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APPLICATIONS OF THE GENERALIZED HOMOGENEOUS q-SHIFT OPERATOR IN q-POLYNOMIALS

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ABSTRACT. In this paper, we construct the generalized homogeneous q-shift operator ${}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)$. Then, we apply this operator to derive some q-identities such as: the generating function and its extension, Rogers formula and its extension, Mehler's formula and its extension, Srivastava-Agarwal type bilinear generating functions for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$. Also, we obtain a transformation formula involving generating functions for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$. We provide some special values for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$ in order to establish identities for the polynomials $\phi_n^{(a)}(x)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

Keywords: the homogeneous q-difference operator, the homogeneous q-shift operator, q-Hahn polynomials, the generalized Al-Salam-Carlitz q-polynomials, generating function, Rogers formula, Mehler's formula.

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1. Introduction

In this paper, we will follow common notations and definitions for the q-series that used in [8]. We assume that $|q| < 1$.

For $a \in \mathbb{C}$, the q-shifted factorial is defined by [8]

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),$$

and the multiple q-shifted factorials by:

$$(a_1,a_2,\ldots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m,$$

where $m \in \mathbb{Z}$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is presented as follows [8]:

$${}_r\phi_s\left(\begin{array}{c} \alpha_1,\ldots,\alpha_r \\ \beta_1,\ldots,\beta_s \end{array};q,x\right) = \sum_{n=0}^{\infty}\frac{(\alpha_1,\ldots,\alpha_r;q)_n}{(q,\beta_1,\ldots,\beta_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n,$$

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where $q \neq 0$ when $r > s + 1$. Note that

$$ {}_{r+1}\phi_r \left(\begin{matrix} \alpha_1, \ldots, \alpha_{r+1} \\ \beta_1, \ldots, \beta_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_{r+1}; q)_n}{(q, \beta_1, \ldots, \beta_r; q)_n} x^n, \quad |x| < 1. $$

The q-binomial coefficient is defined as [8]:

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k \, (q;q)_{n-k}} \quad for \quad 0 \leqslant k \leqslant n, $$

where $n, k \in \mathbb{N}$.

The Cauchy identity is given by:

$$ \sum_{m=0}^{\infty} \frac{(a;q)_m}{(q;q)_m} \, x^m = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1} $$

For $a = 0$, Cauchy identity becomes Euler's identity [8]:

$$ \sum_{m=0}^{\infty} \frac{x^m}{(q;q)_m} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \tag{2} $$

which has the following inverse relation:

$$ \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} x^m}{(q;q)_m} = (x;q)_\infty. \tag{3} $$

The q-Chu-Vandermonde's sum is given by [8]:

$$ {}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{4} $$

Heine's transformation of ${}_2\phi_1$ series [8, Appendix III, equation (III.1)] is:

$$ {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty} \; {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix} ; q, b \right). \tag{5} $$

Heine's transformation of ${}_2\phi_1$ series [8, Appendix III, equation (III.3)] is:

$$ {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} \; {}_2\phi_1 \left(\begin{matrix} c/a, c/b \\ c \end{matrix} ; q, abz/c \right). \tag{6} $$

Transformation of ${}_3\phi_2$ series [8, Appendix III, equation (III.12)] is:

$$ {}_3\phi_2 \left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix} ; q, q \right) = \frac{(e/c;q)_n}{(e;q)_n} c^n \; {}_3\phi_2 \left(\begin{matrix} q^{-n}, c, d/b \\ d, q^{1-n}c/e \end{matrix} ; q, bq/e \right). \tag{7} $$

Transformation of ${}_2\phi_1$ [2, equation (5.3)] is:

$$ {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(abz/c;q)_\infty}{(az/c;q)_\infty} \; {}_3\phi_2 \left(\begin{matrix} b, c/a, 0 \\ qc/az, c \end{matrix} ; q, q \right). \tag{8} $$

We will use the following identities in this paper:

$$ (aq^{-n};q)_n = (q/a;q)_n (-a)^n q^{-n-\binom{n}{2}}. \tag{9} $$

$$ (q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}. \tag{10} $$

In 1949, Hahn [10] defined the following polynomials:

$$ \phi_n^{(\alpha)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha;q)_k \, x^k. \tag{11} $$

In 1965, Al-Salam and Carlitz [1] provided the generating function and Mehler's formula for $\phi_n^{(a)}(x)$ as follows:

The generating function for $\phi_n^{(a)}(x)$ is [1, 12, 14].

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x) \frac{\tau^n}{(q;q)_n} = \frac{(ax\tau;q)_\infty}{(\tau, x\tau;q)_\infty}, \quad \max\{|\tau|, |x\tau|\} < 1. \tag{12}$$

Mehler's formula for $\phi_n^{(a)}(x)$ is [1, 12, 14]:

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)\phi_n^{(b)}(y) \frac{\tau^n}{(q;q)_n} = \frac{(ax\tau, by\tau;q)_\infty}{(\tau, x\tau, y\tau;q)_\infty} \, {}_3\phi_2\left(\begin{matrix} a, b, \tau \\ ax\tau, by\tau \end{matrix} ; q, xy\tau\right), \tag{13}$$

where $\max\{|\tau|, |x\tau|, |y\tau|, |xy\tau|\} < 1$.

In 1972, Carlitz [4] extended the generating function for $\phi_n^{(a)}(x)$ as follows:

$$\sum_{n=0}^{\infty} \phi_{n+k}^{(\alpha)}(x) \frac{t^n}{(q;q)_n} = \frac{(\alpha xt;q)_\infty}{(t, xt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(t,\alpha;q)_j}{(\alpha xt;q)_j} x^j, \quad \max\{|t|, |xt|\} < 1. \tag{14}$$

In 1989, Srivastava and Agarwal [17] proposed the following generating function for $\phi_n^{(a)}(x)$:

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x)(\lambda;q)_n \frac{\tau^n}{(q;q)_n} = \frac{(\lambda\tau;q)_\infty}{(\tau;q)_\infty} \, {}_2\phi_1\left(\begin{matrix} \lambda, a \\ \lambda\tau \end{matrix} ; q, x\tau\right), \quad \max\{|\tau|, |x\tau|\} < 1. \tag{15}$$

The Cauchy polynomials are defined as follows [6, 13, 15, 16]:

$$P_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n \, x^n,$$

which has the following generating function:

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{16}$$

Goulden and Jackson in [9] gave the following identity:

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. \tag{17}$$

In 2003, Chen et al. [5] presented the homogeneous q-difference operator D_{xy}, which performs on functions in two variables as follows:

$$D_{xy}\{f(x,y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}.$$

The homogeneous q-shift operator was built as the q-exponential of the homogeneous q-shift operator, as shown below:

$$\mathbb{E}(D_{xy}) = \sum_{k=0}^{\infty} \frac{D_{xy}^k}{(q;q)_k}. \tag{18}$$

Proposition 1.1. [5]. *We have*

$$D_{xy}^k\{P_n(x,y)\} = \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y). \tag{19}$$

$$D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = t^k \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{20}$$

In 2010, Chen et al. [7] constructed the following homogeneous q-shift operator:

$$\mathbb{F}(aD_{xy}) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (aD_{xy})^k}{(q;q)_k}. \tag{21}$$

In 2014, Cao [3] defined the homogeneous q-difference operator

$$T(a, zD_{xy}) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (zD_{xy})^k. \tag{22}$$

In 2014, Zhou and Luo [19] discovered some new generating functions for q-Hahn polynomials and proved them using the homogeneous q-difference operator $\mathbb{E}(D_{xy})$.

