TWMS J. App. and Eng. Math. V.14, N.1, 2024, pp. 402-409

CERTAIN EXPANSION FORMULAE INVOLVING INCOMPLETE I-FUNCTIONS

S. D. PUROHIT[1], D. L. SUTHAR[2], ALI A. AL-JARRAH[3], V. K. VYAS[3], K. S. NISAR[4*], §

Abstract. The aim of this paper is to derive the expansion formulae for the incomplete I-function. Furthermore, their special cases are illustrated in terms of various types of special functions (incomplete $\overline{I}$-function, incomplete $\overline{H}$-function, and incomplete H-function) that are common in nature and very useful for further analysis.

Keywords: I-function, Incomplete I-function, Taylor's series.

AMS Subject Classification: 26A33, 33B15, 33C05, 33C20.

1. Introduction and Preliminaries

The incomplete I-functions are very useful for deriving new and known results due to its wide applications in science and engineering. Recently Jangid et al. [4] defined a new family of incomplete I-functions ${}^{(\gamma)}I_{p,q}^{m,n}$ and ${}^{(\Gamma)}I_{p,q}^{m,n}$. Incomplete I-functions are the natural generalization of the I-function defined by Rathie [13], $\overline{I}$-function, $\overline{H}$-function, H-function, Meijer G-function, hypergeometric function, and many other functions (see details; [1, 7, 9, 14, 17, 18]). So keeping this in mind, we derive expansion formulae of incomplete I-functions to make these more useful which are general in nature and helpful for further studies.

[1] Department of HEAS (Mathematics), Rajasthan Technical University, Kota, India.
e-mail: sunil_a_purohit@yahoo.com, sdpurohit@rtu.ac.in;
ORCID: https://orcid.org/0000-0002-1098-5961.
[2] Department of Mathematics, Wollo University, P.O. Box: 1145, Dessie, Ethiopia.
e-mail: dlsuthar@gmail.com; ORCID: https://orcid.org/0000-0001-9978-2177.
[3] Sur University College, P.O. Box: 440, P.C. 411, Sur, Sultanate of Oman.
e-mail: aljarrah@suc.edu.om; ORCID: https://orcid.org/0000-0002-5054-8547
e-mail: vmathsvyas@gmail.com; ORCID: https://orcid.org/0000-0002-5748-7070.
[4] Department of Mathematics, College of Arts and Sciences, Wadi Aldawser, Prince Sattam bin Abdulaziz University, Saudi Arabia.
e-mail: n.sooppy@psau.edu.sa; ORCID: https://orcid.org/0000-0001-5769-4320.
[*] Corresponding author.
§ Manuscript received: December 21, 2020; accepted: May 29, 2021.

Recently fractional integral formulas for the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by Parmar and Saxena [11] (see also; [3]).

$$\gamma(s,x)=\int_0^x e^{-t}t^{s-1}dt,\ \ (\Re(s)>0;x\geq 0)\,, \tag{1}$$

$$\Gamma(s,x)=\int_x^\infty e^{-t}t^{s-1}dt,\ \ (\Re(s)>0;x\geq 0)\,. \tag{2}$$

These incomplete gammas functions $\gamma\,(s,x)$ and $\Gamma\,(s,x)$ satisfy the following decomposition formula:

$$\gamma(s,x)+\Gamma(s,x)=\Gamma(s);\ \ (\Re(s)>0)\,.$$

In the study of incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}$ and ${}^{\Gamma}I_{p,q}^{m,n}$, Jangid et al. [4] define the following pair of Mellin-Barnes type contour integral representation

$$\begin{aligned}{}^{(\gamma)}I_{p,q}^{m,n}(z)&={}^{(\gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_j,D_j;B_j)_{1,q}\end{array}\right.\right]\\&={}^{(\gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_2,C_2;A_2),\cdots,(c_p,C_p;A_p)\\(d_1,D_1;B_1),\cdots,(d_q,D_q;B_q)\end{array}\right.\right]\\&=\frac{1}{2\pi i}\int_{\mathfrak{L}}g(s,x)z^s ds,\end{aligned} \tag{3}$$

