\
\quad + \frac{\pi - d_2}{2a_1 b_1 a_2 b_2}\left(\alpha_1^2 \alpha_2^2 + \alpha_1'^2 \alpha_1^2 + \alpha_1'^2 \alpha_2'^2 + 2\alpha_1 \alpha_2 \alpha_1' \alpha_2 \cos \rho_n(2d_1)\right. \\
\quad + \alpha_1^2 \alpha_2 \alpha_2' \cos \rho_n(2d_2) + 2\alpha_1 \alpha_2 \alpha_1' \alpha_2' \cos \rho_n(2d_2 - 2d_1) \\
\quad \left. \left. + \alpha_1'^2 \alpha_2 \alpha_2' \cos \rho_n(2d_2 - 4d_1)\right) \right] \qquad \text{for } m = 3, \\
\qquad \vdots \\[2mm]
\rho_n^4 \left[\frac{d_1}{2} + \frac{d_2 - d_1}{2a_1 b_1}\left(\alpha_1^2 + \alpha_1'^2 + \alpha_1 \alpha_1' \cos \rho_n(2d_1)\right) \right. \\
\quad + \frac{d_3 - d_2}{2a_1 b_1 a_2 b_2}\left(\alpha_1^2 \alpha_2^2 + \alpha_1'^2 \alpha_1^2 + \alpha_1'^2 \alpha_2'^2 + 2\alpha_1 \alpha_2 \alpha_1' \alpha_2 \cos \rho_n(2d_1)\right. \\
\quad + 2\alpha_1 \alpha_2 \alpha_1' \alpha_2' \cos \rho_n(2d_2 - 2d_1) + \alpha_1'^2 \alpha_2 \alpha_2' \cos \rho_n(2d_2 - 4d_1) \\
\quad \left. + \alpha_1^2 \alpha_2 \alpha_2' \cos \rho_n(2d_2)\right) + \ldots \\
\quad + \frac{1}{2a_1 b_1 \ldots a_{m-1} b_{m-1}}\left(\alpha_1^2 \alpha_2^2 \ldots \alpha_{m-1}^2 + \alpha_1'^2 \alpha_2^2 \ldots \alpha_{m-1} + \cdots \alpha_1^2 \alpha_2^2 \ldots \alpha_{m-1}'^2\right. \\
\quad + \ldots + \alpha_1'^2 \alpha_2'^2 \ldots \alpha_{m-1}'^2 + \alpha_1 \alpha_1' \alpha_2^2 \ldots \alpha_{m-1}^2 \cos \rho_n(2d_1) \\
\quad + \alpha_1^2 \alpha_2 \alpha_2' \alpha_3^2 \ldots \alpha_{m-1}^2 \cos \rho_n(2d_2) \\
\quad + \alpha_1^2 \alpha_2^2 \ldots \alpha_{m-1} \alpha_{m-1}' \cos \rho_n(2d_{m-1}) + \cdots \\
\quad + \alpha_1'^2 \alpha_2 \alpha_2' \ldots \alpha_{m-1} \alpha_{m-1}' \cos \rho_n(2[(-1)^{m-1} + 1]d_1 + 2(-1)^{m-2}d_2 + \cdots - 2d_{m-1}) \\
\quad + \alpha_1 \alpha_1' \alpha_2' \ldots \alpha_{m-1} \alpha_{m-1}' \cos \rho_n(2(-1)^{m-1}d_1 + 2[(-1)^{m-2} + 1]d_2 + \cdots - 2d_{m-1}) \\
\quad + \cdots + \alpha_1 \alpha_1' \alpha_2'^2 \ldots \alpha_{m-1}'^2 \cos \rho_n(2d_1) \\
\quad \left. \left. + \cdots + \alpha_1'^2 \alpha_2'^2 \ldots \alpha_{m-1} \alpha_{m-1}' \cos \rho_n(2d_{m-1})\right) \right] \qquad \text{for } m \geq 4.
\end{cases}
$$

Proof. Using the inner product (5), initial conditions (6), and the asymptotic form of $\varphi(x, \lambda)$ in (8), we get

$$
\gamma_n = \int_0^\pi \varphi^2(x, \lambda_n) w(x) dx + r_1 + w(\pi) r_2
$$

$$
= \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right] + r_1 + w(\pi) r_2.
$$

