TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 683-695

BINARY $\check{C}$ECH SOFT CLOSURE SPACES

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ABSTRACT. In this paper the notion of binary $\check{C}$ech soft closure space which is defined over two initial universe sets with fixed sets of parameters is introduced and studied. This space extends and generalizes $\check{C}$ech soft closure space. The main and basic notions for this space such as closed (open) binary soft sets, binary soft interior, and dense binary soft sets are defined and studied. Relationships between binary $\check{C}$ech soft closure space and $\check{C}$ech soft closure space are deduced. Examples and counterexamples are presented to illustrate some of our results. Finally, some operations on binary $\check{C}$ech soft closure operators are defined.

Keywords: Binary soft sets, binary soft topology, soft closure space, binary soft closure space, binary $\check{C}$ech soft closure space.

AMS Subject Classification: 54A05, 97E60.

1. Introduction

$\check{C}$ech closure space (K, C) introduced first by $\check{C}$ech [4]. In this space, the mapping $C : P(K) \longrightarrow P(K)$ is called $\check{C}$ech closure operator on K and is satisfying the conditions $C(\emptyset) = \emptyset, F \subseteq C(F)$, and $C(F \cup G) = C(F) \cup C(G)$. In general, $\check{C}$ech closure spaces have a more general structure than topological spaces. Inspiring by $\check{C}$ech initial results, other researchers are studied further, improved, generalized, and extended $\check{C}$ech closure spaces (see, e.g., [3, 11, 21, 22], among others).

In 1999, Molodstov [19] introduced the concept of soft set theory to solve some complicated problems in mathematics and some other fields. In particular, closure spaces are introduced and studied in a soft set setting. For instance, $\check{C}$ech soft closure spaces were introduced and discussed by Gowri and Jegadeesan [8] and Krishnaveni and Sekar [12]. Majeed [14] established $\check{C}$ech fuzzy soft closure spaces. The later space is investigated further in [13, 15, 16]. The concept of soft closure spaces and their essential features are discussed in detail in [6, 7, 17].

The concept of binary structure between two universal sets K_1 and K_2 was first defined and studied by Jothi and Thangavelu [20], because in real-world situations there may

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§ Manuscript received: May 08, 2022; accepted: August 25, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.14, No.2

be two or more universal sets. A binary structure from K_1 to K_2 is mathematically described as a collection of ordered pairs (F, G), with $F \subseteq K_1$ and $G \subseteq K_2$. Jothi and Thangavelu [20] developed the idea of binary topology which is a single structure denoted by (K_1, K_2, M) where $M \subseteq P(K_1) \times P(K_2)$ and satisfying the three axioms of ordinary topology. The notion of binary Čech closure spaces was proposed by Chacko and Susha [5]. Ackgoz and Tas [1] studied the properties of a binary soft set created from two initial universal sets and a parameter set. Benchalli et al. [2] introduced the notion binary soft topological spaces which are defined over two initial universe sets with a fixed set of parameters. Also, Hussain [9, 10] proposed binary soft topological spaces, which are extensions of soft topological space, and investigated binary soft connectedness in binary soft topological spaces. The structure of binary soft sets is employed, in this paper, to propose the notion of binary Čech soft closure spaces, which is an extension of the binary Čech closure spaces established in [5]. The prerequisites are listed in Section 2. We present the notions of binary soft closure operator, binary Čech soft closure operator, and induced binary soft closure operators in Section 3 and we show how they are related. The connections between soft closure spaces and binary Čech soft closure spaces are discussed in Section 4. The operations union, composition, and intersection of binary Čech soft closure spaces are covered in Section 5.

2. Preliminaries

In this section, we recall several definitions that will be used in the next sections.