The Rogers formula for $\phi_n^{(a)}(x|q)$ is [19]*:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \phi_{n+k}^{(a)}(x) \frac{t^n}{(q;q)_n} \frac{\ell^k}{(q;q)_k} = \frac{(ax\ell;q)_\infty}{(\ell, x\ell, xt;q)_\infty} \, {}_2\phi_1 \left(\begin{array}{c} xa, x\ell \\ xa\ell \end{array} ; q, t \right), \tag{23}$$

where $\max\{|t|, |\ell|, |x\ell|, |xt|\} < 1$.

The extension of Mehler's formula for $\phi_n^{(a)}(x)$ is [19]*:*

$$\sum_{n=0}^{\infty} \phi_{m+n}^{(\alpha)}(x) \phi_n^{(\beta)}(y) \frac{t^n}{(q;q)_n}$$

$$= \frac{(\alpha xt;q)_\infty}{(t, xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(\beta;q)_k (xyt)^k}{(q;q)_k} \sum_{j=0}^{m+k} \begin{bmatrix} m+k \\ j \end{bmatrix} \frac{(xt, x\alpha;q)_j}{(\alpha xt;q)_j} \, x^{m-j}, \tag{24}$$

provided $\max\{|t|, |xt|\} < 1$.

In 2020, Srivastava and Arjika [18] established the generalized Al-Salam-Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$ in the following form:

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1, a_2, \ldots, a_{s+1};q)_k}{(b_1, b_2, \ldots, b_s;q)_k} \, x^k y^{n-k}. \tag{25}$$

Srivastava and Arjika [18] gave the following results:

Theorem 1.1. [18]. *We have*

The generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y|q)$ *is*:

$$\sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(yt;q)_\infty} \, {}_{s+1}\phi_s \left(\begin{array}{c} a_1, \cdots, a_{s+1} \\ b_1, \cdots, b_s \end{array} ; q, xt \right), \tag{26}$$

where $\max\{|xt|, |yt|\} < 1$.

The Srivastava-Agarwal type generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y|q)$ *is*:

$$\sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y|q)(\lambda;q)_n \frac{t^n}{(q;q)_n} = \frac{(\lambda yt;q)_\infty}{(yt;q)_\infty} \, {}_{s+2}\phi_{s+1} \left(\begin{array}{c} a_1, \cdots, a_{s+1}, \lambda \\ b_1, \cdots, b_s, \lambda yt \end{array} ; q, xt \right), \tag{27}$$

where $\max\{|xt|, |yt|\} < 1$.

In 2022, the generalised q-hypergeometric polynomials were defined by Reshem and Saad [11] as follows*:*

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1, \cdots, a_r;q)_k}{(b_1, \ldots, b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} c^k P_{n-k}(x, y), \tag{28}$$

where $\mathbf{a} = (a_1, a_2, \cdots, a_r)$, $\mathbf{b} = (b_1, b_2, \ldots, b_s)$.

- Setting $r = 1$, $s = 0$, $a_1 = a$, $x = 1$, $y = 0$, $c = x$ in (28), we get the q-Hahn polynomials $\phi_n^{(a)}(x)$ defined in equation (11).
- Letting $r = s + 1$, $y = 0$, $x = y$, $c = x$ in (28), we get the generalized Al-Salam-Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y)$ defined in (25).

This paper is arranged as follows: In Section 2, we'll present the generalized homogeneous q-shift operator ${}_r\Phi_s$ and then find some of its identities. The generating function and its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ are presented in Section 3 using the operator representation (30). In Section 4, the operator approach to Rogers' formula and its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ will be driven. For the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$, we establish Mehler's formula and its extension in Section 5. In Section 6, for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$, we develop a Srivastava-Agarwal type bilinear generating function. We use the homogeneous q-difference operator ${}_r\Phi_s$ in Section 7 to obtain the transformation formula for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$.

2. Generalized homogeneous q-shift operator and some of its Operator Identities

In this section, we will begin by introducing the generalised homogeneous q-shift operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$ and then we will proceed to find some of its identities.

Definition 2.1. *We define the generalized homogeneous q-shift operator as follows*:

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right) = \sum_{k=0}^{\infty} \frac{W_k}{(q;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (cD_{xy})^k, \tag{29} $$

where $W_k = \dfrac{(a_1, \cdots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k}$.

By providing specific values to the generalized homogeneous q-shift operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$, several previously stated operators can be obtained, for more information, see [3, 5, 7].

The following lemma is straightforward to prove:

Lemma 2.1. *Let the operator* ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$ *be defined as in* (29), *then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right) \{P_n(x, y)\} = \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q). \tag{30} $$

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right) \left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(yt;q)_\infty}{(xt;q)_\infty}\, {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ct\right), \tag{31} $$

where $|xt| < 1$.

Lemma 2.2. *Let the operator* ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$ *be defined as in* (29), *then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right) \left\{\frac{P_k(x, y)}{(yt;q)_k} \frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} $$

$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, xt;q)_j}{(q, yt;q)_j} q^j \ {}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, ctq^j\right), \quad |xt| < 1. \tag{32}$$

Proof. By using q-Chu-Vandermonde sum (4), we find that

$$\begin{aligned} \frac{P_k(x,y)}{(yt;q)_k} \frac{(yt;q)_\infty}{(xt;q)_\infty} &= \frac{(yt;q)_\infty}{(xt;q)_\infty} \frac{(y/x;q)_k x^k}{(yt;q)_k} \\ &= \frac{(yt;q)_\infty}{(xt;q)_\infty} t^{-k} \ {}_2\phi_1(q^{-k}, xt; yt; q, q) \\ &= t^{-k} \sum_{j=0}^{k} \frac{(q^{-k};q)_j}{(q;q)_j} \frac{(q^j yt;q)_\infty}{(q^j xt;q)_\infty} q^j. \end{aligned} \tag{33}$$

By using (33) and (31), we obtain

$$\begin{aligned} &{}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_{xy}\right) \left\{ \frac{P_k(x,y)}{(yt;q)_k} \frac{(yt;q)_\infty}{(xt;q)_\infty} \right\} \\ &= t^{-k} \sum_{j=0}^{k} \frac{(q^{-k};q)_j}{(q;q)_j} q^j \ {}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_{xy}\right) \left\{ \frac{(q^j yt;q)_\infty}{(q^j xt;q)_\infty} \right\} \\ &= t^{-k} \sum_{j=0}^{k} \frac{(q^{-k};q)_j}{(q;q)_j} q^j \frac{(ytq^j;q)_\infty}{(xtq^j;q)_\infty} \ {}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, ctq^j\right). \end{aligned}$$

□

3. The Generating Function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$

In this section, we use the operator representation (30) to drive the generating function and its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$. We give some special values for the parameters in the generating function as well as its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ to get the generating function and its extension for both polynomials $\phi_n^{(a)}(x)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$.