The second term i.e. $r_1 + w(\pi) r_2$, merged in the leading term, so we get Eq. (12). $\qquad \square$

Suppose that the functions $\tilde{\varphi}(x, \lambda)$ and $\tilde{\psi}(x, \lambda)$ are solutions of

$$
\tilde{\ell} y := -y'' + \tilde{q} y = \lambda y
$$

under the initial conditions

$$
\tilde{\varphi}(0, \lambda) = \lambda - h_2, \qquad \tilde{\varphi}'(0, \lambda) = h_3 - \lambda h_1,
$$

and

$$
\tilde{\psi}(\pi, \lambda) = H_2 - \lambda, \qquad \tilde{\psi}'(\pi, \lambda) = \lambda H_1 - H_3
$$

as well as the jump conditions (3), respectively. So we get

$$
\tilde{\psi}(x, \tilde{\lambda}_n) = \tilde{\beta}_n \tilde{\varphi}(x, \tilde{\lambda}_n),
$$

where

$$\tilde{\beta}_n = \frac{\tilde{\psi}'(0, \tilde{\lambda}_n) + h_1 \tilde{\psi}(0, \tilde{\lambda}_n)}{r_1},$$

where $\tilde{\varphi}(x, \tilde{\lambda}_n)$ and $\tilde{\psi}(x, \tilde{\lambda}_n)$ are eigenfunctions of $\tilde{L} := L(\tilde{q}(x); h_j; H_j; d_i)$ corresponding to the eigenvalue $\tilde{\lambda}_n$ and

$$\tilde{\Phi}_n(x) := \tilde{\Phi}(x, \tilde{\lambda}_n) = \begin{pmatrix} \tilde{\varphi}_n(x) = \tilde{\varphi}(x, \tilde{\lambda}_n) \\ R_1(\tilde{\varphi}_n) \\ R_2(\tilde{\varphi}_n) \end{pmatrix}. \tag{13}$$

Let $L(q(x); h_j; \mathcal{H}_j; d_i)$ be another eigenvalue problem such that by $H_1 - \mathcal{H}_1 \neq 0$. The boundary condition for the problem $L(q(x); h_j; \mathcal{H}_j; d_i)$ in the end point π is

$$L_3(y) := \lambda\Big(y'(\pi) + \mathcal{H}_1 y(\pi)\Big) - \mathcal{H}_2 y'(\pi) - \mathcal{H}_3 y(\pi) = 0.$$

Suppose that $\theta(x, \lambda)$ is the solution of (1) satisfying in the initial conditions $\theta(\pi, \lambda) = \mathcal{H}_2 - \lambda$, $\theta'(\pi, \lambda) = \lambda \mathcal{H}_1 - \mathcal{H}_3$ and the jump conditions (3). It is clear that $\phi(\lambda) := W(\varphi(\lambda), \theta(\lambda)) = -w(\pi) L_3(\varphi(\lambda))$ is the characteristic function of $L(q(x); h_j; \mathcal{H}_j; d_i)$ and the zeros of $\phi(\lambda)$ are eigenvalues of $L(q(x); h_j; \mathcal{H}_j; d_i)$, say $\{\mu_n\}_{n=1}^{\infty}$, are real and simple. This is a new operator and a new spectrum. Define $\tilde{\phi}(\lambda)$ by an analogous manner.

Lemma 2.4. *If $L(q(x); h_j; \mathcal{H}_j; d_i)$ and $L(\tilde{q}(x); h_j; \mathcal{H}_j; d_i)$ have the same eigenvalues then $\phi = \tilde{\phi}$.*

Proof. Using the Hadamard's factorization theorem for entire functions $\phi(\lambda)$ and $\tilde{\phi}(\lambda)$ of order $1/2$, we have

$$\phi(\lambda) = C \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right) \quad \text{and} \quad \tilde{\phi}(\lambda) = \tilde{C} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\tilde{\mu}_n}\right).$$