where

$$g(s,x)=\frac{\{\gamma(1-c_1+C_1s,x)\}^{A_1}\prod_{j=1}^m\{\Gamma(d_j-D_js)\}^{B_j}\prod_{j=2}^n\{\Gamma(1-c_j+C_js)\}^{A_j}}{\prod_{j=m+1}^q\{\Gamma(1-d_j+D_js)\}^{B_j}\prod_{j=n+1}^p\{\Gamma(c_j-C_js)\}^{A_j}}, \tag{4}$$

and

$$\begin{aligned}{}^{(\Gamma)}I_{p,q}^{m,n}(z)&={}^{(\Gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_j,D_j;B_j)_{1,q}\end{array}\right.\right]\\&={}^{(\Gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_2,C_2;A_2),\cdots,(c_p,C_p;A_p)\\(d_1,D_1;B_1),\cdots,(d_q,D_q;B_q)\end{array}\right.\right]\\&=\frac{1}{2\pi i}\int_{\mathfrak{L}}G(s,x)z^s ds,\end{aligned} \tag{5}$$

where

$$G(s,x)=\frac{\{\Gamma(1-c_1+C_1s,x)\}^{A_1}\prod_{j=1}^m\{\Gamma(d_j-D_js)\}^{B_j}\prod_{j=2}^n\{\Gamma(1-c_j+C_js)\}^{A_j}}{\prod_{j=m+1}^q\{\Gamma(1-d_j+D_js)\}^{B_j}\prod_{j=n+1}^p\{\Gamma(c_j-C_js)\}^{A_j}}. \tag{6}$$

where $\gamma(\cdot,x)$ and $\Gamma(\cdot,x)$ are the lower and upper incomplete gamma functions defined in (1) and (2). These incomplete I-functions are exists for all $x\geq 0$ under the same conditions. These incomplete I-functions are symmetric in the set of pair of parameters. The equation is valid only for $A_1=1$, In this case

$${}^{(\gamma)}I_{p,q}^{m,n}(z)+{}^{(\Gamma)}I_{p,q}^{m,n}(z)=I_{p,q}^{m,n}(z).$$

In general, we have

$$[\gamma(1-c_1+C_1s,x)]^{A_1}+[\Gamma(1-c_1+C_1s,x)]^{A_1}\neq[\Gamma(1-c_1+C_1s)]^{A_1}.$$

Some important special cases of incomplete I-functions are enumerated below:

(1) If we set $B_j(j = 1, \cdots, m) = 1$ in (3); we obtain incomplete $\overline{I}$-function ${}^{(\gamma)}\overline{I}^{m,n}_{p,q}$ as:

$$ {}^{(\gamma)}\overline{I}^{m,n}_{p,q}(z) = {}^{(\gamma)}\overline{I}^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_1, D_1; 1)_{1,m}, (d_j, D_j; B_j)_{m+1,q} \end{array} \right] $$

$$ = \frac{1}{2\pi i} \int_{\mathfrak{L}} \overline{g}(s, x) z^s ds, \tag{7} $$

where

$$ \overline{g}(s, x) = \frac{\{\gamma(1 - c_1 + C_1 s, x)\}^{A_1} \prod_{j=1}^{m} \Gamma(d_j - D_j s) \prod_{j=2}^{n} \{\Gamma(1 - c_j + C_j s)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - d_j + D_j s)\}^{B_j} \prod_{j=n+1}^{p} \{\Gamma(c_j - C_j s)\}^{A_j}}. \tag{8} $$

(2) If we put $A_j(j = n+1, \cdots, p) = 1$ and $B_j(j = 1, \cdots, m) = 1$ in (3), incomplete I-function is the incomplete $\overline{H}$-function, defined by Srivastava et al. [19]:

$$ \overline{\gamma}^{m,n}_{p,q}(z) = \overline{\gamma}^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; \beta_j)_{m+1,q} \end{array} \right] $$

$$ = \frac{1}{2\pi i} \int_{\mathfrak{L}} \overline{\phi}(s, x) z^s ds, \tag{9} $$

where

$$ \overline{\phi}(s, x) = \frac{[\gamma(1 - c_1 + C_1 s, x)]^{A_1} \prod_{j=1}^{m} \Gamma(d_j - D_j s) \prod_{j=2}^{n} [\Gamma(1 - c_j + C_j s)]^{A_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - d_j + D_j s)]^{B_j} \prod_{j=n+1}^{p} \Gamma(c_j - C_j s)}. \tag{10} $$