Theorem 3.1. (Generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$). *We have*

$$\sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty} \ {}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, ct\right), \quad |xt| < 1. \tag{34}$$

Proof. By using (30), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q) \frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} \ {}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_{xy}\right) \{P_n(x,y)\} \frac{t^n}{(q;q)_n} \\ &= \ {}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_{xy}\right) \left\{ \sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q;q)_n} \right\} \\ &= \ {}_r\Phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, cD_{xy}\right) \left\{ \frac{(yt;q)_\infty}{(xt;q)_\infty} \right\}, \quad |xt| < 1 \\ &= \frac{(yt;q)_\infty}{(xt;q)_\infty} \ {}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, ct\right). \end{aligned}$$

□

- Setting $r=1$, $s=0$, $a_1=a$, $x=1$, $y=0$, $c=x$ in (3.1), we recover the generating function for the polynomials $\phi_n^{(a)}(x)$ (equation (12)).
- Letting $r=s+1$, $y=0$, $x=y$, $c=x$ in (3.1), we regain the generating function for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ (equation (26)).

Theorem 3.2. (An extension to generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_{n+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}\, t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},xt;q)_j}{(q,yt;q)_j}\, q^j\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,ctq^j\right),\quad |xt|<1. \tag{35}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\phi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\\
&= \sum_{n=0}^{\infty}{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\{P_{n+k}(x,y)\}\frac{t^n}{(q;q)_n} \quad \text{(by using (30))}\\
&= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_{n+k}(x,y)\frac{t^n}{(q;q)_n}\right\}\\
&= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{P_k(x,y)\sum_{n=0}^{\infty}P_n(x,q^ky)\frac{t^n}{(q;q)_n}\right\}\\
&= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{P_k(x,y)\frac{(q^kyt;q)_\infty}{(xt;q)_\infty}\right\} \quad \text{(by using (16))}\\
&= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{\frac{P_k(x,y)}{(yt;q)_k}\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}\\
&= \frac{(yt;q)_\infty}{(xt;q)_\infty}\, t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},xt;q)_j}{(q,yt;q)_j}q^j\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,ctq^j\right). \quad \text{(by using (32))}
\end{aligned}$$

□

- Setting $k=0$ in equation (35), we obtain the generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$ (equation (34)).

Lemma 3.1. *We have*

$${}_3\phi_2\left(\begin{matrix} q^{-n},b,c \\ d,0\end{matrix};q,q\right) = c^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}\frac{(c,d/b;q)_k}{(d;q)_k}\ (b/c)^k. \tag{36}$$

Proof. Setting $e=0$ in (7), we obtain

$$\begin{aligned}
{}_3\phi_2\left(\begin{matrix} q^{-n},b,c \\ d,0\end{matrix};q,q\right) &= c^n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\frac{(c,d/b;q)_k}{(d;q)_k}\frac{(bq)^k}{(-1)^kq^{\binom{k}{2}}q^{-nk}(qc)^k}\\
&= c^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}\frac{(c,d/b;q)_k}{(d;q)_k}\ (b/c)^k. \quad \text{(by using (10))}
\end{aligned}$$

□

- By giving special values to the parametrs in equation (35), we retrieve an extension to the generating function for the polynomials $\phi_n^{(a)}(x)$ (equation (14)).

Proof. Setting $r=1$, $s=0$, $a_1=a$, $x=1$, $y=0$, $c=x$ in (35), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_{n+k}^{(a)}(x|q)\frac{t^n}{(q;q)_n}&=\frac{t^{-k}}{(t;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},t;q)_j}{(q;q)_j}\;q^j\;{}_1\phi_0\left(\begin{matrix}a\\0\end{matrix};q,xtq^j\right)\\&=\frac{t^{-k}}{(t;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},t;q)_j}{(q;q)_j}\;q^j\frac{(axtq^j;q)_\infty}{(xtq^j;q)_\infty}\quad\text{(by using (1))}\\&=\frac{(axt;q)_\infty}{(t,xt;q)_\infty}t^{-k}\sum_{j=0}^{k}\frac{(q^{-k},t,xt;q)_j}{(q,axt;q)_j}\;q^j\\&=\frac{(axt;q)_\infty}{(t,xt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(t,a;q)_j}{(axt;q)_j}\;x^j,\quad\text{(by using (36))}\end{aligned}$$

where $\max\{|t|,|xt|\}<1$. □

- If $r=s+1$, $y=0$, $x=y$, $c=x$ in equation (35), we obtain an extension to the generating function for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ as follows:

Corollary 3.1. (An extention to generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_{n+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{t^{-k}}{(yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_{s+1};q)_n}{(b_1,\cdots,b_s;q)_n}(xt)^n$$
$$\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(yt;q)_j\;(-1)^jq^{\binom{j}{2}+(n-k+1)j},\quad |yt|<1.$$

4. Rogers Formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$

In this section, we will present an operator approach to Rogers formula and its extension for the generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$. The Rogers formula and its extension for the q-Hahn polynomials $\phi_n^{(a)}(x)$ and the extension of the Rogers formula for the generalized Al-Salam-Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are obtained by incorporating special values for the parameters in the Rogers formula and its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$.

Theorem 4.1. (Rogers formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}=\frac{(y\ell;q)_\infty}{(t/\ell,x\ell;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(x\ell;q)_k}{(q,y\ell,q\ell/t;q)_k}q^k\;{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right),\quad\max\{|t/\ell|,|\ell x|\}<1.\tag{37}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+k}^{(\mathbf{a},\mathbf{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{xy}\right)\{\,P_{n+k}(x,y)\}\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\sum_{k=0}^{\infty}P_k(x,q^ny)\frac{\ell^k}{(q;q)_k}\right\}$$

(by using (16))

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{(q^ny\ell;q)_\infty}{(x\ell;q)_\infty}\frac{t^n}{(q;q)_n}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\,{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{xy}\right)\left\{\frac{P_n(x,y)}{(y\ell;q)_n}\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\right\} \quad \text{(by using (32))}$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\,\ell^{-n}\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},x\ell;q)_k}{(q,y\ell;q)_k}\,q^k\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)$$

$$= \frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{n=0}^{\infty}\frac{(t/\ell)^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},x\ell;q)_k}{(q,y\ell;q)_k}\,q^k\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)$$

$$= \frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k}{(q,y\ell;q)_k}\,q^k\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)\sum_{n=k}^{\infty}\frac{(t/\ell)^n(-1)^kq^{\binom{k}{2}-nk}}{(q;q)_{n-k}}$$