Definition 2.1. [19] *A soft set $(\mathcal{F}, A)$ over an initial universe set K and a set of parameters Q is a mapping $\mathcal{F} : A \longrightarrow P(K)$ where A is a nonempty subset of Q and $P(K)$ denotes the power set of K.*

Definition 2.2. [18] *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over an initial universe set K and a set of parameters Q. Then, $(\mathcal{F}, A)$ is soft subset of $(\mathcal{G}, B)$, denoted by $(\mathcal{F}, A) \sqsubseteq (\mathcal{G}, B)$, if (1) $A \subseteq B$, and (2) $\mathcal{F}(\omega) \subseteq \mathcal{G}(\omega)$, for all $\omega \in A$.*

Definition 2.3. [18] *The union of two soft sets $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ over the common universe K is the soft set $(\mathcal{H}, C) = (\mathcal{F}, A) \sqcup (\mathcal{G}, B)$, where $C = A \cup B$ and for all $\omega \in C$,*

$$\mathcal{H}(\omega) = \begin{cases} \mathcal{F}(\omega) & \text{if } \omega \in A - B, \\ \mathcal{G}(\omega) & \text{if } \omega \in B - A, \\ \mathcal{F}(\omega) \cup \mathcal{G}(\omega) & \text{if } \omega \in A \cap B. \end{cases}$$

Definition 2.4. [18] *The intersection of two soft sets $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ over the common universe K is the soft set $(\mathcal{H}, C) = (\mathcal{F}, A) \sqcap (\mathcal{G}, B)$, where $C = A \cap B$ and for all $\omega \in C$, $\mathcal{H}(\omega) = \mathcal{F}(\omega) \cap \mathcal{G}(\omega)$.*

Definition 2.5. [6] *An operator $u : \mathcal{SS}(K, Q) \longrightarrow \mathcal{SS}(K, Q)$ is called a soft closure operator on K, if for all $\mathcal{F}_Q, \mathcal{G}_Q \in \mathcal{SS}(K, Q)$ the following axioms are satisfied:*
$(C1)$ $\widetilde{\Phi}_Q = u(\widetilde{\Phi}_Q)$,
$(C2)$ $\mathcal{F}_Q \sqsubseteq u(\mathcal{F}_Q)$,
$(C3)$ $\mathcal{F}_Q \sqsubseteq G_Q \Longrightarrow u(\mathcal{F}_Q) \sqsubseteq u(G_Q)$.
The triple (K,u,Q) is called a soft closure space. If in addition (C4) $u(\mathcal{F}_Q \sqcup G_Q) = u(\mathcal{F}_Q) \sqcup (u(G_Q)$. The space (K, u, Q) is called a Čech soft closure space [12].

Consider K_1 and K_2 be two initial universal sets, Q be a collection of parameters, and $P(K_1)$ and $P(K_2)$ be the K_1 and K_2 power sets respectively. Also, let $A, B, C \subseteq Q$.

Definition 2.6. [1] *$\tilde{\tilde{\mathcal{F}}}_A$ is said to be a binary soft set ($\mathfrak{B}\mathfrak{S}$-set, for short) over K_1, K_2 where $\tilde{\tilde{\mathcal{F}}} : A \longrightarrow P(K_1) \times P(K_2)$, $\tilde{\tilde{\mathcal{F}}}(\omega) = (N, M)$, for each $\omega \in A$ such that $N \subseteq K_1, M \subseteq K_2$.*

Definition 2.7. [1] *Let $\tilde{\tilde{\mathcal{F}}}_A, \tilde{\tilde{\mathcal{G}}}_B$ are two $\mathfrak{B}\mathfrak{S}$-sets over the universes K_1, K_2. $\tilde{\tilde{\mathcal{F}}}_A$ is called a $\mathfrak{B}\mathfrak{S}$-subset of $\tilde{\tilde{\mathcal{G}}}_B$, if $A \subseteq B$ and $N_1 \subseteq N_2, M_1 \subseteq M_2$ such that $\tilde{\tilde{\mathcal{F}}}(\omega) = (N_1, M_1)$, $\tilde{\tilde{\mathcal{G}}}(\omega) = (N_2, M_2)$ for each $\omega \in A$ such that $N_1, N_2 \subseteq K_1, M_1, M_2 \subseteq K_2$.*

We denote it by $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{\mathcal{G}}}_B$. $\tilde{\tilde{\mathcal{F}}}_A$ is called a $\mathfrak{B}\mathfrak{S}$-superset of $\tilde{\tilde{\mathcal{G}}}_B$, if $\tilde{\tilde{\mathcal{G}}}_B$ is a binary soft subset of $\tilde{\tilde{\mathcal{F}}}_A$. We write $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqsupseteq}} \tilde{\tilde{\mathcal{G}}}_B$.