Define $M(\lambda) := \frac{\phi(\lambda)}{\tilde{\phi}(\lambda)}$. Note that the function $M(\lambda)$ is an entire function. Using the asymptotic forms of $\phi(\lambda)$ and $\tilde{\phi}(\lambda)$, we get

$$M(\lambda) = 1 + o(1), \quad \text{for } |\lambda| \to \infty.$$

Using Liouville's theorem for the entire function $M(\lambda)$, we get

$$\phi(\lambda) = \tilde{\phi}(\lambda).$$

$\square$

Lemma 2.5. *Let $\Lambda_0 \subset \mathbb{N}$ be a finite set and $\Lambda = \mathbb{N}\backslash\Lambda_0$. If $L(q(x); h_j; \mathcal{H}_j; d_i)$, $L(\tilde{q}(x); h_j; \mathcal{H}_j; d_i)$ have the same eigenvalues and, as well as, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of $L(q(x); h_j; H_j; d_i)$ and $L(\tilde{q}(x); h_j; H_j; d_i)$, then $\beta_n = \tilde{\beta}_n$ for all $n \in \Lambda$.*

Proof. From definition of φ, θ, ψ and $\tilde{\varphi}$, $\tilde{\theta}$, $\tilde{\psi}$, we get

$$W(\psi_n, \theta_n) = \beta_n W(\varphi_n, \theta_n) = \beta_n \phi(\lambda_n) \quad \text{and} \quad W(\tilde{\psi}_n, \tilde{\theta}_n) = \tilde{\beta}_n W(\tilde{\varphi}_n, \tilde{\theta}_n) = \tilde{\beta}_n \tilde{\phi}(\tilde{\lambda}_n)$$

So

$$\begin{aligned} \beta_n &= \frac{w(\pi)(\psi_n(\pi)\theta'_n(\pi) - \psi'_n(\pi)\theta_n(\pi))}{\phi(\lambda_n)} \\ &= \frac{\lambda_n^2(H_1 - \mathcal{H}_1) + \lambda_n(\mathcal{H}_2 H_1 - H_1 \mathcal{H}_2 + \mathcal{H}_3 - H_3) + H_3 \mathcal{H}_2 - H_2 \mathcal{H}_3}{\phi(\lambda_n)} \end{aligned} \tag{14}$$

and

$$\tilde{\beta}_n = \frac{w(\pi)(\tilde{\psi}_n(\pi)\tilde{\theta}'_n(\pi) - \tilde{\psi}'_n(\pi)\tilde{\theta}_n(\pi))}{\tilde{\phi}(\tilde{\lambda}_n)}$$

$$= \frac{\tilde{\lambda}_n^2(H_1 - \mathcal{H}_1) + \tilde{\lambda}_n(H_2\mathcal{H}_1 - H_1\mathcal{H}_2 + \mathcal{H}_3 - H_3) + H_3\mathcal{H}_2 - H_2\mathcal{H}_3}{\tilde{\phi}(\tilde{\lambda}_n)} \tag{15}$$

From $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and Lemma 2.4, we have $\phi(\lambda_n) = \tilde{\phi}(\tilde{\lambda}_n)$. Using the similar proof of ([5, Lemma:1.1.3]), we get $\phi(\lambda_n) \neq 0$. So, Eqs. (14)–(15) conclude $\beta_n = \tilde{\beta}_n$. $\square$

Assume that λ is not in the spectrum of (1)–(3) and let

$$S_\lambda := (A - \lambda I)^{-1}\big|_{\text{dom}(A)}.$$

Replace A by $\tilde{A}$ and define $\tilde{S}_\lambda$ analogously. We consider the following spaces

$$U := \text{dom}(A) \ominus \{\Phi_n : n \in \Lambda_0\}, \quad \text{and} \quad \tilde{U} := \text{dom}(\tilde{A}) \ominus \{\tilde{\Phi}_n : n \in \Lambda_0\}. \tag{16}$$

Define the transformation operator $T : U \to \tilde{U}$ by

$$T\Phi_n = \tilde{\Phi}_n, \tag{17}$$

for all $n \in \Lambda$. Note that by $\text{dom}(A) \ominus \{\Phi_n : n \in \Lambda_0\}$ we mean $\text{dom}(A)$ contains all of $\{\Phi_n\}_{n=1}^\infty$ except $\{\Phi_n\}_{n \in \Lambda_0}$, where Φ_n and $\tilde{\Phi}_n$ are defined in (11) and (13), respectively.