(by using (10))

$$= \frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k}{(q,y\ell;q)_k}\,q^k\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)\sum_{n=0}^{\infty}\frac{(t/\ell)^{n+k}(-1)^kq^{\binom{k}{2}-(n+k)k}}{(q;q)_n}$$

$$= \frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k(t/\ell)^k(-1)^kq^{-\binom{k}{2}}}{(q,y\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)\sum_{n=0}^{\infty}\frac{(q^{-k}t/\ell)^n}{(q;q)_n}$$

$$= \frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k(t/\ell)^k(-1)^kq^{-\binom{k}{2}}}{(q,y\ell;q)_k(q^{-k}t/\ell;q)_\infty}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)$$

$$= \frac{(y\ell;q)_\infty}{(x\ell,t/\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k(t/\ell)^k(-1)^kq^{-\binom{k}{2}}}{(q,y\ell;q)_k(q^{-k}t/\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right)$$

$$= \frac{(y\ell;q)_\infty}{(x\ell,t/\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k\;q^k}{(q,y\ell,q\ell/t;q)_k}\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right). \quad \text{(by using (9))}$$

□

- Rogers formula for polynomials $\phi_n^{(a)}(x)$ (equation (23)) can be preserved by using special values for the parameters in equation (37).

Proof. Substitute $r = 1$, $s = 0$, $a_1 = a$, $x = 1$, $y = 0$, $c = x$ in equation (37), we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+k}^{(a)}(x|q)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}$$
$$= \frac{1}{(\ell,t/\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(\ell;q)_k}{(q,q\ell/t;q)_k}\,q^k\,{}_1\phi_0\left(\begin{matrix} a \\ 0 \end{matrix};q,x\ell q^k\right)$$

$$
\begin{aligned}
&= \frac{1}{(\ell, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(\ell; q)_k}{(q, q\ell/t; q)_k} \; q^k \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \; (x\ell q^k)^n \\
&= \frac{1}{(\ell, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(\ell; q)_k}{(q, q\ell/t; q)_k} \; q^k \frac{(q^k a x\ell)_\infty}{(q^k x\ell)_\infty} \\
&= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(\ell, x\ell; q)_k}{(q, ax\ell, q\ell/t; q)_k} \; q^k \\
&= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t/\ell; q)_\infty} \; {}_3\phi_2 \left(\begin{matrix} \ell, x\ell, 0 \\ ax\ell, q\ell/t \end{matrix} ; q, q \right) \\
&= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, t; q)_\infty} \; {}_2\phi_1 \left(\begin{matrix} a, \ell \\ ax\ell \end{matrix} ; q, xt \right) \quad \text{(by using(8))} \\
&= \frac{(ax\ell; q)_\infty}{(\ell, x\ell, xt; q)_\infty} \; {}_2\phi_1 \left(\begin{matrix} ax, x\ell \\ ax\ell \end{matrix} ; q, t \right), \quad \text{(by using (6))}
\end{aligned}
$$

where $\max\{|\ell|, |x\ell|, |xt|, |t|\} < 1$. □

Theorem 4.2. (An extention to Rogers formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$). *we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+m+k}^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
&\quad = \frac{(y\ell; q)_\infty}{(x\ell, \tau/t, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(x\ell; q)_k \; q^k}{(q, y\ell, q\ell/t; q)_k} \; {}_r\phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, c\ell q^k \right),
\end{aligned} \tag{38}
$$

where $\max\{|x\ell|, |\tau/t|, |t/\ell|\} < 1$.

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \phi_{n+m+k}^{(\mathbf{a},\mathbf{b})}(x, y, c|q) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \; {}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_{xy} \right) \{P_{n+m+k}(x, y)\} \; \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{\ell^k}{(q; q)_k} \\
&= \; {}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_{xy} \right) \Bigg\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \; P_{n+m}(x, y) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \\
&\qquad\qquad \times \sum_{k=0}^{\infty} P_k(x, q^{n+m} y) \frac{\ell^k}{(q; q)_k} \Bigg\} \\
&= \; {}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \; P_{n+m}(x, y) \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \frac{(q^{n+m} y\ell, q)_\infty}{(x\ell; q)_\infty} \right\} \\
&\qquad \text{(by using (16))} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \; {}_r\Phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, cD_{xy} \right) \left\{ \frac{P_{n+m}(x, y)}{(y\ell; q)_{n+m}} \frac{(y\ell; q)_\infty}{(x\ell; q)_\infty} \right\} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{(q; q)_{n+m}} \frac{\tau^m}{(q; q)_m} \; \ell^{-(n+m)} \frac{(y\ell; q)_\infty}{(x\ell; q)_\infty} \sum_{k=0}^{n+m} \frac{(q^{-(n+m)}, x\ell, q)_k q^k}{(q, y\ell; q)_k}
\end{aligned}
$$

$$\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)\quad\text{(by using (32))}$$

$$=\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\ell)^n}{(q;q)_{n+m}}\frac{(\tau/\ell)^m}{(q;q)_m}\sum_{k=0}^{n+m}\frac{(q^{-(n+m)},x\ell,q)_k q^k}{(q,y\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)$$

$$=\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\ell)^n(\tau/\ell)^m}{(q;q)_m}\sum_{k=0}^{n+m}\frac{(x\ell,q)_k q^k}{(y\ell;q)_k}\frac{(-1)^k q^{\binom{k}{2}}q^{-(n+m)k}}{(q;q)_{n+m-k}(q;q)_k}$$

$$\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)\quad\text{(by using (10))}$$

$$=\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n+m=k}^{\infty}\frac{(t/\ell)^n(\tau/\ell)^m}{(q;q)_m}\frac{(x\ell,q)_k q^k}{(y\ell;q)_k}\frac{(-1)^k q^{\binom{k}{2}}q^{-(n+m)k}}{(q;q)_{n+m-k}(q;q)_k}$$

$$\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)$$

$$=\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\tau/t)^m(t/\ell)^{k+i}}{(q;q)_m(q;q)_i}\frac{(-1)^k q^{\binom{k}{2}}q^{-(k+i)k+k}}{(q;q)_k}\frac{(x\ell,q)_k}{(y\ell;q)_k}$$

$$\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)$$

$$=\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{m=0}^{\infty}\frac{(\tau/t)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(t/\ell)^k}{(q;q)_k}\frac{(x\ell,q)_k}{(y\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)$$

$$\times\sum_{i=0}^{\infty}\frac{(q^{-k}t/\ell)^i}{(q;q)_i}$$

$$=\frac{(y\ell;q)_\infty}{(x\ell,\tau/t;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(t/\ell)^k}{(q;q)_k}\frac{(x\ell,q)_k}{(y\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)\frac{1}{(q^{-k}t/\ell;q)_\infty}$$

$$=\frac{(y\ell;q)_\infty}{(x\ell,\tau/t,t/\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(t/\ell)^k}{(q;q)_k}\frac{(x\ell,q)_k}{(y\ell,q^{-k}t/\ell;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right)$$

$$=\frac{(y\ell;q)_\infty}{(x\ell,\tau/t,t/\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(x\ell;q)_k q^k}{(q,y\ell,q\ell/t;q)_k}\,{}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right).\quad\text{(by using (9))}$$

□

- Letting $\tau=0$ in equation (38), we get Rogers formula for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$ (equation (4.1)).
- When $r=1$, $s=0$, $a_1=a$, $x=1$, $y=0$, $c=x$ in equation (38), we get an extension to the Rogers formula for the polynomials $\phi_n^{(a)}(x|q)$.