Definition 2.8. [1] *Let $\tilde{\tilde{\mathcal{F}}}_A, \tilde{\tilde{\mathcal{G}}}_B$ are two $\mathfrak{B}\mathfrak{S}$-sets over the universes K_1, K_2. $\tilde{\tilde{\mathcal{F}}}_A$ is called binary soft equal to $\tilde{\tilde{\mathcal{G}}}_B$, if $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{\mathcal{G}}}_B$ and $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqsupseteq}} \tilde{\tilde{\mathcal{G}}}_B$. We denote it by $\tilde{\tilde{\mathcal{F}}}_A = \tilde{\tilde{\mathcal{G}}}_B$.*

Definition 2.9. [1] *A $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{F}}}_A$ over K_1, K_2 is characterized as binary null soft set denoted by $\tilde{\tilde{\emptyset}}$, if $\tilde{\tilde{\mathcal{F}}}(\omega) = (\emptyset, \emptyset)$, for all $\omega \in A$.*

Definition 2.10. [1] *A $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{F}}}_A$ over K_1, K_2 binary absolute soft set denoted by $\tilde{\tilde{\mathcal{A}}}$, $\tilde{\tilde{\mathcal{F}}}(\omega) = (K_1, K_2)$, for all $\omega \in A$.*

Definition 2.11. [1] *A $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{H}}}_C$ is the union of two $\mathfrak{B}\mathfrak{S}$-sets $\tilde{\tilde{\mathcal{F}}}_A$ and $\tilde{\tilde{\mathcal{G}}}_B$ over the universes K_1, K_2 where $C = A \cup B$, and for each $\omega \in C$ such that $N_1, N_2 \subseteq K_1$, $M_1, M_2 \subseteq K_2$,*

$$\tilde{\tilde{\mathcal{H}}}(\omega) = \begin{cases} (N_1, M_1) & \text{if} \quad \omega \in A - B \\ (N_2, M_2) & \text{if} \quad \omega \in B - A \\ (N_1 \cup N_2, M_1 \cup M_2) & \text{if} \quad \omega \in A \cap B \end{cases}$$

such that $\tilde{\tilde{\mathcal{F}}}(\omega) = (N_1, M_1)$ for each $\omega \in A$ and $\tilde{\tilde{\mathcal{G}}}(\omega) = (N_2, M_2)$ for each $\omega \in B$. We denote it by $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_B = \tilde{\tilde{\mathcal{H}}}_C$.

Definition 2.12. [1] *The $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{H}}}_C$ is the intersection of two $\mathfrak{B}\mathfrak{S}$-sets $\tilde{\tilde{\mathcal{F}}}_A$ and $\tilde{\tilde{\mathcal{G}}}_B$ over the universes K_1, K_2, where $C = A \cap B$ and $\tilde{\tilde{\mathcal{H}}}(\omega) = (N_1 \cap N_2, M_1 \cap M_2)$, for each $\omega \in C$ such that $\tilde{\tilde{\mathcal{F}}}(\omega) = (N_1, M_1)$ for each $\omega \in A$ and $\tilde{\tilde{\mathcal{G}}}(\omega) = (N_2, M_2)$ for each $\omega \in B$, such that $N_1, N_2 \subseteq K_1$, $M_1, M_2 \subseteq K_2$. We denote it $\tilde{\tilde{\mathcal{F}}}_A \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_B = \tilde{\tilde{\mathcal{H}}}_C$.*

Definition 2.13. [9] *The $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{H}}}_Q$ is the difference of two $\mathfrak{B}\mathfrak{S}$-sets $\tilde{\tilde{\mathcal{F}}}_Q$ and $\tilde{\tilde{\mathcal{G}}}_Q$ over the universes K_1, K_2,denoted by $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\backslash}} \tilde{\tilde{\mathcal{G}}}_Q$ and is defined as $\tilde{\tilde{\mathcal{H}}}(\omega) = (N_1 - N_2, M_1 - M_2)$ for each $\omega \in Q$ such that $\tilde{\tilde{\mathcal{F}}}(\omega) = (N_1, M_1)$ and $\tilde{\tilde{\mathcal{G}}}(\omega) = (N_2, M_2)$.*