(3) If we take $A_j(j = 1, \cdots, p) = 1$ and $B_j(j = 1, \cdots, q) = 1$ in (3), incomplete I-function is the incomplete H-function, defined by Srivastava et al. [19]:

$$ \gamma^{m,n}_{p,q}(z) = \gamma^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (c_1, C_1 : x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array} \right] $$

$$ = \frac{1}{2\pi i} \int_{\mathfrak{L}} \Phi(s, x) z^s ds, \tag{11} $$

where

$$ \Phi(s, x) = \frac{\gamma(1 - c_1 + C_1 s, x) \prod_{j=1}^{m} \Gamma(d_j - D_j s) \prod_{j=2}^{n} \Gamma(1 - c_j + C_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - d_j + D_j s) \prod_{j=n+1}^{p} \Gamma(c_j - C_j s)}. \tag{12} $$

(4) Additionally, if we set $x = 0$ in (3), then we obtain the I-function [13].

Remark 1.1. *Similarly, one can easily obtain another class of incomplete functions as special cases of the incomplete I-function* ${}^{(\Gamma)}I^{m,n}_{p,q}(z)$

In this paper, we derive certain expansions of incomplete I-functions by using the generalized Tailor's series formula given by Osler [10] as follows:

$$ f(z) = \sum_{n=-\infty}^{\infty} \frac{\rho\, \mathfrak{D}_z^{\rho n + \eta} f(z)|_{z=w} (z - w)^{\rho n + \eta}}{\Gamma(\rho n + \eta + 1)}, \tag{13} $$

where η is the arbitrary complex number so the order of the derivative is arbitrary and $0 < \rho \leq 1$ and n is the integer over the summation.

2. Main Results

In this section, we have established certain expansion formulae of incomplete I-function by using the Taylor's series formula defined in Osler [10] and our results are presented in Theorem 2.1 and Theorem 2.2 follow as:

Theorem 2.1. *Let $h > 0, m-1 \leq \Re(\rho n + \eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$ {}^{(\gamma)}I_{p,q}^{m,n}\left[(aw)^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} a^{-\rho n-\eta} $$

$$ \times {}^{(\gamma)}I_{p+1,q+1}^{m,n+1}\left[(aw)^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (0, h; 1), (c_j, C_j, A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q}, (\rho n+\eta, h; 1) \end{array}\right]. $$

Proof. We start from the L.H.S, by using the generalized Taylor's series (13). For this in our investigation, we consider

$$ f(z) = {}^{(\gamma)}I_{p,q}^{m,n}\left[z^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right], $$

and then, we obtain

$$ {}^{(\gamma)}I_{p,q}^{m,n}\left[z^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right] $$

$$ = \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} \mathfrak{D}_z^{\rho n+\eta} \left\{ {}^{(\gamma)}I_{p,q}^{m,n}\left[z^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right]\right\}, \tag{14} $$

for solving the above fractional derivative, we consider the fractional derivative formula which is given by Raina ([12], Eq. 2.1), then we obtain

$$ \mathfrak{D}_z^{\rho n+\eta} \left\{ {}^{(\gamma)}I_{p,q}^{m,n}\left[z^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right]\right\} $$

$$ = z^{-\rho n-\eta}\ {}^{(\gamma)}I_{p+1,q+1}^{m,n+1}\left[z^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (0, h; 1), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j, B_j)_{1,q}, (\rho n+\eta, h; 1) \end{array}\right], \tag{15} $$

by substitution $z = aw$ in (15) and using (3), we get the required result. □

In the similar way, we can derive Theorem 2.2 for ${}^{(\Gamma)}I_{p,q}^{m,n}$.

Theorem 2.2. *Let $h > 0, m-1 \leq \Re(\rho n + \eta) \leq m, \eta \in \mathbb{C}, 0 < \rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$ {}^{(\Gamma)}I_{p,q}^{m,n}\left[(aw)^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q} \end{array}\right] = \sum_{n=-\infty}^{\infty} \frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} a^{-\rho n-\eta} $$

$$ \times {}^{(\Gamma)}I_{p+1,q+1}^{m,n+1}\left[(aw)^h \middle| \begin{array}{c} (c_1, C_1; A_1 : x), (0, h; 1), (c_j, C_j, A_j)_{2,p} \\ (d_j, D_j; B_j)_{1,q}, (\rho n+\eta, h; 1) \end{array}\right]. $$

3. Special cases

In this section, we present certain expansion formulae for other well known incomplete functions (see details; [2, 5, 6, 8, 15, 16]) as special cases of the main results, in the form of incomplete $\overline{I}$-functions, incomplete $\overline{H}$-functions, incomplete H-functions and I-function by substituting particular value to the parameters in the below results.

