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SHIFTED LEGENDRE POLYNOMIAL SOLUTIONS OF NONLINEAR STOCHASTIC ITÔ - VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this article, we propose the shifted Legendre polynomial-based solution for solving a stochastic integral equation. The properties of shifted Legendre polynomials are discussed. Also, the stochastic operational matrix required for our proposed methodology is derived. This operational matrix is capable of reducing the given stochastic integral equation into simultaneous equations with N+1 coefficients, where N is the number of terms in the truncated series of function approximation. These unknowns can be found by using any well-known numerical method. In addition to the capability of the operational matrices, an essential advantage of the proposed technique is that it does not require any integration to compute the constant coefficients. This approach may also be used to solve stochastic differential equations, both linear and nonlinear, as well as stochastic partial differential equations. We also prove the convergence of the solution obtained through the proposed method in terms of the expectation of the error function. The upper bound of the error in L^2 norm between exact and approximate solutions is also elaborately discussed. The applicability of this methodology is tested with a few numerical examples, and the quality of the solution is validated by comparing it with other methods with the help of tables and figures.

Keywords: Nonlinear stochastic Itô - Volterra integral equation; shifted Legendre polynomial, stochastic operational matrix, convergence analysis; error estimation.

AMS Subject Classification: 65C30, 60G42, 60H35, 60H10, 65C20, 60H20, 68U20.

1. Introduction

The addition of one or more random elements, which is often considered as the noise term, to the deterministic models results in stochastic models like the stochastic differential equation, stochastic integral equation, etc. Such models are used to study various physical or biological phenomena in multiple fields like biology, medicine, population dynamics, mechanics, and finance [1, 2, 3, 4]. The Numerical solution of stochastic quadratic

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integral equations using operational matrices is studied by F. Mirzaee and N. Samadyar [5]. The Numerical solution for stochastic Itô - Volterra integral equations of fractional order is studied in [6] and for such an equation driven by fractional Brownian motion is studied in [7]. Quintic B-spline collocation method, Cubic and bicubic B-spline collocation methods are applied to obtain the numeric solution for linear and nonlinear stochastic integral equations as seen in [8, 9]. The Spectral collocation method together with the moving least square scheme to solve stochastic Volterra type equations is studied as in [10, 11]. Orthonormal Bernoulli polynomials collocation approach for stochastic Itô Volterra integral equations can be seen in [12] and for nonlinear Stratonovich Volterra integral equations in [13, 14]. Motivated by the preceding works, in this paper, we consider the following stochastic model to study the random effects of population growth in the form of a stochastic integral equation [15, 16].

$$X(t) = f(t) + \int_0^t k_1(s,t)N_1(s,X(s))ds + \int_0^t k_2(s,t)N_2(s,X(s))dW(s), t \in [0,1], \quad (1)$$

where $f(t), k_1(s,t), k_2(s,t)$, $N_1(s,X)$ and $N_2(s,X)$ are linear or nonlinear and $X(t)$ is to be determined. All the above processes, including $X(t)$ are the stochastic processes defined on the probability space. Here, $W(t)$ is a standard Brownian motion whose detailed information is discussed in Section 2. The stochastic non-autonomous logistic equation and the population growth model in a closed system are the variants of the existing model. In the financial market, Eq.(1.1) is used to study the behaviour of the stock price with risky assets $X(t)$, the spot price $f(t) = X_0$ at time 0, $k_1(s,t) = \mu(s), k_2(s,t) = \sigma(s)$, $N_1 = N_2 = X(s)$ and $W(t)$ is the standard Brownian motion with $W(0) = 0$. This model is valid on [0,T], T is the maturity of the option and the resultant model is linear. The system of integral equations in the stochastic form is used to study stochastic linear and nonlinear pendulum problems with damping, frequency and excitation in a stochastic sense. The stochastic delay differential equations are used to study the problems arising in the field of reactor dynamics and the theory of automatic systems [17, 18].

Handling the nonlinear terms N_1 and N_2, in terms of the unknown stochastic process $X(s)$, is not an easy task. To find the approximate solutions of these stochastic equations, several numerical methods with their variations have been utilised by various researchers [19].The approximate or the analytical solutions of stochastic Volterra integral equations based on various polynomials have been handled by many researchers [20, 21, 22, 23, 24, 25, 26, 27].

In recent years, to find an approximate solution, good approximation methods based on the orthogonal basis of the polynomials have attracted the attention of mathematicians' interest. The Jacobi polynomial, which arises as Eigen functions of the singular Sturm-Liouville problem, is one such polynomial [28]. A collection of polynomials like Legendre, Chebyshev, and other spherical polynomials on [-1,1] are the solutions to the above problem, and these polynomials are generated from the Jacobi polynomials by assigning particular values to their parameters.

In the case of [0,1], we use the shifted version of Legendre polynomials called shifted Legendre polynomials [29] to find an approximate solution of Eq(1.1) numerically. The salient features of these polynomials, together with the operational matrices of integration and stochastic integration, jointly guide us to convert the given equation into a system of simultaneous algebraic equations. Solving this system of equations by any known numerical method leads us to the numerical solution of the problem under consideration. Some of the most important advantages of the proposed method are listed as follows:

- The proposed method reduces the solution of the problem considered to a system of algebraic equations, which is solved using an appropriate numerical method
- We have used shifted Legendre polynomials that have orthogonal property. This property is very useful in numerical methods and is more convenient than the other non orthogonal polynomials.
- The proposed method provides a more accurate solution and is easy to implement as it involves sparse matrices.