Definition 2.14. [9] *The binary soft relative complement of a $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{F}}}_Q$ is denoted by $\tilde{\tilde{\mathcal{F}}}_Q{}' = \tilde{\tilde{\mathcal{F}}}'_Q$ where $\tilde{\tilde{\mathcal{F}}}' : Q \longrightarrow P(K_1) \times P(K_2)$ is a mapping given by $\tilde{\tilde{\mathcal{F}}}'(\omega) = (K_1 - N, K_2 - M)$ where $\tilde{\tilde{\mathcal{F}}}(\omega) = (N, M)$, for all $\omega \in Q$ such that $N \subseteq K_1, M \subseteq K_2$.*

Definition 2.15. [9] *Let τ be the collection of $\mathfrak{B}\mathfrak{S}$-sets over K_1 and K_2 and Q denotes the set of parameters. Then τ is said to be binary soft topology on K_1 and K_2 if*

(1) $\tilde{\tilde{\emptyset}}, \tilde{\tilde{Q}} \in \tau$

(2) *The union of any numbers of $\mathfrak{B}\mathfrak{S}$-sets in τ belongs to τ.*

(3) *The intersection of any two $\mathfrak{B}\mathfrak{S}$-sets in τ belongs to τ.*

3. Binary Čech soft closure spaces

In this section, we define binary (resp. binary Čech) soft closure operator and discuss their basic properties. We also introduce the notion of morphism between soft closure space and binary Čech soft closure space.

Definition 3.1. *Let K_1 and K_2 are two initial universal sets and Q be a set of parameters. A mapping ∂ from the family of all $\mathfrak{BS}$-sets over K_1, K_2 to itself (i.e., $\partial : SS(K_1, K_2, Q) \longrightarrow SS(K_1, K_2, Q)$) is called a binary soft closure operator ($\mathcal{BSCO}$, for short) if*
1. $\partial(\tilde{\tilde{\emptyset}}) = \tilde{\tilde{\emptyset}}$,
2. $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q)$,
3. $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{\mathcal{G}}}_Q \Longrightarrow \partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)$.
The space (K_1, K_2, ∂, Q) is then referred to as a binary soft closure space ($\mathcal{BSCS}$, for short).

To explain Definition 3.1, we'll provide an example.

Example 3.1. *Let $K_1 = \{a_1, a_2, a_3\}$, $K_2 = \{d_1, , d_2\}$ and $Q = \{\omega_1, \omega_2\}$. Let $\partial : SS(K_1, K_2, Q) \to SS(K_1, K_2, Q)$ be a mapping defined as follows:*

$$\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \begin{cases} \tilde{\tilde{\emptyset}} & \text{if} \quad \tilde{\tilde{\mathcal{F}}}_Q = \tilde{\tilde{\emptyset}}, \\ \tilde{\tilde{\mathcal{F}}}_Q & \text{if} \quad \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (N, \emptyset)) : N \in P(K_1)\}, \\ \tilde{\tilde{\mathcal{F}}}_Q & \text{if} \quad \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_2, (\emptyset, M)) : M \in P(K_2)\}, \\ \tilde{\tilde{Q}} & \quad \text{otherwise.} \end{cases}$$

Then, ∂ is a $\mathcal{BSCO}$. Hence, (K_1, K_2, ∂, Q) is $\mathcal{BSCS}$.

Definition 3.2. *The $\mathcal{BSCO}$ is a binary Čech soft closure operator ($\mathcal{B\check{C}SCO}$, for short) if it satisfies the property $\partial(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q) = \partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)$. Then, (K_1, K_2, ∂, Q) is called a binary Čech soft closure space ($\mathcal{B\check{C}SCS}$, for short).*

The next example illustrates Definition 3.2.

Example 3.2. *Let $K_1 = \{a_1, a_2, a_3\}$, $K_2 = \{d_1, , d_2\}$ and $Q = \{\omega_1, \omega_2\}$. Let $\partial : SS(K_1, K_2, Q) \to SS(K_1, K_2, Q)$ be a mapping defined as follows, for $i = 1, 2$:*