Lemma 2.6. *The operator $T : U \to \tilde{U}$ defined by (17) is bounded.*

Proof. From Lemma 2.3 we see that

$$\gamma_n = \| \Phi_n \|^2 = \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right] \tag{18}$$

and

$$\tilde{\gamma}_n = \| \tilde{\Phi}_n \|^2 = \mu(\rho_n; d_i; a_i; b_i)\left[1 + O\left(\frac{1}{n}\right)\right] \tag{19}$$

for all $n \in \Lambda$. Thus by (19) and (18), we get

$$\frac{\| T\Phi_n \|^2}{\| \Phi_n \|^2} = \frac{\| \tilde{\Phi}_n \|^2}{\| \Phi_n \|^2} = 1 + O\left(\frac{1}{n}\right).$$

$\square$

By using the similar proof of theorem ([5, Thm:1.2.1]) we get:

Theorem 2.2. *The system of eigenfunctions $\{\Phi_n(x)\}_{n \geq 0}$ of the boundary value problem A is complete in $L_2((0,\pi); w) \oplus \mathbb{C}^2$.*

Lemma 2.7. *The relation $\tilde{S}_\lambda T = TS_\lambda$ holds for $\lambda \neq \lambda_n, \tilde{\lambda}_n$ and $n \in \mathbb{N}$.*

Proof. Let $F \in U$, then we can expand F in terms of the set Φ_n

$$F(x) = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} = \sum_\Lambda f_n\Phi_n(x), \tag{20}$$

for $n \in \Lambda$, where $f_n = \frac{\langle F, \Phi_n \rangle_{\mathcal{H}}}{\langle \Phi_n, \Phi_n \rangle_{\mathcal{H}}}$. Let λ be in complex plane which is not an eigenvalue of $L(q; h_j; H_j; d_i)$, then the operator S_λ exists and can be written as

$$-S_\lambda F(x) = \sum_\Lambda \frac{f_n}{\lambda - \lambda_n}\Phi_n(x).$$

If we apply T to the above relation, we obtain

$$-TS_\lambda F(x) = \sum_\Lambda \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

If we apply $\tilde{S}_\lambda$ and T to (20) respectively, we obtain

$$-\tilde{S}_\lambda TF(x) = \sum_\Lambda \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

Then we get

$$\tilde{S}_\lambda T = TS_\lambda.$$

$\square$

3. Main result

In this section, we examine a different representation for T, in a general case when there are m discontinuous parameter-dependent boundary conditions. We generalize the well-known results of [8, 17, 4] to the finite number of jump conditions. Denote

$$G(x, y; \lambda) := \begin{cases} \frac{\varphi(x,\lambda)\psi(y,\lambda)}{\Delta(\lambda)}, & 0 < x < y < \pi, \\ \frac{\varphi(y,\lambda)\psi(x,\lambda)}{\Delta(\lambda)}, & 0 < y < x < \pi, \end{cases} \qquad (21)$$

where $x, y \neq d_i$. For simplicity, we can write

$$G(x, y; \lambda) = \frac{\varphi(x <)\psi(x >)}{\Delta(\lambda)}$$

where $x <:= \min\{x, y\}$ and $x >:= \max\{x, y\}$. Consider the function

$$Y(x, \lambda) = \int_0^\pi G(x, y; \lambda) f(y) w(y) dy.$$

The function $G(x, y; \lambda)$ is called the Green's function for L. $G(x, y; \lambda)$ is the kernel of the inverse operator for the Sturm–Liouville operator, i.e. $Y(x, \lambda)$ is the solution of the boundary value problem

$$\ell Y - \lambda Y = f(x), \quad U(Y) = V(Y) = 0;$$

and the jump conditions (3), this is easily verified by differentiation. Let C_n be a sequence of circles about the origin intersecting the positive λ-axis between λ_n and λ_{n+1}. By using the Green's function from (21), we get

$$\lim_{n \to \infty} \int_{C_n} \frac{G(x, y; \mu)}{\lambda - \mu} d\mu = 0, \quad \lambda \in \text{int } C_n. \qquad (22)$$