The overview of this paper comprises the following: The fundamental definitions and theorems that are required for our subsequent study are given in Section 2. The fundamentals of shifted Legendre polynomials and their properties are discussed in Section 3. Various operational matrices required for the proposed method are also derived. In Section 4, the convergence theorems and the error estimates are presented in detail. The accuracy and applicability of the scheme are tested on several examples, and the comparative results are also presented in Section 5. The superiority of this method is also highlighted in that section. Concluding remarks are given in the final section.

2. Mathematical Background

In this section, we provide the fundamental definitions of stochastic calculus and information pertaining to our subsequent study [30, 31, 32]. We start by defining Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathcal{F}, P)$ can be constructed on the space $\Omega = C_0(R_+)$ of continuous real-valued functions on R_+ starting at 0. Next, we introduce the idea of Hilbert space and Banach space, where the concept of defining a norm has been established in the probability space $(\Omega, \mathcal{F}, P)$. The idea of convergence of a sequence X_n in the given space, where the function is defined, is also discussed. The basic properties of Itô integral and Itô isometry are also elucidated for our subsequent development.

Definition 2.1. [33] *Let $(\Omega, \mathcal{F}, P)$be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$ -adapted process $\{B_t\}_{t\geq 0}$ with the following properties:*
(i) $B_0 = 0$ a.s.;
(ii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;
(iii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\{\mathcal{F}_s\}$.

Definition 2.2. [30] *Let $p \geq 2$ and $L^p(\Omega, H)$ be the collection of all strongly measurable random variables and if $\|V\|_{L^p} = \{E\,|V|^p\}^{1/p} = \left(\int_\Omega |V|^p\, dP\right)^{1/p}$, for each $V \in L^P(\Omega, H)$ then $L^p(\Omega, H)$ is a Banach space.*

Definition 2.3. [30] *Let $A, B \in [0, T] \to \mathbb{R}$ and if $A(t) \leq \lambda + \int_0^t B(s)A(s)ds$ for $t \in [0, T]$ then*
$A(t) \leq \lambda \left(\int_0^t B(s)ds\right)$ for all $t \in [0, T]$ with $\lambda \geq 0$.

Definition 2.4. [23] *The sequence X_n converges to X in L^2 if $E(|X_n|^2) < \infty$ and $E(\|X_n - X\|)^2 \longrightarrow 0$ when $n \to \infty$.*

Definition 2.5. [31] *The Itô integral of $f \in v(s, T)$ is defined by $\int_s^T f(t, w)dB(t)(w)$ $= lim_{n\to\infty} \int_s^T \varphi_n(t, w)dB$, where φ_n is the sequence of elementary functions such that $E\left(\int_s^T (f - \varphi_n)^2 dt\right) \to 0$ as $n \to \infty$.*

Lemma 2.1. [31] *The Itô isometry of* $f \in v(s,T)$ *is given by* $E\left((\int_s^T (f(t,w)dB(t)(w))^2\right) = E\left(\int_s^T (f^2(t,w)dt)\right)$.

3. Shifted Legendre Polynomials

3.1. **Preliminaries and properties.** The Legendre polynomials $P_n(z)$ are the solutions of Legendre's Differential Equation [34]. The orthogonal property of Legendre polynomials is defined as $\int_{-1}^{1} P_n(z)P_m(z)dz = \frac{2}{2n+1}\delta_{nm}$, where δ_{nm} is the Kronecker delta. The shifted Legendre polynomials are derived from $P_n(z)$ by replacing z by 2t-1, denoted by $L_n(t)$ thereby refined interval is [0,1]. The orthogonal property of $L_n(t)$ with Kronecker delta in [0,1] is defined by

$$\int_0^1 L_n(t)L_m(t)dt = \frac{1}{2n+1}\delta_{nm}.$$

Then (i) the recurrence relation of $L_n(t)$ is defined as

$$L_{i+1}(t) = \frac{(2i+1)(2t-1)}{i+1}L_i(t) - \frac{i}{i+1}L_{i-1}(t), i = 1, 2..., \tag{2}$$

where $L_0(t) = 1$ and $L_1(t) = 2t - 1$.
(ii) The analytic form of the shifted Legendre polynomials $L_n(t)$ of degree n is given by

$$L_n(t) = \sum_{i=0}^{n} (-1)^{n+i} \frac{(n+i)!}{(n-i)!} \frac{t^i}{(i!)^2}. \tag{3}$$

Note that $L_n(0) = (-1)^n$ and $L_n(1) = 1$.
(iii) The shifted Legendre vector L(t) is normally defined as

$$L(t) = [L_0(t) \quad L_1(t) \quad . \quad . \quad . \quad L_N(t)]^T. \tag{4}$$

(iv) the matrix form of L(t) which is of degree N can be represented as

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ (-1)^{1+0}\frac{(1+0)!}{(1-0)!(0!)^2} & (-1)^{1+1}\frac{(1+1)!}{(1-1)!(1!)^2} & \dots & 0 \\ (-1)^{2+0}\frac{(2+0)!}{(2-0)!(0!)^2} & (-1)^{2+1}\frac{(2+1)!}{(2-1)!(1!)^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{N+0}\frac{(N+0)!}{(N-0)!(0!)^2} & (-1)^{N+1}\frac{(N+1)!}{(N-1)!(1!)^2} & \dots & (-1)^{N+N}\frac{(N+N)!}{(N-N)!(N!)^2} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix}. \tag{5}$$