Corollary 4.1 (An extention to Rogers formula for $\phi_n^{(a)}(x|q)$)**.** *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+m+k}^{(a)}(x|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\ell^k}{(q;q)_k}$$
$$=\frac{(ax\ell;q)_\infty}{(\ell,x\ell,xt,\tau/\ell;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}ax,x\ell\\ ax\ell\end{matrix};q,t\right),$$

where $\max\{|\ell|, |x\ell|, |xt|, \tau/\ell|\} < 1$.

- Setting $r = s+1$, $y = 0$, $x = y$, $c = x$ in equation (38), we get an extension to the Rogers formula for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$.

Corollary 4.2 (An extention to Rogers formulaa for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$)**.**

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+m+k}^{(\boldsymbol{a},\boldsymbol{b})}(x, y|q)\frac{t^n}{(q;q)_{n+m}}\frac{\tau^m}{(q;q)_m}\frac{\ell^k}{(q;q)_k}$$
$$= \frac{1}{(yt, \tau/t, t/\ell; q)_\infty}\sum_{k=0}^{\infty}\frac{(y\ell;q)_k}{(q, \ell q/t;q)_k}q^k \; {}_{s+1}\phi_s\left(\begin{matrix} a_1, \cdots, a_{s+1} \\ b_1, \cdots, b_s \end{matrix}; q, x\ell q^k\right),$$

where $\max\{|y\ell|, |\tau/t|, |t/\ell|\} < 1$.

5. Mehler's Formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$

In this section, we construct Mehler's formula and its extension for the generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ using the operator representation (30). Miller's formula and its extension for both q-Hahn polynomials $\phi_n^{(a)}(x|q)$ and generalized Al-Salam-Carlitz polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$ are derived by providing special values for the variables in the Mehler's formula and its extension for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$.

Theorem 5.1. (Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)\phi_n^{(\boldsymbol{d},\boldsymbol{e})}(u, v, f|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(d_1, ..., d_r;q)_k}{(e_1, ..., e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u, v)\frac{f^k}{(q;q)_k}$$
$$\times\sum_{j=0}^{k}\frac{(q^{-k}, xt;q)_j}{(q, yt;q)_j}q^j \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ctq^j\right), \quad |xt| < 1. \tag{39}$$

Proof.

$$\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)\phi_n^{(\mathbf{d},\mathbf{e})}(u, v, f|q)\frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\{P_n(x, y)\}\,\phi_n^{(\mathbf{d},\mathbf{e})}(u, v, f|q)\frac{t^n}{(q;q)_n} \quad \text{(by using (30))}$$
$$= {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_n(x, y)\frac{t^n}{(q;q)_n}\right\}$$
$$\times\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(d_1, \ldots, d_r;q)_k}{(e_1, \ldots, e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} f^k P_{n-k}(u, v) \quad \text{(by using (28))}$$
$$= {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(d_1, ..., d_r;q)_k}{(e_1, ..., e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\right.$$
$$\left.\times\frac{f^k t^n}{(q;q)_k(q;q)_{n-k}}P_{n-k}(u, v)P_n(x, y)\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{f^k t^{n+k}}{(q;q)_k(q;q)_n}\right.$$
$$\left.\times P_k(u,v)P_{n+k}(x,y)\right\}$$
$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}P_k(u,v)\frac{(ft)^k}{(q;q)_k}\right.$$
$$\left.\times P_k(x,y)\sum_{n=0}^{\infty}P_n(x,q^k y)\frac{t^n}{(q;q)_n}\right\}$$
$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}P_k(u,v)\frac{(ft)^k}{(q;q)_k}\right.$$
$$\left.\times P_k(x,y)\frac{(q^k yt;q)_\infty}{(xt;q)_\infty}\right\} \quad \text{(by using (16))}$$
$$=\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}P_k(u,v)\frac{(ft)^k}{(q;q)_k}$$
$$\times\ {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\frac{P_k(x,y)}{(yt;q)_k}\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}$$
$$=\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}P_k(u,v)\frac{(ft)^k}{(q;q)_k}t^{-k}\frac{(yt;q)_\infty}{(xt;q)_\infty}$$
$$\times\sum_{j=0}^{k}\frac{(q^{-k},xt;q)_j}{(q,yt;q)_j}q^j\,{}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,ctq^j\right) \quad \text{(by using (32))}$$
$$=\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}P_k(u,v)\frac{f^k}{(q;q)_k}$$
$$\times\sum_{j=0}^{k}\frac{(q^{-k},xt;q)_j}{(q,yt;q)_j}q^j\ {}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,ctq^j\right).$$

□

- By using special values to the parameters in equation (39), we revive Mehler's formula for the polynomials $\phi_n^{(a)}(x|q)$ (equation (13)).

Proof. Setting $r=1$, $s=0$, $a_1=a$, $x=1$, $y=0$, $c=x$, $d_1=b$, $u=1$, $v=0$, $f=y$, and $t=\tau$ in equation (39), we get

$$\sum_{n=0}^{\infty}\phi_n^{(a)}(x|q)\phi_n^{(b)}(y|q)\frac{\tau^n}{(q;q)_n}$$
$$=\frac{1}{(\tau;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k}{(q;q)_k}\ y^k\sum_{j=0}^{k}\frac{(q^{-k},\tau;q)_j}{(q;q)_j}\ q^j\ {}_1\phi_0\left(\begin{array}{c} a \\ 0 \end{array};q,x\tau q^j\right)$$
$$=\frac{1}{(\tau;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k}{(q;q)_k}\ y^k\sum_{j=0}^{k}\frac{(q^{-k},\tau;q)_j}{(q;q)_j}\ q^j\frac{(ax\tau q^j;q)_\infty}{(x\tau q^j;q)_\infty}$$