From residue integration, it follows that

$$\frac{1}{2\pi i} \int_{C_n} \frac{G(x, y; \mu)}{\lambda - \mu} d\mu = -G(x, y; \lambda) + \sum_{i=0}^n \frac{\varphi_i(x <)\psi_i(x >)}{\dot{\Delta}(\lambda_i)(\lambda - \lambda_i)}, \qquad (23)$$

where $\varphi_i(x) = \varphi(x, \lambda_i)$ and $\psi_i(x) = \psi(x, \lambda_i)$. By applying Mittag-Leffler expansion for $G(x, y; \lambda)$ and using (22) and (23), we obtain

$$G(x, y; \lambda) = \sum_{i=0}^\infty \frac{\varphi_i(x <)\psi_i(x >)}{\dot{\Delta}(\lambda_i)(\lambda - \lambda_i)},$$

where φ_k, ψ_k are eigenfunctions corresponding to the eigenvalues λ_k.

Theorem 3.1. *If $L(q(x); h_j; \mathcal{H}_j; d_i)$ and $L(\tilde{q}(x); h_j; \mathcal{H}_j; d_i)$ have the same spectrum and $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, then*

$$q - \tilde{q} = \sum_{\Lambda_0} (\tilde{y}_n \varphi_n)' w,$$

a.e. on $[0, d_1) \cup_{i=1}^{m-2} (d_i, d_{i+1}) \cup (d_{m-1}, \pi]$, where w is defined in (4), $\tilde{y}_n$ and φ_n are suitable solutions of $\tilde{\ell} y_n = -\tilde{y}_n'' - \tilde{q} \tilde{y}_n = \tilde{\lambda}_n \tilde{y}_n$ and $\ell \varphi_n = -\varphi_n'' - q \varphi_n = \lambda_n \varphi_n$, respectively.

Proof. By using the same techniques of [1] for $-S_\lambda \Phi_n = \mathcal{G}_n$, where $\mathcal{G}_n(x) := \mathcal{G}(x, \lambda_n) = (g_n(x) := g(x, \lambda_n), R_1(g_n), R_2(g_n))^T \in \mathcal{H}$, by simple calculation we can show that the relation

$$g_n''(x) + (\lambda - q(x)) g_n(x) = \varphi_n(x), \quad x \in \cup_{i=0}^{m-1} (d_i, d_{i+1}), \tag{24}$$

$$\lambda(g_n'(0) + h_1 g_n(0)) - h_2 g_n'(0) - h_3 g_n(0) = 0,$$
$$\lambda(g_n'(\pi) + H_1 g_n(\pi)) - H_2 g_n'(\pi) - H_3 g_n(\pi) = 0, \tag{25}$$

and

$$U_i(g_n) = 0, \text{ and } V_i(g_n) = 0, \quad i = 1, 2, \ldots, m-1 \tag{26}$$

are satisfied. The equation (24) with (25) and (26) has the unique solution (i.e. $g_n(x)$), which can be represented as

$$g_n(x) := \int_0^x G(x, t; \lambda_n) \varphi_n(t) w(t) dt. \tag{27}$$

The formula (27) reduces to

$$\mathcal{G}_n(x) = \begin{pmatrix} g_n(x) \\ R_1(g_n) \\ R_2(g_n) \end{pmatrix} = \begin{pmatrix} \int_0^x G(x, t; \lambda_n) \varphi_n(t) w(t) dt \\ \frac{R_1(\varphi_n)}{\lambda - \lambda_n} \\ \frac{R_2(\varphi_n)}{\lambda - \lambda_n} \end{pmatrix} \tag{28}$$

and the function $G(x, t; \lambda)$ is as defined in (21). Using the asymptotic forms of $\varphi(x, \lambda)$, $\psi(x, \lambda)$, $\Delta(\lambda)$ for sufficiently large ρ and $\rho \neq \rho_n$, we deduce that the Green's function $G(x, t; \lambda)$ is bounded. $G(x, t; \lambda)$ is a meromorphic function with the eigenvalues λ_k as its poles [1]. Let $(f(x), R_1(f), R_2(f))^T \in U$, from (16), (21), (28), and Lemma 2.5 we have