Thus

$$L(t) = DY(t) \tag{6}$$

The dual matrix Q_1 is given by

$$\begin{aligned} Q_1 = \int_0^1 L(t)L^T(t)dt &= \int_0^1 DY(t)(DY(t))^T dt \\ &= D\left(\int_0^1 Y(t)Y^T(t)dt\right)D^T \\ &= DHD^T, \end{aligned} \tag{7}$$

where H, a Hilbert matrix of order (N+1) is given by

$$H = \int_0^1 Y(t)Y^T(t)dt = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1} \end{bmatrix}.$$

Theorem 3.1. [34] *Any function $u(t) \in L^2[0,1]$ can be approximated in terms of $L_n(t)$ as*

$$u(t) = \sum_{n=0}^{\infty} u_n L_n(t), \tag{8}$$

from which the coefficients u_j are given by

$$u_j = (2j+1) \int_0^1 u(x) L_j(x) dx, j = 0, 1, ... \tag{9}$$

If we approximate $u(t)$ by the first $N+1$ terms, then we can write

$$u(t) \simeq \sum_{n=0}^{N} u_n L_n(t) = U^T L(t) = L^T(t) U,$$

where U is the shifted Legendre coefficient vector given by

$$U = [u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T.$$

We approximate the kernel function by truncating the Taylor series of degree N in the form

$$k(s,t) = \sum_{m=0}^{N} \sum_{n=0}^{N} k_{mn} s^m t^n,$$

where $k_{mn} = \frac{1}{m!n!} \frac{\partial^{m+n} k(0,0)}{\partial s^m \partial t^n}, \ n, m = 0, 1, ..., N$.
The matrix form of the above expression is given by $k(s,t) = Y(s) K Y^T(t)$. Additionally, the kernel function $k(s,t)$ can be expanded approximately by $L_m(s)$ and $L_n(t)$ of degree N in the form

$$k_N(s,t) = \sum_{m=0}^{N} \sum_{n=0}^{N} L_{k_{mn}} L_m(s) L_n(t),$$

and the matrix form of $k(s,t)$ in terms of $L(s)$ and $L^T(t)$ is

$$k(s,t) = L(s) K_L L^T(t), K_L = L_{k_{mn}}.$$

3.2. **Operational Matrices.** In the subsequent parts of this section, we construct the operational matrices as follows. We define the product matrix $Q(t)$, as

$$Q(t) = L(t) L^T(t), \tag{10}$$

where $Q(t)$ is a matrix of order $(N+1)$. Let $U = [u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T$, then

$$Q(t) U \simeq \hat{U} L(t). \tag{11}$$

$\hat{U}$ is called the product operational matrix of shifted Legendre polynomial which is calculated as

$$Q(t)U = D\left[\sum_{i=0}^{N} u_i L_i(t) \quad \sum_{i=0}^{N} u_i t L_i(t) \quad . \quad . \quad . \quad \sum_{i=0}^{N} u_i t^n L_i(t)\right]^T. \tag{12}$$

By approximating each $t^k L_i(t)$ by $L^T(t)C_{k,i}$, we get

$$C_{k,i} = [C_0^{k,i} \quad C_1^{k,i} \quad . \quad . \quad . \quad C_N^{k,i}]^T.$$

From Eq.(7) we have

$$\int_0^t t^k L_i(t)L(t)dt \simeq \left[\int_0^t L(t)L^T(t)dt\right] C_{k,j} = Q_1 C_{k,j}.$$

Therefore, for each i and k , we get

$$\begin{aligned} C_{k,i} &\simeq Q_1^{-1}\int_0^t t^k L(t)L_i(t)dt \\ &= Q_1^{-1}\left[\int_0^t t^k L_0(t)L_i(t)dt \quad \int_0^t t^k L_1(t)L_i(t)dt \quad . \quad . \quad . \quad \int_0^t t^k L_N(t)L_i(t)dt\right]^T. \end{aligned}$$

Now the term $\sum_{i=0}^{N} u_i t^k L_i(t)$ can be computed as follows

$$\begin{aligned} \sum_{i=0}^{N} u_i t^k L_i(t) &\simeq \sum_{i=0}^{N} u_i L^T(t) C_{k,i} \\ &= \sum_{i=0}^{N} u_i \sum_{j=0}^{N} L_j(t) C_j^{k,i} \\ &= \sum_{j=0}^{N} L_j(t) \sum_{j=0}^{N} u_i C_j^{k,i} \\ &= L^T(t)\left[\sum_{i=0}^{N} u_i C_0^{k,i} \quad \sum_{i=0}^{N} u_i C_1^{k,i} \quad . \quad . \quad . \quad \sum_{i=0}^{N} u_i C_N^{k,i}\right]^T \\ &= L^T(t)[C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]^T U \end{aligned}$$

$$= L^T(t)\hat{C}_k. \tag{13}$$

where $\hat{C}_k = [C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]U$, $k = 0, 1, 2 \ldots N$.
From Eqs.(12) and (13), we obtain $\hat{U} = D\hat{L}^T$. The integrals of $L_n(s)$ are evaluated with the aid of recurrence property of $L_n(t)$

$$\int_0^t L_n(s)ds = \frac{1}{2(2n+1)}[L_{n+1}(t) - L_{n-1}(t)]. \tag{14}$$

Therefore,

$$\int_0^t L(s)ds = PL(t) - \frac{1}{2(2n+1)}L_{n+1}(t), \tag{15}$$

where P is the matrix, which denotes the integration matrix of polynomials, given by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0\ldots & 0 & 0 \\ \frac{-1}{6} & 0 & \frac{1}{6} & 0\ldots & 0 & 0 \\ 0 & \frac{1}{10} & 0 & \frac{1}{10}\ldots & 0 & 0 \\ 0 & 0 & \frac{-1}{14} & 0\ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0\ldots & 0 & \frac{1}{2(2m-3)} \\ 0 & 0 & 0 & 0\ldots & \frac{-1}{2(2m-3)} & 0 \end{bmatrix}. \tag{16}$$