$$
\begin{aligned}
&= \frac{(ax\tau;q)_\infty}{(\tau,x\tau;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k}\ y^k \sum_{j=0}^{k} \frac{(q^{-k},\tau,x\tau;q)_j}{(q,ax\tau;q)_j}\ q^j \\
&= \frac{(ax\tau;q)_\infty}{(\tau,x\tau;q)_\infty} \sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k}\ (y\tau)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(\tau,a;q)_j}{(ax\tau;q)_j}\ x^j \quad \text{(by using (36))} \\
&= \frac{(ax\tau;q)_\infty}{(\tau,x\tau;q)_\infty} \sum_{j=0}^{\infty} \frac{(\tau,a;q)_j}{(q,ax\tau;q)_j} x^j \sum_{k=j}^{\infty} \frac{(b;q)_k}{(q;q)_{k-j}}\ (y\tau)^k \\
&= \frac{(ax\tau;q)_\infty}{(\tau,x\tau;q)_\infty} \sum_{j=0}^{\infty} \frac{(\tau,a,b;q)_j}{(q,ax\tau;q)_j} (xy\tau)^j \sum_{k=0}^{\infty} \frac{(bq^j;q)_k}{(q;q)_k}\ (y\tau)^k \\
&= \frac{(ax\tau;q)_\infty}{(\tau,x\tau;q)_\infty} \sum_{j=0}^{\infty} \frac{(\tau,a,b;q)_j}{(q,ax\tau;q)_j} (xy\tau)^j \frac{(q^j by\tau;q)_\infty}{(y\tau;q)_\infty} \\
&= \frac{(ax\tau,by\tau;q)_\infty}{(\tau,x\tau,y\tau;q)_\infty} \sum_{j=0}^{\infty} \frac{(\tau,a,b;q)_j}{(q,ax\tau,by\tau;q)_j} (xy\tau)^j \\
&= \frac{(ax\tau,by\tau;q)_\infty}{(\tau,x\tau,y\tau;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} \tau,a,b \\ ax\tau,by\tau \end{matrix} ;q,xy\tau \right).
\end{aligned}
$$

□

- Setting $r=s+1$, $y=0$, $x=y$, $c=x$, $v=0$, $u=v$, $f=u$ in equation (39), we get Mehler's formula for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

  Corollary 5.1.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\phi_n^{(\boldsymbol{d},\boldsymbol{e})}(u,v|q)\frac{t^n}{(q;q)_n} \\
&\quad = \frac{1}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{(d_1,\cdots,d_{s+1};q)_k}{(e_1,\cdots,e_s;q)_k}(uv)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (yt;q)_j\ (-1)^j q^{\binom{j}{2}-j(k-1)} \\
&\quad\times\ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\cdots,a_{s+1} \\ b_1,\cdots,b_s \end{matrix} ;q,xtq^j \right), \quad |yt|<1. \qquad (40)
\end{aligned}
$$

Theorem 5.2. (An extention to Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$). *We have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \phi_{n+m}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\phi_n^{(\boldsymbol{d},\boldsymbol{e})}(u,v,f|q)\frac{t^n}{(q;q)_n} \\
&\quad = \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{f^k}{(q;q)_k} \\
&\quad\times\ t^{-m} \sum_{j=0}^{m+k} \frac{(q^{-(m+k)},xt;q)_j}{(q,yt;q)_j} q^j\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix} ;q,ctq^j \right), \quad |xt|<1. \qquad (41)
\end{aligned}
$$

Proof. From (30), we have

$$
\sum_{n=0}^{\infty} \phi_{m+n}^{(\mathbf{a},\mathbf{b})}(x,y,c|q)\phi_n^{(\mathbf{d},\mathbf{e})}(u,v,f|q)\frac{t^n}{(q;q)_n}
$$

$$= \sum_{n=0}^{\infty} {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\{P_{m+n}(x,y)\}\,\phi_n^{(\mathbf{d},\mathbf{e})}(u,v,f|q)\frac{t^n}{(q;q)_n}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty} P_{m+n}(x,y)\frac{(t)^n}{(q;q)_n}\right\}$$

$$\times \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_{n-k}(u,v)f^k \quad \text{(by using (28))}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\right.$$

$$\left.\times \frac{f^k t^n}{(q;q)_k(q;q)_{n-k}} P_{n-k}(u,v)P_{m+n}(x,y)\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{f^k t^{n+k}}{(q;q)_k(q;q)_n}\right.$$

$$\left.\times P_k(u,v)P_{m+n+k}(x,y)\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{(ft)^k}{(q;q)_k}\right.$$

$$\left.\times P_{m+k}(x,y)\sum_{n=0}^{\infty}P_n(x,q^{m+k}y)\frac{t^n}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{(ft)^k}{(q;q)_k}\right.$$

$$\left.\times P_{m+k}(x,y)\frac{(q^{m+k}yt;q)_\infty}{(xt;q)_\infty}\right\} \quad \text{(by using (16))}$$

$$= \sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{(ft)^k}{(q;q)_k}$$

$$\times\ {}_r\Phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,cD_{xy}\right)\left\{\frac{P_{m+k}(x,y)}{(yt;q)_{m+k}}\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{(ft)^k}{(q;q)_k}$$

$$\times t^{-(m+k)}\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{j=0}^{m+k}\frac{(q^{-(m+k)},xt;q)_j}{(q,yt;q)_j}q^j\ {}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,ctq^j\right)$$

$$\text{(by using (32))}$$

$$= \frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(d_1,...,d_r;q)_k}{(e_1,...,e_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} P_k(u,v)\frac{f^k}{(q;q)_k}$$

$$\times\ t^{-m}\sum_{j=0}^{m+k}\frac{(q^{-(m+k)},xt;q)_j}{(q,yt;q)_j}q^j\ {}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array};q,ctq^j\right).$$

□

- Setting $m = 0$ in equation (41), we recover Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ (equation (40)).
- Letting $r = 1$, $s = 0$, $a_1 = a$, $x = 1$, $y = 0$, $c = x$, $d_1 = b$, $u = 1$, $v = 0$, $f = y$ in equation (41), we reobtain an extension of Mehler's formula for $\phi_n^{(a)}(x|q)$ (equation (24)).
- Setting $r = s + 1$, $y = 0$, $x = y$, $c = x$, $v = 0$, $u = v$, $f = u$ in (41), we obtain an extension of Mehler's formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$.

Corollary 5.2. *We have*

$$\sum_{n=0}^{\infty}\phi_{m+n}^{(\boldsymbol{a},\boldsymbol{b})}(x,y|q)\phi_n^{(\boldsymbol{d},\boldsymbol{e})}(u,v|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{1}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(d_1,\cdots,d_{s+1};q)_k}{(e_1,\cdots,e_s;q)_k}(uv)^k\ t^{-m}\sum_{j=0}^{m+k}\begin{bmatrix} m+k \\ j\end{bmatrix}(yt;q)_j$$
$$\times\ (-1)^j q^{\binom{j}{2}-j(m+k-1)}\ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\cdots,a_{s+1} \\ b_1,\cdots,b_s\end{matrix};q,xtq^j\right),\quad |yt|<1.$$

6. The Srivastava-Agarwal type bilinear generating functions for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$

In this section, we create the Srivastava-Agarwal type bilinear generating function for the generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$. We use specific parameter values to reobtain Mehler's formula for q-Hahn polynomials $\phi_n^{(a)}(x|q)$. In addition, we build the Srivastava-Agarwal type generating function for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$. We use given parameter values to reconstruct the Srivastava-Agarwal type generating function for both $\phi_n^{(a)}(x|q)$ and $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$.

Theorem 6.1. (Srivastava-Agarwal type bilinear generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(u,v,c|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,ut;q)_n}{(q,vt,q/x;q)_n}q^n\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,ctq^n\right),\tag{42}$$

provided $\max\{|ut|, |x|\} < 1$.