$$\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \begin{cases} \tilde{\tilde{\emptyset}} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \tilde{\tilde{\emptyset}}, \\ \{(\omega_i, (\{a_1, a_2\}, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_1\}, \emptyset))\}, \\ \{(\omega_i, (\{a_2\}, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_2\}, \emptyset))\}, \\ \{(\omega_i, (\{a_3\}, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_3\}, \emptyset))\}, \\ \{(\omega_i, (\{a_1, a_2\}, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_1, a_2\}, \emptyset))\}, \\ \{(\omega_i, (K_1, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_1, a_3\}, \emptyset))\}, \\ \{(\omega_i, (\{a_2, a_3\}, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\{a_2, a_3\}, \emptyset))\}, \\ \{(\omega_i, (K_1, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (K_1, \emptyset))\}, \\ \{(\omega_i, (\emptyset, \{d_1\}))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\emptyset, \{d_1\}))\}, \\ \{(\omega_i, (\{a_1\}, K_2))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\emptyset, \{d_2\}))\}, \\ \{(\omega_i, (\{a_1\}, K_2))\} & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_i, (\emptyset, K_2))\}, \\ \partial(\{(\omega_1, (N_1, \emptyset))\}) \tilde{\tilde{\sqcup}} \partial(\{(\omega_1, (\emptyset, M_1))\}) \tilde{\tilde{\sqcup}} & \\ \partial((\{(\omega_2, (N_2, \emptyset))\})) \tilde{\tilde{\sqcup}} \partial((\{(\omega_2, (\emptyset, M_2))\})) & \text{if } \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (N_1, M_1)), (\omega_2, \\ & (N_2, M_2)) : N_1, N_2 \subseteq K_1, \\ & M_1, M_2 \subseteq K_2\}. \end{cases}$$

Then, ∂ is a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCO}$. Therefore, (K_1, K_2, ∂, Q) is $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$.

Remark 3.1. *Every $\mathcal{B}\check{\mathcal{C}}\mathcal{SCO}$ is $\mathcal{BSCO}$ but not conversely, In Example 3.1, ∂ is not $\mathcal{B}\check{\mathcal{C}}\mathcal{SCO}$ since there exist $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (\{a_1, a_2\}, \emptyset))\}$ and $\tilde{\tilde{\mathcal{G}}}_Q = \{(\omega_1, (\emptyset, \{d_1\}))\}$ such that $\partial(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q) = \tilde{\tilde{Q}} \neq \partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{\mathcal{G}}}_Q) = \{(\omega_1, (\{a_1, a_2\}, \{d_1\}))\}$.*

Definition 3.3. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$. Any $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ is said to be ∂-closed $\mathfrak{BS}$-set if $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{F}}}_Q$ and a $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{G}}}_Q$ is ∂-open $\mathfrak{BS}$-set if $\tilde{\tilde{\mathcal{F}}}_Q{}'$ is ∂-closed $\mathfrak{BS}$-set.*

Proposition 3.1. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$. Then, $\tilde{\tilde{\emptyset}}$ and $\tilde{\tilde{Q}}$ are both ∂-open(resp., ∂-closed) $\mathfrak{BS}$-set.*

Proof. Since $\partial(\tilde{\tilde{\emptyset}}) = \tilde{\tilde{\emptyset}}$, then $\tilde{\tilde{\emptyset}}$ is ∂-closed $\mathfrak{BS}$-set and hence $\tilde{\tilde{\emptyset}}' = \tilde{\tilde{Q}}$ is ∂-open $\mathfrak{BS}$-set. Now, since $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q)$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$, then $\tilde{\tilde{Q}} \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{Q}})$. On the other hand, since $\partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{Q}}$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$, then $\partial(\tilde{\tilde{Q}}) \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{Q}}$. This implies $\tilde{\tilde{Q}} = \partial(\tilde{\tilde{Q}})$ which is ∂-closed $\mathfrak{BS}$-set. Hence, $\tilde{\tilde{Q}}' = \tilde{\tilde{\emptyset}}$ is ∂-open $\mathfrak{BS}$-set. □

Definition 3.4. *A $\mathcal{B}\check{\mathcal{C}}\mathcal{SCO}$ ∂_1 is said to be finer than a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCO}$ ∂_2 on the same K_1 and K_2 and the set of parameters Q if $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$. Then, we write $\partial_2 \tilde{\tilde{<}} \partial_1$.*

Remark 3.2. *The discrete binary soft closure operator given by $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{F}}}_Q$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ is the finest binary soft closure operator over K_1 and K_2. The indiscrete binary soft closure operator is given by $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\emptyset}}$ for all $\tilde{\tilde{\mathcal{F}}}_Q \neq \tilde{\tilde{\emptyset}}$ is the coarsest binary soft closure operator over K_1 and K_2.*