The integration of the vector $L(t)$ can be approximated from Eq.(15)

$$\int_0^t L(s)ds \simeq PL(t). \tag{17}$$

Hence any function $f(t)$ can be approximated as

$$\int_0^t f(s)ds \simeq \int_0^t F^T L(s)ds = F^T PL(t). \tag{18}$$

3.3. **Stochastic operational matrix of shifted Legendre polynomials.** For the the vector $L(t)$, we define its Itô integral with stochastic operational matrix of integration P_s

$$\int_0^t L(s)dW(s) = P_s L(t) \tag{19}$$

$$\int_0^t L(s)dW(s) = \int_0^t DX(s)dW(s) \tag{20}$$

$$= D\left[\int_0^t dW(s) \quad \int_0^t sdW(s) \quad . \quad . \quad . \quad \int_0^t s^N dW(s)\right]^T$$

$$= D\left[W(t)Y(t) - \left[0 \quad \int_0^t dW(s) \quad . \quad . \quad . \quad N\int_0^t s^{N-1}dW(s)\right]^T\right]$$

$$= D\vartheta(t) = D(\lambda_i), i = 0, 1, ..., N,$$

where $\lambda_i = t^i W(t) - \int_0^t s^{i-1}W(s)ds$, $i = 0, 1, ..., N$.
Evaluating the integral for each i, we get
$\lambda_i = t^i W(t) - \frac{t^i}{4}(2(\frac{t}{2})^{i-1}W(\frac{t}{2}) + t^{i-1}W(t)) = [(1 - \frac{i}{4})W(t) - \frac{i}{2}W(\frac{t}{2})]t^i$.
We assume that $W(0.5)$ and $W(0.25)$ are the approximate value of $W(t)$ and $W(\frac{t}{2})$ respectively for any value of $t \in [0, 1]$. Hence $D\vartheta(t)$ is given by

$$D\vartheta(t) = D\ \Gamma_s \begin{pmatrix} 1 \\ t \\ \vdots \\ t^N \end{pmatrix},$$

where

$$\Gamma_s = \begin{pmatrix} W(0.5) & 0 & \ldots & 0 \\ 0 & \frac{3}{4}W(0.5) - \frac{1}{2}W(0.25) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (1 - \frac{N}{4})W(0.5) - \frac{N}{2^N}W(0.25) \end{pmatrix}.$$

Hence, $D\vartheta(t) = D\Gamma_s Y(t)= D\Gamma_s D^{-1}L(t)= P_s L(t)$.
where $P_s= D\Gamma_s D^{-1}$.
By using Eqs.(8) and(17), the Itô integral of $u(t)$ is defined as

$$\int_0^t u(s)dW(s) = \int_0^t U^T L(s)dW(s) = U^T P_s L(t). \tag{21}$$

Let

$$\phi_i(t) = N_i(t, X(t)), i = 1, 2. \tag{22}$$

Using Eq.(1) in Eq.(22),

$$\phi_i(t) = N_i(t, f(t) + \int_0^t k_1(s,t)\phi_1(s)ds + \int_0^t k_2(s,t)\phi_2(s)dW(s)), i = 1, 2. \tag{23}$$

Approximating the above mentioned functions in terms of $L(s)$ and $L^T(t)$ in the following manner

$$f(t) \simeq L^T(t)F \tag{24}$$

$$k_i(s,t) \simeq L^T(t)K_{iL}^T L(s), i = 1, 2. \tag{25}$$

$$\phi_i(t) \simeq L^T(t)\Phi_i, i = 1, 2. \tag{26}$$

Here, F, Φ_1, Φ_2 are $(N+1)$ column vectors and K_{1L} and K_{2L} are
$(N+1)\times(N+1)$ matrices.
By substituting Eqs.(24) - (26) in Eq.(23), we have

$$\begin{aligned} L^T(t)\Phi_i(t) = N_i(t, L^T(t)F + \int_0^t L^T(t)K_{1L}^T L(s)L^T(s)\Phi_1(s)ds + \\ \int_0^t L^T(t)K_{2L}^T L(s)L^T(s)\Phi_2(s)dW(s)), i = 1, 2. \end{aligned} \tag{27}$$

By using Eqs.(11, 17 and 19), Eq.(27) becomes

$$L^T(t)\Phi_i(t) = N_i(t, L^T(t)F + L^T(t)k_{1L}^T\hat{\Phi}_1 PL(t) + L^T(t)k_{2L}^T\hat{\Phi}_2 P_s L(t)), i = 1, 2. \tag{28}$$

We collocate Eq.(28) at $(N+1)$ points using the formula

$$t_s = \frac{2s+1}{2(N+1)}, s = 0, 1, \ldots, N. \tag{29}$$

Therefore,

$$L^T(t_s)\Phi_i(t) = N_i(t_s, L^T(t_i)F + L^T(t_s)K_{1L}^T\hat{\Phi}_1 PL(t_s) + L^T(t_s)K_{2L}^T\hat{\Phi}_2 P_s L(t_s)), \tag{30}$$

where $i = 1, 2$. By collocating Eq.(30) at these N+1 points, we get a nonlinear system of $2(N+1)$ algebraic equations from which the coefficients can be obtained by using Newton's method.
Hence the approximate solution of Eq.(1) is obtained as

$$X(t) \simeq L^T(t)F + L^T(t)K_{1L}^T\hat{\Phi}_1 PL(t) + L^T(t)K_{2L}^T\hat{\Phi}_2 P_s L(t). \tag{31}$$

4. Theoretical Analysis

Let $e_N(t) = X(t) - X_N(t)$ be the error function where $X_N(t)$ is the Nth degree approximation of the exact solution $X(t)$. The error bound and convergence theorem for the proposed method in terms of the function approximation and the error function are discussed here.