Proof.

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\phi_n^{(\mathbf{a},\mathbf{b})}(u,v,c|q)\frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}(\alpha;q)_k x^k\phi_n^{(\mathbf{a},\mathbf{b})}(u,v,f|q)\frac{t^n}{(q;q)_n}\quad \text{(by using (11))}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}(\alpha;q)_k x^k\ {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{uv}\right)\{P_n(u,v)\}\frac{t^n}{(q;q)_n}\quad \text{(by using (30))}$$
$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{uv}\right)\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}(\alpha;q)_k x^k P_n(u,v)\frac{t^n}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(\alpha;q)_k x^k P_n(u,v)t^n}{(q;q)_k(q;q)_{n-k}}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)\left\{\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_k x^k P_{n+k}(u,v)t^{n+k}}{(q;q)_k(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)\left\{\sum_{k=0}^{\infty}P_k(u,v)\frac{(\alpha;q)_k (xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}P_n(u,q^kv)\frac{t^n}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)\left\{\sum_{k=0}^{\infty}P_k(u,v)\frac{(\alpha;q)_k (xt)^k}{(q;q)_k}\frac{(q^kvt;q)_\infty}{(ut;q)_\infty}\right\}$$

(by using (16))

$$=\sum_{k=0}^{\infty}\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\,{}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)\left\{\frac{P_k(u,v)}{(vt;q)_k}\frac{(vt;q)_\infty}{(ut;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha;q)_k(xt)^k}{(q;q)_k}\frac{(vt;q)_\infty}{(ut;q)_\infty}t^{-k}\sum_{n=0}^{k}\frac{(q^{-k},ut;q)_n}{(q,vt;q)_n}q^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)$$

(by using (32))

$$=\frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{k=0}^{\infty}\frac{(\alpha;q)_k}{(q;q)_k}x^k\sum_{n=0}^{k}\frac{(q^{-k},ut;q)_n}{(q,vt;q)_n}q^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)$$

$$=\frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(ut;q)_n}{(q,vt;q)_n}q^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)\sum_{k=n}^{\infty}\frac{(\alpha;q)_kx^kq^{-nk}}{(q;q)_{k-n}}$$

$$=\frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{-\binom{n}{2}}(ut,\alpha;q)_n}{(q,vt;q)_n}x^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)\sum_{k=0}^{\infty}\frac{(q^n\alpha;q)_k}{(q;q)_k}(q^{-n}x)^k$$

$$=\frac{(vt;q)_\infty}{(ut;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{-\binom{n}{2}}(ut,\alpha;q)_n}{(q,vt;q)_n}x^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)\frac{(\alpha x;q)_\infty}{(q^{-n}x;q)_\infty}$$

$$=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{-\binom{n}{2}}(ut,\alpha;q)_n}{(q,vt;q)_n(q^{-n}x;q)_n}x^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right)$$

$$=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(ut,\alpha;q)_n}{(q,vt,q/x;q)_n}q^n\,{}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ctq^n\right). \quad \text{(by using (9))}$$

□

- We give special values for the parameters in equation (42) to reobtain Mehler's formula for the q-Hahn polynomials $\phi_n^{(a)}(x|q)$ (equation (13)).

  Proof. Setting $r=1$, $s=0$, $\alpha=a$, $\mathbf{a}=b$, $\mathbf{b}=0$, $u=1$, $v=0$, $c=y$, and $t=\tau$ in equation (42), we get

$$\sum_{n=0}^{\infty}\phi_n^{(a)}(x|q)\phi_n^{(b)}(y|q)\frac{t^n}{(q;q)_n}$$

$$=\frac{(ax;q)_\infty}{(t,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(t,a;q)_nq^n}{(q,q/x;q)_n}\,{}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)$$

$$
\begin{aligned}
&= \frac{(ax;q)_\infty}{(t,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(t,a;q)_n}{(q,q^{-n}x;q)_n}x^n\ {}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)\\
&= \frac{1}{(t;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(t,a;q)_n}{(q,q^{-n}x;q)_n}x^n\ {}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)\frac{(ax;q)_\infty}{(q^{-n}x;q)_\infty}\\
&= \frac{1}{(t;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(t;q)_n}{(q;q)_n}x^n\ {}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)\sum_{k=0}^{\infty}\frac{(a;q)_{n+k}}{(q;q)_k}(q^{-n}x)^k\\
&= \frac{1}{(t;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}(t;q)_n}{(q;q)_n}x^n\ {}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)\sum_{k=n}^{\infty}\frac{(a;q)_k}{(q;q)_{k-n}}(q^{-n}x)^{k-n}\\
&= \frac{1}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k\sum_{n=0}^{k}\frac{(q^{-k},t;q)_n q^n}{(q;q)_n}\ {}_1\phi_0\left(\begin{matrix} b \\ 0 \end{matrix};q,ytq^n\right)\\
&= \frac{(byt;q)_\infty}{(t,yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k\sum_{n=0}^{k}\frac{(q^{-k},t,yt;q)_n}{(q,byt;q)_n}\ q^n \quad \text{(by using (1))}\\
&= \frac{(byt;q)_\infty}{(t,yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(xt)^k\sum_{n=0}^{k}\begin{bmatrix} k \\ n \end{bmatrix}\frac{(t,b;q)_n}{(byt;q)_n}\ y^n \quad \text{(by using (36))}\\
&= \frac{(byt;q)_\infty}{(t,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(t,b;q)_n}{(q,byt;q)_n}y^n\sum_{k=n}^{\infty}\frac{(a;q)_k}{(q;q)_{k-n}}\ (xt)^k\\
&= \frac{(byt;q)_\infty}{(t,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(t,a,b;q)_n}{(q,byt;q)_n}(xyt)^n\sum_{k=0}^{\infty}\frac{(aq^n;q)_k}{(q;q)_k}\ (xt)^k\\
&= \frac{(byt;q)_\infty}{(t,yt;q)_\infty}\sum_{n=0}^{\infty}\frac{(t,a,b;q)_n}{(q,byt;q)_n}(xyt)^n\frac{(q^n axt;q)_\infty}{(xt;q)_\infty}\\
&= \frac{(axt,byt;q)_\infty}{(t,xt,yt;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} a,b,t \\ axt,byt \end{matrix};q,xyt\right).
\end{aligned}
$$

□

Theorem 6.2. *We have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)P_n(u,v)\frac{t^n}{(q;q)_n}\\
&\quad = \frac{(v/u,yut;q)_\infty}{(xut;q)_\infty}\sum_{j=0}^{\infty}\frac{(xut;q)_j(v/u)^j}{(q,yut;q)_j}\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cutq^j\right),
\end{aligned}
\tag{43}
$$

provided $|xut| < 1$.