Definition 3.5. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$. Then, the binary Čech soft interior operator associated with ∂, denoted by Int_∂ is a mapping from $SS(K_1, K_2, Q)$ to itself given by $Int_\partial(\tilde{\tilde{\mathcal{F}}}_Q) = (\partial(\tilde{\tilde{\mathcal{F}}}_Q{}'))'$. A $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂-open $\mathfrak{BS}$-set if and only if $Int_\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{F}}}_Q$.*

Now, we show for each $\mathcal{B\check{C}SCS}$ (K_1, K_2, ∂, Q), there exists a binary soft topological space $(K_1, K_2, \tau_\partial, Q)$ which is defined naturally. That is $\tau_\partial = \{\tilde{\tilde{\mathcal{F}}}_Q{}' : \partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{F}}}_Q\}$.

Theorem 3.1. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$. Then the set of all ∂-open $\mathfrak{BS}$-sets is a binary soft topology over K_1 and K_2.*

Proof. Let $\tau_\partial = \{\tilde{\tilde{\mathcal{F}}}_Q{}' : \partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{F}}}_Q\}$ be the family of all ∂-open $\mathfrak{BS}$-sets over K_1 and K_2. We must show τ_∂ satisfies the three conditions of Definition 2.15.

(1) Since $\tilde{\tilde{\emptyset}}$ and $\tilde{\tilde{Q}}$ are ∂-open $\mathfrak{BS}$-sets, then $\tilde{\tilde{\emptyset}}$ and $\tilde{\tilde{Q}}$ are in τ_∂.

(2) Let $\tilde{\tilde{\mathcal{F}}}_Q, \tilde{\tilde{\mathcal{G}}}_Q \in \tau_\partial$. Then, $\partial(\tilde{\tilde{\mathcal{F}}}_Q{}') = \tilde{\tilde{\mathcal{F}}}_Q{}' = \{(\omega, (K_1 - \mathcal{F}^1(\omega), K_2 - \mathcal{F}^2(\omega))) : \omega \in Q\}$ and $\partial(\tilde{\tilde{\mathcal{G}}}_Q{}') = \tilde{\tilde{\mathcal{G}}}_Q{}' = \{(\omega, (K_1 - \mathcal{G}^1(\omega), K_2 - \mathcal{G}^2(\omega))) : \omega \in Q\}$. To prove $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_Q$ is an ∂-open $\mathfrak{BS}$-set. That means to prove $\partial((\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_Q)') = (\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_Q)'$.

$$\begin{aligned}
\partial((\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_Q)') &= \partial(\{(\omega, (K_1 - (\mathcal{F}^1(\omega) \cap \mathcal{G}^1(\omega)), K_2 - (\mathcal{F}^2(\omega) \cap \mathcal{G}^2(\omega)))) : \omega \in Q\}) \\
&= \partial(\{(\omega, ((K_1 - \mathcal{F}^1(\omega)) \cup (K_1 - \mathcal{G}^1(\omega)), ((K_2 - \mathcal{F}^2(\omega)) \cup (K_2 - \mathcal{G}^2(\omega)))) \\
&\quad : \omega \in Q\}) \\
&= \partial(\{(\omega, (K_1 - \mathcal{F}^1(\omega), K_2 - \mathcal{F}^2(\omega))) : \omega \in Q\} \tilde{\tilde{\sqcup}} \{(\omega, (K_1 - \mathcal{G}^1(\omega), K_2 - \\
&\quad \mathcal{G}^2(\omega))) : \omega \in Q\}) \\
&= \partial(\{(\omega, (K_1 - \mathcal{F}^1(\omega), K_2 - \mathcal{F}^2(\omega))) : \omega \in Q\}) \tilde{\tilde{\sqcup}} \partial(\{(\omega, (K_1 - \mathcal{G}^1(\omega), \\
&\quad K_2 - \mathcal{G}^2(\omega))) : \omega \in Q\}) \\
&= \{(\omega, (K_1 - \mathcal{F}^1(\omega), K_2 - \mathcal{F}^2(\omega))) : \omega \in Q\} \tilde{\tilde{\sqcup}} \{(\omega, (K_1 - \mathcal{G}^1(\omega), K_2 - \\
&\quad \mathcal{G}^2(\omega))) : \omega \in Q\} \\
&= \{(\omega, (K_1 - (\mathcal{F}^1(\omega) \cap \mathcal{G}^1(\omega)), K_2 - (\mathcal{F}^2(\omega) \cap \mathcal{G}^2(\omega)))) : \omega \in Q\}.
\end{aligned}$$