Theorem 4.1. *Let $f_N(t)$ be the function approximation of $f(t)$ then the error bound is given by $\|f(t) - f_N(t)\|_{L^2} \leq C\hat{F}(2)^{-N}, t \in [0,1]$, where $\hat{F} =^{sup}_{t} \left\|f^{(N)}(t)\right\|_{L^2}$, C being a constant.*

Proof.

$$\begin{aligned} \|f(t) - f_N(t)\|^2 &= \int_0^t (f(t) - f_N(t))^2 dt \\ &\leq \int_0^t \left(\frac{1}{N!2^N}\hat{F}dt\right)^2 \\ &= \left(\frac{1}{N!2^N}\hat{F}\right)^2 \\ &= (C\hat{F}2^{-N})^2, \end{aligned}$$

where $C = \frac{1}{N!}$ and $\hat{F} =^{sup}_{t} \left\|f^{(N)}(t)\right\|, t \in [0,1]$. □

Theorem 4.2. *Let $k_N(s,t)$ be the shifted Legendre approximation of the function $k(s,t)$ then we have,*
$\|k(s,t) - k_N(s,t)\| \leq \hat{C}\hat{K}(2)^{-2N}$,
where $\hat{C}$ is a positive constant, $\hat{K} =^{sup}_{(s,t)} \left\|\frac{\partial^{2n}k(s,t)}{\partial s^n \partial t^n}\right\|$, $(s,t) \in [0,1] \times [0,1]$.

Proof. Proof of this theorem is based on the assumptions and the steps followed in Theorem 4.1. □

Theorem 4.3. *Let $X_N(t)$ be the approximate solution of the exact solution $X(t)$ with $N_1(s,t)$, $N_2(s,t)$ satisfying the Lipschitz condition*

$$\|N_1(s,t_1) - N_1(s,t_2)\| + \|N_2(s,t_1) - N_2(s,t_2)\| \leq L\|t_1 - t_2\|. \tag{32}$$

Also assume that
i)$\|\Phi_i(t)\| \leq \rho_i, t \in [0,1]$
ii)$\|k_i(s,t)\| \leq M_i$ *, for every (s,t) defined in the domain* $[0,1] \times [0,1]$
iii) $G(N) < 1$ for i=1,2
Then we have
I) $\|X(t) - X_N(t)\| \leq \frac{\eta(N) + ((M_1+\psi(N))\beta_1(N) + \psi(N)\rho_1) + \|W(t)\|((M_2+\gamma(N))\beta_2(N)+\gamma(N)\rho_2)}{1-G(N)}$
II) $X_n(t) \to X(t)$ in L^2 when $E\left(|e_N(t)|^2\right) \to 0$ where $\eta(N) = C\hat{F}(2)^{-N}$
$\lambda(N) = \hat{C}_1(2)^{-2N}$
$\gamma(N) = \hat{C}_2(2)^{-2N}$
$\beta_i(N) = C\hat{\Phi}_i(2)^{-N}$.
$\hat{\Phi}_i = sup\left\|\Phi_i^{(N)}(t)\right\|$ $n = 0,1,2,\ldots$.

Proof. Proof of I) : Let $\hat{\phi}_i(s)$ be the approximate solution of $\phi_i(s)$ of Eq.(22) and Eq.(23) respectively. Then we have

$$\hat{\phi}_i(s) = \hat{N}_i(s, X_N(s)), i = 1,2 \tag{33}$$

and

$$\phi_i^N(s) = N_i(s, X_N(s)), i = 1, 2. \tag{34}$$

Hence by the above theorems, we have

$$\begin{aligned} \left\|\phi_i(s) - \hat{\phi}_i(s)\right\| &\leq \left\|\phi_i(s) - \phi_i^N(s)\right\| + \left\|\phi_i^N(s) - \hat{\phi}_i(s)\right\| \\ &\leq L\left\|X(s) - X_N(s)\right\| + \beta_i(N), i = 1, 2. \end{aligned} \tag{35}$$

Also the approximate of Eq.(1) is given as

$$X_N(t) = f_N(t) + \int_0^t k_{1N}(s,t)\hat{\phi}_1(s)ds + \int_0^t k_{2N}(s,t)\hat{\phi}_2(s)dW(s).$$

Hence, the norm of the error function is given by

$$\begin{aligned} \left\|X(t) - X_N(t)\right\| \leq \left\|f(t) - f_N(t)\right\| + \left\|k_1(s,t)\phi_1(s) - k_{1N}(s,t)\hat{\phi}_1(s)\right\| + \\ \left\|W(t)\right\| \left\|k_2(s,t)\phi_2(s) - k_{2N}(s,t)\hat{\phi}_2(s)\right\|. \end{aligned} \tag{36}$$