(i) If we put $B_j(j=1,\cdots,m)=1$, $h=1$ and using the relation

$$
{}^{(\gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_1,D_1;1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right]
= {}^{(\gamma)}\overline{I}_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right] \tag{16}
$$

in Theorems 2.1 and 2.2, we obtain the following Corollaries.

Corollary 3.1. *Let $m-1\le\Re(\rho n+\eta)\le m,\eta\in\mathbb{C},0<\rho\le1,$ where a is the arbitrary constant and n is the integer over the summation, then*

$$
{}^{(\gamma)}\overline{I}_{p,q}^{m,n}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right]=\sum_{n=-\infty}^{\infty}\frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}a^{-\rho n-\eta}
$$
$$
\times{}^{(\gamma)}\overline{I}_{p+1,q+1}^{m,n+1}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(0,h;1),(c_j,C_j,A_j)_{2,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q},(\rho n+\eta,h;1)\end{array}\right.\right].
$$

Corollary 3.2. *Let $m-1\le\Re(\rho n+\eta)\le m,\eta\in\mathbb{C},0<\rho\le1,$ where a is the arbitrary constant and n is the integer over the summation, then*

$$
{}^{(\Gamma)}\overline{I}_{p,q}^{m,n}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right]=\sum_{n=-\infty}^{\infty}\frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}a^{-\rho n-\eta}
$$
$$
\times{}^{(\Gamma)}\overline{I}_{p+1,q+1}^{m,n+1}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(0,h;1),(c_j,C_j,A_j)_{2,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q},(\rho n+\eta,h;1)\end{array}\right.\right].
$$

(ii) If we put $A_j(j=n+1,\cdots,p)=1$, $B_j(j=1,\cdots,m)=1$, $h=1$ and using the relation

$$
{}^{(\gamma)}I_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,n},(c_j,C_j;1)_{n+1,p}\\(d_1,D_1;1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right]
=\overline{\gamma}_{p,q}^{m,n}\left[z\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,n},(c_j,C_j)_{n+1,p}\\(d_1,D_1)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right] \tag{17}
$$

in Theorems 2.1 and 2.2, we obtain the following Corollaries.

Corollary 3.3. *Let $m-1\le\Re(\rho n+\eta)\le m,\eta\in\mathbb{C},0<\rho\le1,$ where a is the arbitrary constant and n is the integer over the summation, then*

$$
\overline{\gamma}_{p,q}^{m,n}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(c_j,C_j;A_j)_{2,n},(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;B_j)_{m+1,q}\end{array}\right.\right]
$$
$$
=\sum_{n=-\infty}^{\infty}\frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)}a^{-\rho n-\eta}
$$
$$
\times\,\overline{\gamma}_{p+1,q+1}^{m,n+1}\left[aw\left|\begin{array}{c}(c_1,C_1;A_1:x),(0,h;1),(c_n,C_n;A_n),(c_j,C_j)_{n+1,p}\\(d_j,D_j)_{1,m},(d_j,D_j;B_j)_{m+1,q},(\rho n+\eta,h;1)\end{array}\right.\right].
$$

Corollary 3.4. *Let $m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0<\rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$\overline{\Gamma}_{p,q}^{m,n}\left[aw \left| \begin{array}{c} (c_1, C_1; A_1 : x), (c_j, C_j; A_j)_{2,n}, (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; B_j)_{m+1,q} \end{array} \right.\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\rho(z-w)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} z^{-\rho n-\eta}$$

$$\times \overline{\Gamma}_{p+1,q+1}^{m,n+1}\left[aw \left| \begin{array}{c} (c_1, C_1; A_1 : x), (0, h; 1), (c_n, C_n; A_n), (c_j, C_j)_{n+1,p} \\ (d_j, D_j)_{1,m}, (d_j, D_j; B_j)_{m+1,q}, (\rho n+\eta, h; 1) \end{array} \right.\right].$$