Thus, $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcap}} \tilde{\tilde{\mathcal{G}}}_Q$ is an ∂-open $\mathfrak{BS}$-set.

(3) Consider an arbitrary collection of ∂-open $\mathfrak{BS}$-sets $\{(\tilde{\tilde{\mathcal{F}}}_Q)_\alpha : \alpha \in J\}$. For each $\alpha \in J$, $(\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha$ is an ∂-closed $\mathfrak{BS}$-set and $\tilde{\tilde{\sqcap}}_{\alpha \in J}(\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha \tilde{\tilde{\sqsubseteq}} (\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha$. So, $\partial(\tilde{\tilde{\sqcap}}_{\alpha \in J}(\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha) \tilde{\tilde{\sqsubseteq}} \partial((\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha) \tilde{\tilde{=}} (\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha$ for all $\alpha \in J$. Hence, $\partial(\tilde{\tilde{\sqcap}}_{\alpha \in J}(\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha) \tilde{\tilde{\sqsubseteq}} (\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha$. Thus, $\tilde{\tilde{\sqcap}}_{\alpha \in J}(\tilde{\tilde{\mathcal{F}}}_Q)'_\alpha$ is an ∂-closed $\mathfrak{BS}$-set. Hence, $\tilde{\tilde{\sqcup}}(\tilde{\tilde{\mathcal{F}}}_Q)_\alpha$ is an ∂-open $\mathfrak{BS}$-set.

□

Definition 3.6. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$ and τ_∂ be the induced binary soft topology of $((K_1, K_2, \partial, Q)$. Then, $(K_1, K_2, \tau_\partial, Q)$ is called the induced binary soft topological space.*

Proposition 3.2. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$. Then,*

(1) *The union of any two ∂-closed $\mathfrak{BS}$-sets is an ∂-closed $\mathfrak{BS}$-set.*
(2) *The intersection of any family of ∂-closed $\mathfrak{BS}$-sets is an ∂-closed $\mathfrak{BS}$-set.*

Proof. Obvious. □

Definition 3.7. *Let $SS(V, R)$ and $SS(K_1, K_2, Q)$ be the families of all soft sets over V and $\mathfrak{BS}$-sets over K_1 and K_2 respectively. Let $f : V \longrightarrow K_1 \times K_2$, and $p : R \longrightarrow Q$ are* *mappings, such that if $A \subset V$, then $f(A) = (C, D) \in P(K_1) \times P(K_2)$, where $C = \{x : (x, y) = f(a)$ for some $a \in A\}$ and $D = \{y : (x, y) = f(a)$ for some $a \in A\}$. Then, a binary soft mapping $f_p : SS(V, R) \longrightarrow SS(K_1, K_2, Q)$ is defined as:*

(1) *Let $\mathcal{F}_R \in SS(V, R)$, then $f_p(\mathcal{F}_R)$ is a $\mathfrak{BS}$-set over K_1 and K_2 given by*

$$f_p(\mathcal{F}_R)(\omega) = f(\cup_{r \in p^{-1}(\omega)} \mathcal{F}(r)),\ \text{for all}\ \omega \in Q$$

(2) *Let $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$. Then, ${f_p}^{-1}(\tilde{\tilde{\mathcal{F}}}_Q)$ is a soft set over V given by*

$${f_p}^{-1}(\tilde{\tilde{\mathcal{F}}}_Q)(r) = f^{-1}(\tilde{\tilde{\mathcal{F}}}(p(r))),\ \text{for}\ r \in R$$