By using Theorems 4.1, 4.2 and assumptions (i) and (ii) of Theorem 4.3, we have

$$\begin{aligned} \left\|k_1(s,t)\phi_1(s) - k_{1N}(s,t)\hat{\phi}_1(s)\right\| \leq \left\|k_1(s,t)\right\| \left\|\phi_1(s) - \hat{\phi}_1(s)\right\| + \\ \left\|k_1(s,t) - k_{1N}(s,t)\right\| \left(\left\|\phi_1(s) - \hat{\phi}_1(s)\right\| + \left\|\phi_1(s)\right\|\right). \end{aligned} \tag{37}$$

$$\begin{aligned} \left\|k_1(s,t)\phi_1(s) - k_{1N}(s,t)\hat{\phi}_1(s)\right\| \leq (M_1 + \lambda(N))L\left\|X(t) - X_N(t)\right\| + \\ (M_1 + \lambda(N))\beta_1(N) + \lambda(N)\rho_1. \end{aligned} \tag{38}$$

and

$$\begin{aligned} \left\|k_2(s,t)\phi_2(s) - k_{2N}(s,t)\hat{\phi}_2(s)\right\| \leq \left\|k_2(s,t)\right\| \left\|\phi_2(s) - \hat{\phi}_2(s)\right\| + \\ \left\|k_2(s,t) - k_{2N}(s,t)\right\| \left(\left\|\phi_2(s) - \hat{\phi}_2(s)\right\| + \left\|\phi_2(s)\right\|\right) \\ \left\|k_2(s,t)\phi_2(s) - k_{2N}(s,t)\hat{\phi}_2(s)\right\| \leq (M_2 + \gamma(N))L\left\|X(t) - X_N(t)\right\| + \\ (M_2 + \gamma(N))\beta_2(N) + \gamma(N)\rho_2. \end{aligned} \tag{39}$$

Using Eq.(38) - Eq.(40) and assumption (iii) of Theorem 4.3, we have

$$\left\|X(t) - X_N(t)\right\| \leq \frac{\eta(N) + H(N) + \left\|W(t)\right\| I(N)}{1 - G(N)}, \tag{40}$$

where $G(N) = L(M_1 + \lambda(N)) - \|W(t)\| L(M_2 + \gamma(N))$
$H(N) = ((M_1 + \lambda(N))\beta_1(N) + \lambda(N)\rho_1)$
$I(N) = ((M_2 + \gamma(N))\beta_2(N) + \gamma(N)\rho_2)$.

Proof of II) :

$$E\left(|e_N(t)|^2\right) = E\left(|X(t) - X_N(t)|^2\right).$$

By using Theorems 8,9,10 and [21], we get

$$E\left(|X(t) - X_N(t)|^2\right) \leq P(N) + T(N)E\left(|X(t) - X_N(t)|^2\right), \tag{41}$$

where

$$P(N) = 3\eta^2(N) + 9(M_1 + \lambda(N))^2\beta_1^2(N) + 9\lambda^2(N)\rho_1^2 + \\ 9\,|W(t)|^2\,((M_2 + \gamma(N))^2\beta_2^2(N) + \gamma^2(N)\rho_2^2$$

and

$$T(N) = 9(M_1 + \lambda(N))^2L^2 + 9\,|W(t)|^2\,\gamma^2(N)\rho_2^2).$$

Hence from Eq.(41) and Gronwall inequality, we have $E\left(|e_N(t)|^2\right) \to 0.$ □

4.1. **Time complexity.** This proposed method deals with matrix multiplication and solving a system of equations.

Theorem 4.4. *Suppose that N and k are the degree of the approximate function $X(t)$ and the number of simulations, respectively, then the time complexity of this proposed method is $O(k(N+1)^2)$.*

Proof. The key steps of the proposed method is presented as follows:
Step 1: Construct the approximate vector $L(t)$.
Step 2: Compute the matrices $D, K_{1L}, K_{2L}, \hat{U}, P, \Gamma_s, P_s$.
Step 3: Compute column vectors $F, \Phi_1, \Phi_2, \hat{\Phi_1}, \hat{\Phi_2}$.
Step 4: Solve the nonlinear system of algebraic equations with respect to $\hat{\Phi_1}, \hat{\Phi_2}$.
This proposed method has 3 major steps of computation. Step 2 computes various matrix multiplications which require $O\left((N+1)^2\right)$ time. Step 3 computes the column vectors which require $O\,(N+1)$ time. Step 4 computes the system of equations and display the approximate solution numerically. They require $O\left((N+1)^2\right)$ time. These steps are executed k times. Hence, the overall time complexity of this proposed method is $O\left(k\,(N+1)^2\right)$. □

5. Numerical examples

To illustrate the efficiency, effectiveness, and reliability of the proposed method, three examples are carried out in this section. N and k represent the degree of the approximate function and the number of simulations, respectively. The absolute error function is defined by $e_N(t) = |X(t) - X_N(t)|$. All numerical computations have been performed on a PC by running some programmes written in MATLAB software.

Example 1: We consider Eq.(1) with $f(t) = 0.5, k_1(s,t) = 1; k_2(s,t) = 0.25; N_1(s, X(s)) = X(s)(1-X(s)); N_2(s, X(s)) = X(s)$ which governs the population growth model with random variations[27].

The solution for X(t) is obtained by the method described in Section 3. The computational results for k=100 and N=12 together with exact solution

$$X(t) = \frac{0.5exp(0.96875t + 0.25W(t))}{1 + 0.5\int_0^t exp(0.96875s + 0.25W(s))ds}$$

and solutions obtained by Euler and Bernoulli polynomial methods are shown in Figure 1. In table 1, the mean $\overline{X}_E$ and standard deviation S_E for N = 8 and k = 100 of the absolute errors of X(t) along with their 0.95 confidence intervals are presented.