(iii) If we put $A_j (j = 1, \cdots, p) = 1$, $B_j (j = 1, \cdots, q) = 1$, $h = 1$ and using the relation

$$^{(\gamma)}I_{p,q}^{m,n}\left[z \left| \begin{array}{c} (c_1, C_1; 1 : x), (c_j, C_j; 1)_{2,p} \\ (d_1, D_1; 1)_{1,q} \end{array} \right.\right] = \gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (c_1, C_1 : x), (c_j, C_j)_{2,p} \\ (d_1, D_1)_{1,q} \end{array} \right.\right] \tag{18}$$

in Theorems 2.1 and 2.2, we obtain the following Corollaries.

Corollary 3.5. *Let $m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0<\rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$\gamma_{p,q}^{m,n}\left[aw \left| \begin{array}{c} (c_1, C_1 : x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array} \right.\right] = \sum_{n=-\infty}^{\infty} \frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} a^{-\rho n-\eta}$$

$$\times \gamma_{p+1,q+1}^{m,n+1}\left[aw \left| \begin{array}{c} (c_1, C_1 : x), (0, h), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (\rho n+\eta, h; 1) \end{array} \right.\right].$$

Corollary 3.6. *Let $m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0<\rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$\Gamma_{p,q}^{m,n}\left[aw \left| \begin{array}{c} (c_1, C_1 : x), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q} \end{array} \right.\right] = \sum_{n=-\infty}^{\infty} \frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} a^{-\rho n-\eta}$$

$$\times \Gamma_{p+1,q+1}^{m,n+1}\left[aw \left| \begin{array}{c} (c_1, C_1 : x), (0, h), (c_j, C_j)_{2,p} \\ (d_j, D_j)_{1,q}, (\rho n+\eta, h; 1) \end{array} \right.\right].$$

Remark 3.1. *By taking into account the decomposition formula of incomplete H-functions or by setting $x = 0$, the Corollary 3.5 lead to the known result provided earlier by Raina [12].*

(iv) If we put $x = 0$ in Theorems 2.1, we get the following Corollary:

Corollary 3.7. *Let $h>0, m-1 \leq \Re(\rho n+\eta) \leq m, \eta \in \mathbb{C}, 0<\rho \leq 1$, where a is the arbitrary constant and n is the integer over the summation, then*

$$I_{p,q}^{m,n}\left[(aw)^h \left| \begin{array}{c} (c_j, C_j; A_j)_{1,p} \\ (d_j, D_j; B_j)_{1,q} \end{array} \right.\right] = \sum_{n=-\infty}^{\infty} \frac{\rho(a-1)^{\rho n+\eta}}{\Gamma(\rho n+\eta+1)} a^{-\rho n-\eta}$$

$$\times I_{p+1,q+1}^{m,n+1}\left[(aw)^h \left| \begin{array}{c} (0, h; 1), (c_j, C_j, A_j)_{1,p} \\ (d_j, D_j; B_j)_{1,q}, (\rho n+\eta, h; 1) \end{array} \right.\right].$$

4. Concluding remarks

Moreover, it is important to note that the particular cases of the results obtained in this paper for $x = 0$ given the corresponding results for classical I-function and other special functions. Therefore, we conclude with the remark that, by specializing the parameter in the main results, one can derive number of expansion formulas for variety of special functions.

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this paper

Acknowledgement. The authors thank the referees for their concrete suggestions which resulted in a better organization of this article.

References

[1] Abeye, N. and Suthar, D. L., (2019), The $\overline{H}$-function and Srivastava's polynomials involving the generalized Mellin-Barnes contour integrals, Journal of Fractional Calculus and Applications, 10(2), pp. 290-297.

[2] Bansal, M. K., Kumar, D., Nisar, K. S. and Singh, J., (2019), Application of incomplete H-function in determination of Lambert's law, Journal of Interdisciplinary Mathematics, 22(7), pp. 1205-1212.

[3] Chaudhry, M. A. and Zubair, S. M., (2002), Extended incomplete Gamma functions with applications, J. Math. Anal. Appl., 274, pp. 725-745.