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)P_n(u,v)\frac{t^n}{(q;q)_n}\\
&\quad = \sum_{n=0}^{\infty}\ {}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{xy}\right)\{P_n(x,y)\}\,P_n(u,v)\frac{t^n}{(q;q)_n}
\end{aligned}
$$

$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}P_n(u,v)\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}v^k u^{n-k}\right\}$$

(by using (17))

$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}v^k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{P_n(x,y)u^{n-k}t^n}{(q;q)_{n-k}}\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}v^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_{n+k}(x,y)u^n t^{n+k}}{(q;q)_n}\right\}$$

$$= {}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(vt)^k P_k(x,y)}{(q;q)_k}\sum_{n=0}^{\infty}\frac{P_n(x,q^k y)}{(q;q)_n}(ut)^n\right\}$$

$$= {}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(vt)^k P_k(x,y)}{(q;q)_k}\frac{(q^k yut;q)_\infty}{(xut;q)_\infty}\right\}$$

(by using (16))

$$= {}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(vt)^k}{(q;q)_k}\frac{P_k(x,y)}{(yut;q)_k}\frac{(yut;q)_\infty}{(xut;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(vt)^k}{(q;q)_k}\,{}_r\Phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cD_{xy}\right)\left\{\frac{P_k(x,y)}{(yut;q)_k}\frac{(yut;q)_\infty}{(xut;q)_\infty}\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(vt)^k}{(q;q)_k}(ut)^{-k}\frac{(yut;q)_\infty}{(xut;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},xut;q)_j q^j}{(q,yut;q)_j}$$

$$\times\ {}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cutq^j\right)\quad\text{(by using (32))}$$

$$=\frac{(yut;q)_\infty}{(xut;q)_\infty}\sum_{j=0}^{\infty}\frac{(xut;q)_j(-1)^j q^{\binom{j}{2}}q^j}{(q,yut;q)_j}\,{}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cutq^j\right)$$

$$\times\sum_{k=j}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(v/u)^k q^{-kj}}{(q;q)_{k-j}}$$

$$=\frac{(v/u,yut;q)_\infty}{(xut;q)_\infty}\sum_{j=0}^{\infty}\frac{(xut;q)_j(v/u)^j}{(q,yut;q)_j}\,{}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array};q,cutq^j\right).$$

□

- Setting $v/u=\lambda$ and $ut\to t$ in equation (43), we get the following corollary:

Corollary 6.1. (Srivastava-Agarwal type generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$). *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)(\lambda;q)_n\frac{t^n}{(q;q)_n}$$

$$= \frac{(\lambda, yt;q)_\infty}{(xt;q)_\infty} \sum_{j=0}^{\infty} \frac{(xt;q)_j \lambda^j}{(q, yt;q)_j} \; {}_r\phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, ctq^j \right), \quad |xt| < 1. \tag{44}$$

- We give special values for the parameters in equation (44) to get Srivastava-Agarwal type generating function for $\phi_n^{(a)}(x)$ (equation (15)).
- Letting $r = s+1$, $x = y$, $y = 0$, $c = x$ in equation (44) and using Cauchy identity (1), we recover Srivastava-Agarwal type generating function for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y|q)$ (equation (27)).

7. A Transformation formula involving generating functions for $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$

In this section, we derive the transformation formula for generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x, y, c|q)$ by using the homogeneous q-shift operator ${}_r\Phi_s$.

Theorem 7.1. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relation:*

$$\sum_{n=0}^{\infty} A(n) P_n(u, v) = \sum_{n=0}^{\infty} B(n) \frac{(q^n vt;q)_\infty}{(q^n ut;q)_\infty}. \tag{45}$$

Then

$$\sum_{n=0}^{\infty} A(n) \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(u, v, c|q) = \sum_{n=0}^{\infty} B(n) \frac{(q^n vt;q)_\infty}{(q^n ut;q)_\infty} \; {}_r\phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, ctq^n \right), \tag{46}$$

where each of the series in (45) *and* (46) *are absolutely convergent.*

Proof. Let $f(u, v, c)$ be the right-hand side of (46). By using (45) and (30) we find that

$$\begin{aligned} f(u, v, c) &= \sum_{n=0}^{\infty} B(n) \frac{(q^n vt;q)_\infty}{(q^n ut;q)_\infty} \; {}_r\phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, ctq^n \right) \qquad (47) \\ &= \sum_{n=0}^{\infty} B(n) \; {}_r\Phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, cDuv \right) \left\{ \frac{(q^n vt;q)_\infty}{(q^n ut;q)_\infty} \right\} \\ &= {}_r\Phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, cDuv \right) \left\{ \sum_{n=0}^{\infty} B(n) \frac{(q^n vt;q)_\infty}{(q^n ut;q)_\infty} \right\} \\ &= {}_r\Phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, cDuv \right) \left\{ \sum_{n=0}^{\infty} A(n) P_n(u, v) \right\} \\ &= \sum_{n=0}^{\infty} A(n) \; {}_r\Phi_s \left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} ; q, cDuv \right) \{P_n(u, v)\} \\ &= \sum_{n=0}^{\infty} A(n) \phi_n^{(\mathbf{a},\mathbf{b})}(u, v, c|q), \qquad (48) \end{aligned}$$

which is precisely the left-hand side of (46). The proof of Theorem 7.1 is completed. □

- When $r = 1$, $s = 0$, $a_1 = a$, $u = 1$, $v = 0$, $c = u$ in equations (45)and (46), we get the transformation formula for the polynomials $\phi_n^{(a)}(x|q)$.

Corollary 7.1. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relationship*:

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t;q)_\infty}. \tag{49}$$

Then

$$\sum_{n=0}^{\infty} A(n)\phi_n^{(a)}(u|q) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n t;q)_\infty}\frac{(autq^n;q)_\infty}{(utq^n;q)_\infty}, \tag{50}$$

where each of the series in (49) *and* (50) *are absolutely convergent.*

- Setting $r = s+1$, $v = 0$, $u = v$, $c = u$ in equations (45)and (46), we get the transformation formula for the polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

Corollary 7.2. *Let the coefficients $A(n)$ and $B(n)$ satisfy the following relationship:*

$$\sum_{n=0}^{\infty} A(n)v^n = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n vt;q)_\infty} \tag{51}$$

Then

$$\sum_{n=0}^{\infty} A(n)\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(u,v|q) = \sum_{n=0}^{\infty} \frac{B(n)}{(q^n vt;q)_\infty}\ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\cdots,a_{s+1} \\ b_1,\cdots,b_s\end{matrix};q,utq^n\right), \tag{52}$$

where each of the series in (51) *and* (52) *are absolutely convergent.*

8. Conclusions

(1) Several previously specified operators can be obtained by providing special values to the generalized homogeneous q-shift operator ${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)$, for more information, see [3, 5, 7].

(2) The q-Hahn polynomials $\phi_n^{(a)}(x|q)$ are a special case of the generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$.

(3) The generalized q-hypergeometric polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$ are a generalization for the generalized Al-Salam-Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

(4) The polynomials identities for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,c|q)$ are a generalization of the q-Hahn polynomials $\phi_n^{(a)}(x|q)$ and the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

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