Example 2: The next example of stochastic integral equation [27] is

$$X(t) = 1 + \int_0^t X(s)(\frac{1}{32} - X^2(s))ds + 0.25\int_0^t X(s)dW(s), t \in [0,1], \tag{42}$$

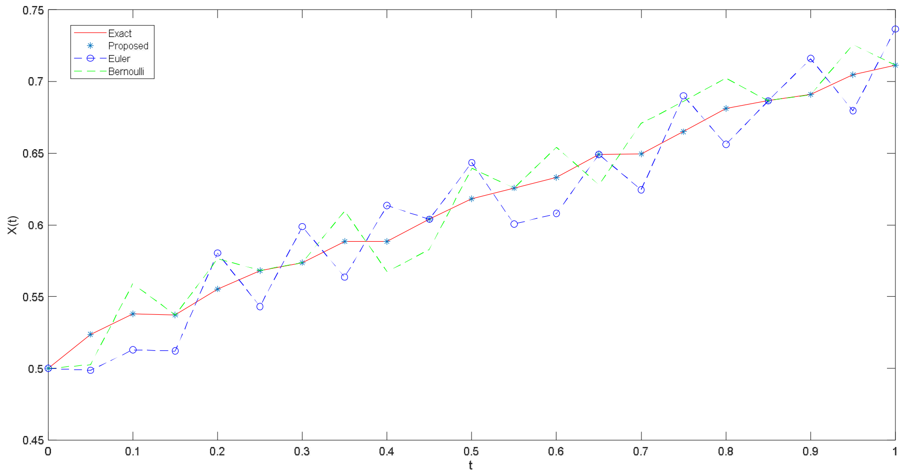

FIGURE 1. The Graph of Exact and Approximate solutions of Example 1.

TABLE 1. Mean, standard deviation and mean confidence interval for error in Example 1

t	$\overline{X}_E$	S_E	0.95 Confidence interval	
			Upper bound	Lower bound
0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.00087763	0.000713461	0.001017468	0.000737792
0.2	0.00334325	0.001171731	0.003572909	0.003113591
0.3	0.00083653	0.000651001	0.000964126	0.000708934
0.4	0.00183873	0.000943614	0.002023678	0.001653782
0.5	0.00069749	0.0041099	0.00150303	0.00010805
0.6	0.01175867	0.00973704	0.01366713	0.00985021
0.7	0.01538657	0.00987085	0.017321257	0.013451883
0.8	0.01758945	0.009792239	0.019508729	0.015670171
0.9	0.02265037	0.009979387	0.02460633	0.02069441
1	0.02315769	0.009966322	0.025111089	0.021204291

The computational results of proposed approximation method along with exact solution $X(t) = \frac{exp(0.25W(t))}{\sqrt{1+2\int_0^t exp(0.5W(s))ds}}$, Euler polynomials method and Bernoulli polynomials method are shown in Figure 2 for N=12. Table 2 shows the absolute error between the exact and Euler [27], Bernoulli [35] and proposed numerical solutions of Eq.(42) for the various values of N.

Example 3: Consider the equation [36]

$$X(t) = \frac{1}{8} - 0.015625 \int_0^t X(s)(1 - X^2(s))ds + 0.125 \int_0^t (1 - X^2(s)dW(s), t \in [0, 1], \quad (43)$$

TABLE 2. Error comparison for Example 2.

	Euler [27]			Bernoulli [35]			Proposed Method		
t \N	4	8	10	4	8	10	4	8	10
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0529	0.0251	0.0631	0.0070	0.0080	0.0043	0.0136	0.0089	0.0091
0.2	0.0289	0.0259	0.0386	0.0815	0.0792	0.0381	0.0087	0.0084	0.0093
0.3	0.0067	0.0306	0.0165	0.0297	0.0345	0.0056	0.0027	0.0085	0.0078
0.4	0.0159	0.0384	0.0043	0.0654	0.0686	0.0297	0.0079	0.0098	0.0009
0.5	0.0412	0.0487	0.0241	0.0371	0.0411	0.0123	0.0094	0.0099	0.0075
0.6	0.0725	0.0608	0.0431	0.0457	0.0098	0.0057	0.0356	0.0150	0.0079
0.7	0.1141	0.0742	0.0612	0.0395	0.0247	0.0083	0.0986	0.0259	0.0067
0.8	0.1714	0.0889	0.0787	0.0407	0.0354	0.0094	0.0998	0.0388	0.0389
0.9	0.2512	0.1055	0.0955	0.0397	0.0325	0.0089	0.1276	0.0543	0.0528
1.0	0.2578	0.1123	0.0989	0.0398	0.0337	0.0089	0.1288	0.0581	0.0531