$$
\begin{aligned}
(y(x), R_1(y), R_2(y))^T &= S_\lambda (f(x), R_1(f), R_2(f))^T \\
&= S_\lambda F(x) \\
&= \begin{pmatrix} \sum_\Lambda \frac{\beta_n \varphi_n(x) \int_0^\pi \varphi_n(t) f(t) w(t) dt}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ \sum_\Lambda \frac{f_n R_1(\varphi_n)}{\lambda - \lambda_n} \\ \sum_\Lambda \frac{f_n R_2(\varphi_n)}{\lambda - \lambda_n} \end{pmatrix}.
\end{aligned}
\tag{29}
$$

By applying T to both sides of (29), we see that

$$
T S_\lambda F(x) = \begin{pmatrix} \sum_\Lambda \frac{\beta_n \tilde{\varphi}_n(x) \int_0^\pi \varphi_n(t) f(t) w(t) dt}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \\ \sum_\Lambda \frac{f_n R_1(\tilde{\varphi}_n)}{\lambda - \lambda_n} \\ \sum_\Lambda \frac{f_n R_2(\varphi_n)}{\lambda - \lambda_n} \end{pmatrix}. \tag{30}
$$

Define
$$\mathfrak{U}(x) := \begin{pmatrix} \frac{1}{\Delta(\lambda)}\tilde{\psi}(x)\int_0^x \varphi(y)f(y)w(y)dy + \tilde{\varphi}(x)\int_x^\pi \psi(y)f(y)w(y)dy \\ R_1(y) \\ R_2(y) \end{pmatrix}.$$

By applying the Mittag–Leffler expansion for $\mathfrak{U}(x)$, we have
$$\begin{pmatrix} \sum_{\Lambda_0} \frac{\tilde{u}_n(x)\int_0^x \varphi_n(y)f(y)w(y)dy + \tilde{z}_n(x)\int_x^\pi \psi_n(y)f(y)w(y)dy}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)} \\ + \sum_{\Lambda} \frac{\tilde{\psi}_n(x)\int_0^x \varphi_n(y)f(y)w(y)dy + \tilde{\varphi}_n(x)\int_x^\pi \psi_n(y)f(y)w(y)dy}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)} \\ \sum_{n\in\mathbb{N}} \frac{f_n R_1(\tilde{\varphi}_n)}{\lambda-\lambda_n} \\ \sum_{n\in\mathbb{N}} \frac{f_n R_2(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}.$$

By using (30) the second term of the above expression is equal to $TS_\lambda F$ and $\tilde{u}_n(x)$ and $\tilde{z}_n(x)$ represents $\tilde{\psi}(x,\tilde{\lambda}_n)$ and $\tilde{\varphi}(x,\tilde{\lambda}_n)$ respectively. Hence
$$\tilde{S}_\lambda TF(x) = \mathfrak{U}(x) - \begin{pmatrix} \sum_{n\in\Lambda_0} \frac{\tilde{u}_n(x)\int_0^x \varphi_n(y)f(y)w(y)dy + \tilde{z}_n(x)\int_x^\pi \psi_n(y)f(y)w(y)dy}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)} \\ \sum_{n\in\Lambda_0} \frac{f_n R_1(\tilde{\varphi}_n)}{\lambda-\lambda_n} \\ \sum_{n\in\Lambda_0} \frac{f_n R_2(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}.$$

By using a simple calculation by applying Eq. (5), and the following relation
$$\int_0^\pi \psi_n \overline{f} w + \frac{w(0)}{r_1}R_1(\psi_n)\overline{f_1} + \frac{w(\pi)}{r_2}R_2(\psi_n)\overline{f_2} = 0,$$

we get
$$Tf(x) = f(x) - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_n(x)\int_0^x \varphi_n(t)f(t)w(t)dt$$
$$+ \sum_{\Lambda_0} \frac{f_n \beta_n \tilde{z}_n(x)}{\dot{\Delta}(\lambda_n)}\left(\frac{w(0)}{r_1}R_1(\psi_n)R_1(\varphi_n) + \frac{w(\pi)}{r_2}R_2(\psi_n)R_2(\varphi_n)\right),$$

where
$$\frac{1}{2}\tilde{y}_n(x) = \frac{\tilde{u}_n(x) - \beta_n \tilde{z}_n(x)}{\dot{\Delta}(\lambda_n)}.$$