[4] Jangid, K., Bhatter, S., Meena, S., Baleanu, D., Qurashi, M. A. and Purohit, S. D., (2020), Some fractional calculus findings associated with the incomplete I-functions, Adv. Diff. Equ., 265, https://doi.org/10.1186/s13662-020-02725-7

[5] Jangid, N. K., Joshi, S., Purohit, S. D. and Suthar, D. L., (2021), Certain expansion formulae involving incomplete H and $\overline{H}$-functions, Journal of Fractional Calculus and Applications, 12(2), pp. 188-196.

[6] Jangid, N. K., Joshi, S., Purohit, S. D. and Suthar, D. L., (2021), Fractional Derivatives and Expansion Formulae of H and $\overline{H}$-functions, Advances in the Theory of Nonlinear Analysis and its Applications, 5(2), pp. 193-202.

[7] Mathai, A. M. and Saxena, R. K., (1978), The H-function with applications in statistics and other disciplines, Wiley Eastern Limited, New Delhi; John Wiley and Sons, New York.

[8] Meena, S., Bhatter, S., Jangid, K. and Purohit, S. D., (2020), Certain expansion formulae of incomplete H functions associated with Leibniz rule. TWMS J. App. and Eng. Math., Accepted.

[9] Meijer, C. S., (1952), Expansion theorems for the G-function-I, Proc. Kon. Ned. Akad. v. Wetensch. Series A, 55, pp. 369-379.

[10] Osler, T. J., (1971), Taylor's series generalized for fractional derivatives and applications, SIAM J. Math. Anal., 2, pp. 115-119.

[11] Parmar, R. K. and Saxena, R. K., (2016), The incomplete generalized τ-hypergeometric and second τ-Appell functions, J. Korean Math. Soc., 53(2), pp. 363-379.

[12] Raina, R. K., (1979), On certain expansions involving H-function, Comment. Math. Univ. St. Pauli XXVIII(2), pp. 37-48.

[13] Rathie, A. K., (1997), A new generalization of generalized hypergeometric functions, Matematiche, 52(2), pp. 297-310.

[14] Saxena, R. K., Ram, J. and Suthar, D. L., (2004), Integral formulas for the H-function generalized fractional calculus, South East Asian J. Math. & Math. Sc., 3(1), pp. 69-74.

[15] Srivastava, R., Agarwal, R. and Jain, S., (2017), A family of the incomplete Hypergeometric functions and associated integral transform and fractional derivative formulas, Filomat, 31, pp. 125-140.

[16] Srivastava, H. M., Chaudhry, M. A. and Agarwal, R. P., (2012), The incomplete Pochhammer symbols and their applications to Hypergeometric and related functions, Integral Transforms Spec. Funct., 23, pp. 659-683.

[17] Srivastava,H. M., Gupta, K. C. and Goyal, S. P., (1982), The H-functions of one and two variables with applications. South Asian Publishers, New Delhi and Madras.

[18] Srivastava, H. M., Harjule, P. and Jain, R., (2015), A general fractional differential equation associated with an integral operator with the H-function in the kernel, Russian J. Math. Phys., 22, pp. 112-126.

[19] Srivastava, H. M., Saxena, R. K. and Parmar, R. K., (2018), Some families of the incomplete H-functions and the incomplete $\overline{H}$-functions and associated integral transforms and operators of fractional calculus with applications, Russian J. Math. Phys., 25(1), pp. 116-138.

Sunil Dutt Purohit for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.3.

D. L. Suthar for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.4.

A. A. Al-Jarrah is an Associate Professor of Applied Mathematics and the Dean of Sur University College in Oman. He formerly worked as an Associate Professor at Yarmouk University in Jordan. His research focuses include Special Functions, Integral Operators, and Vibration Theory.

V.K. Vyas is an Assistant Professor of Mathematics at Sur University College in Oman. His research interests include special functions, fractional calculus, integral transformations, and basic hypergeometric sequence.

Kottakkaran Sooppy Nisar is a full professor of Applied Mathematics at Prince Sattam bin Abdulaziz University, Saudi Arabia. His current research interests are Special functions, Fractional Calculus, Fluid dynamics, Bio Mathematics, Machine Learning, SAC-OCDMA and Multidisciplinary applications of Mathematics.