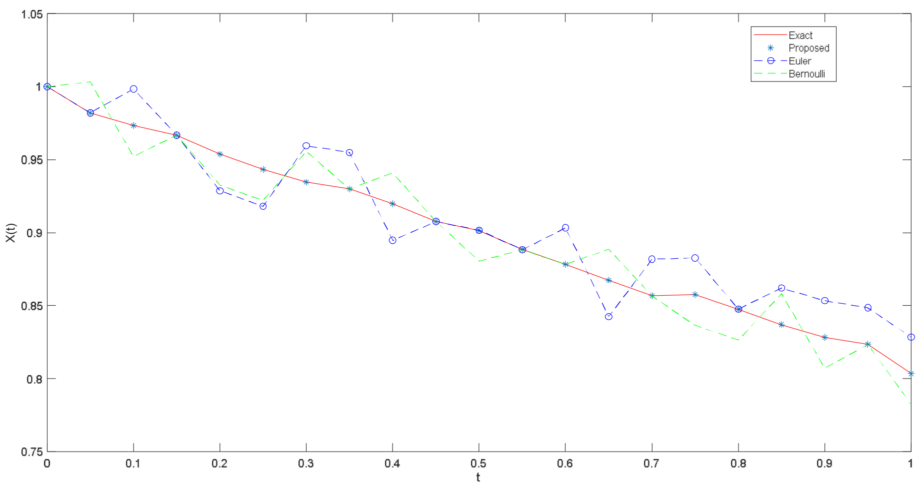

FIGURE 2. Numerical solutions of Example 2.

We compare the proposed numerical results and absolute errors of Eq.(43) with exact solution $X(t) = \frac{\frac{9}{8}exp(0.25W(t))-\frac{7}{8}}{\frac{9}{8}exp(0.125W(t))+\frac{7}{8}}$ and other methods discussed in [27, 35] are shown in Figure 3 and Table 3 respectively.

The key features of our proposed methodology are summarised as follows. The proposed technique provides good approximation solution in less computational time than the other methods reported in the literature. The superiority of the technique stands in the amount of error caused which is very less when compared with other methods and it can be inferred through figures. We also observe from the tables that the error values fall within the upper bound discussed in the theoretical analysis. As the polynomials utilized here are orthogonal, construction of operational matrices and the calculation of connection coefficients involved in function approximation have been carried out in an effortless manner.The various matrices of the approximate function and their nature are

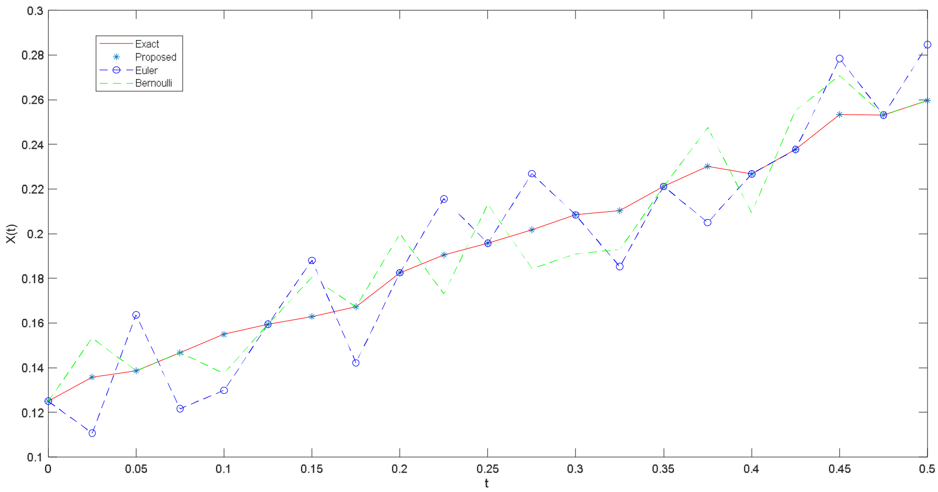

FIGURE 3. The Graph of Exact and Approximate solutions of Example 3.

TABLE 3. The absolute errors of the approximate solution for Example 3.

t \N	4	8	16
0.0	0.000000	0.000000	0.000000
0.1	0.034890	0.000103	0.000001
0.2	0.038752	0.000513	0.000005
0.3	0.035984	0.001103	0.000009
0.4	0.024618	0.000745	0.000007
0.5	0.027598	0.000134	0.000002
0.6	0.029475	0.000435	0.000003
0.7	0.030639	0.000127	0.000002
0.8	0.031736	0.000863	0.000008
0.9	0.032639	0.000574	0.000004
1.0	0.020788	0.000653	0.000006

utilized to convert the given equations into a system of algebraic equations.The advantage of possessing the lower triangular and tridiagonal forms enable us to solve the problem in a more accurate manner whereas when dealing with Euler polynomials, it seeks the help of the Bernoulli polynomials thereby the amount of work involved is huge even though it reduces to lower triangular system of equations. The implementation of shifted Legendre polynomials is superior to the generalized hat functions, Bernoulli and Bernstein polynomials as they have the weak form of sparse matrices which make the calculation process very difficult. Some numerical methods, namely Euler, Euler - Maruyama, R-K method and Milstein methods require the previous iteration values for pointwise solutions, whereas this method does not require any such assigned values. It has the advantage of providing a more accurate solution with a lesser number of basis functions and these polynomials are elementary to handle any type of stochastic differential equations.

6. Conclusion

This article deals with an efficient approximation technique for solving the nonlinear stochastic integral equations that occur in the physical and biological sciences. The proposed methodology is based on approximating the given function in terms of a linear combination of unknown constants and the basis of the polynomials. The stochastic operational matrices for the function approximation have been derived to solve the given equation. The theoretical analysis has been carried out for the proposed methodology, and the applicability of the method has been validated through some numerical examples. The solution quality has been tested with other classical methods mentioned in the literature. From tables and figures, it can be observed that the amount of error gets reduced by increasing the values of N, and at one such stage, it is on par with the original solution obtained through traditional methods. This technique is easy to implement to solve other stochastic differential equations since the original problem gets solved through the system of algebraic equations.

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