From Lemma 2.4 it follows that
$$\tilde{A}TF = TAF. \tag{31}$$

Suppose that $F = \Phi_n$ $(n\in\Lambda)$ then we get $f_m = \frac{\langle\Phi_n,\Phi_m\rangle_{\mathcal{H}}}{\langle\Phi_m,\Phi_m\rangle_{\mathcal{H}}} = 0$, for $m\in\Lambda_0$. For left and right sides of (31) we get
$$\tilde{A}T\Phi_n = \tilde{A}\begin{pmatrix} \varphi_n - \frac{1}{2}\sum_{m\in\Lambda_0}\tilde{y}_m\int_0^x \varphi_m\varphi_n w \\ R_1(\tilde{\varphi}_n) \\ R_2(\tilde{\varphi}_n) \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{\ell}\varphi_n - \frac{1}{2}\sum_{m\in\Lambda_0}\tilde{\ell}\tilde{y}_m\int_0^x \varphi_m\varphi_n w \\ -\frac{1}{2}\sum_{m\in\Lambda_0}2\tilde{y}'_m(\varphi_m\varphi_n)w - \frac{1}{2}\sum_{m\in\Lambda_0}\tilde{y}_m(\varphi_m\varphi_n w)' \\ R'_1(\tilde{\varphi}_n) \\ R'_2(\tilde{\varphi}_n) \end{pmatrix}$$

$$\tag{32}$$

and

$$TA\Phi_n = \begin{pmatrix} T\ell\varphi_n = T(-\varphi_n'' + q\varphi_n) \\ R_1'(\tilde{\varphi}_n) \\ R_2'(\tilde{\varphi}_n) \end{pmatrix}$$

$$= \begin{pmatrix} \ell\varphi_n - \frac{1}{2}\sum\limits_{m\in\Lambda_0}\tilde{y}_m\int_0^x \varphi_m\varphi_n w - \frac{1}{2}\sum\limits_{m\in\Lambda_0}\tilde{y}_m(\varphi_n\varphi_m' - \varphi_m\varphi_n')w \\ R_1'(\tilde{\varphi}_n) \\ R_2'(\tilde{\varphi}_n) \end{pmatrix}.$$

(33)

From (31)–(33), we deduce that

$$q - \tilde{q} = \sum_{\Lambda_0}(\tilde{y}_m\varphi_m)'w.$$

If Λ_0 is empty, then T is a unitary operator and $A = \tilde{A}$. Hence $q = \tilde{q}$. This completes the proof. □

Theorem 3.2. *Suppose that $\lambda_n = \tilde{\lambda}_n$ and $\gamma_n = \tilde{\gamma}_n$, for all $n \in \mathbb{Z}^0$, where γ_n and $\tilde{\gamma}_n$ are defined in (18) and (19), then*

$$q = \tilde{q}.$$

Proof. Applying Lemma 2.4 to $L(q(x); h_j; H_j; d_i)$ and $L(\tilde{q}(x); h_j; H_j; d_i)$ in place of $L(q(x); h_j; \mathcal{H}_j; d_i)$ and $L(\tilde{q}(x); h_j; \mathcal{H}_j; d_i)$ we obtain $\Delta(\lambda) = \tilde{\Delta}(\lambda)$. Hence

$$\dot{\Delta}(\lambda_n) = \dot{\tilde{\Delta}}(\lambda_n)$$

for all $n \in \mathbb{Z}^0$. From Lemma 2.2 and the assumptions we get $\beta_n = \tilde{\beta}_n$. The rest of proof follows form Theorem 3.1. □

4. Conclusion

In this paper, the inverse Sturm–Liouville problems with finite number of transmission and parameter dependent boundary conditions was studied. For this purpose, a new Hilbert space by defining a new inner product for obtaining a self–adjoint operator was defined. So, the asymptotic form of solutions, eigenvalues and eigenfunctions of this problem was obtained. Finally, we formulated the Hochestadt's result based on transformation operator for inverse Sturm–Liouville problems.

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