Example 3.3. *Consider the following sets: $V = \{v_1, v_2\}, K_1 = \{a_1, a_2, a_3\}, K_2 = \{d_1, d_2\}$, $R = \{r_1, r_2\}, Q = \{\omega_1, \omega_2\}$ and $SS(V, R)$, $SS(K_1, K_2, Q)$ are the classes of all soft sets over V and $\mathfrak{BS}$-set over K_1 and K_2 respectively. Define $f : V \longrightarrow K_1 \times K_2$ and $p : R \longrightarrow Q$ as: $f(v_1) = (a_1, d_1), f(v_2) = (a_3, d_2), p(r_1) = \omega_1, p(r_2) = \omega_2$. Choose the soft set $\mathcal{F}_R \in SS(V, R)$ and $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ as:*
$\mathcal{F}_R = \{(r_1, \{v_1\}), (r_2, \{v_1, v_2\})\}, \tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (\{a_1, a_2\}, \{d_2\})), (\omega_2, (\{a_3\}, \{d_1\}))\}$. Therefore, the binary soft mapping $f_p : SS(V, R) \longrightarrow SS(K_1, K_2, Q)$ is defined as:
$f_p(\mathcal{F}_R)(\omega_1) = f(\cup \mathcal{F}(r_1)) = f(\{v_1\}) = (\{a_1\}, \{d_1\})$, $f_p(\mathcal{F}_R)(\omega_2) = f(\cup \mathcal{F}(r_2)) = f(\{v_1, v_2\})$ $= (\{a_1, a_3\}, \{d_1, d_2\})$. Hence, $f_p(\mathcal{F}_R) = \{(\omega_1, (\{a_1\}, \{d_1\})), (\omega_2, (\{a_1, a_3\}, \{d_1, d_2\}))\}$ is a $\mathfrak{BS}$-sets over K_1 and K_2. Moreover, for the inverse image of $\tilde{\tilde{\mathcal{F}}}_Q$ is given as:
${f_p}^{-1}(\tilde{\tilde{\mathcal{F}}}_Q)(r_1) = f^{-1}(\tilde{\tilde{\mathcal{F}}}(p(r_1))) = f^{-1}(\tilde{\tilde{\mathcal{F}}}(\omega_1)) = f^{-1}((\{a_1\}, \{d_1\})) = \{v_1\}$,
${f_p}^{-1}(\tilde{\tilde{\mathcal{F}}}_Q)(r_2) = f^{-1}(\tilde{\tilde{\mathcal{F}}}(p(r_2))) = f^{-1}(\tilde{\tilde{\mathcal{F}}}(\omega_2)) = f^{-1}((\{a_3\}, \{d_1\})) = \emptyset$,
Thus, ${f_p}^{-1}(\tilde{\tilde{\mathcal{F}}}_Q) = \{(r_1, \{v_1\})\}$.

4. Relationships between Čech soft closure spaces and binary Čech soft closure spaces

In this section, we study the relationships between Čech soft closure spaces and binary Čech soft closure spaces, and define the notion of dense binary soft sets in binary Čech soft closure spaces. First, we need to give a notation which will be used to express that every $\mathfrak{BS}$-sets over K_1 and K_2 can be written as two soft sets over K_1 and K_2 respectively.

Notation 4.1. *Let $\tilde{\tilde{\mathcal{F}}}_Q$ be a $\mathfrak{BS}$-set over K_1, K_2. That means $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (N, M)) : \omega \in Q, N \in P(K_1), M \in P(K_2)\}$. Then, $\tilde{\tilde{\mathcal{F}}}_Q$ reduce two soft sets denoted as $\mathcal{F}^1_Q$ and $\mathcal{F}^2_Q$ where $\mathcal{F}^1_Q \in SS(K_1, Q)$ and $\mathcal{F}^2_Q \in SS(K_2, Q)$ defined as: $\mathcal{F}^1_Q = \{(\omega, N) : \omega \in Q, N \in P(K_1)\}$ and $\mathcal{F}^2_Q = \{(\omega, M) : \omega \in Q, M \in P(K_2)\}$. Thus, every $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{F}}}_Q$ can be written as the following: $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (F^1(\omega), F^2(\omega))) : \omega \in Q, F^1(\omega) \in P(K_1), F^2(\omega) \in P(K_2)\}$.*

The following example illustrates Notation 4.1.

Example 4.1. *In Example 3.3, choose $\tilde{\tilde{\mathcal{F}}}_Q$ the $\mathfrak{BS}$-set over K_1, K_2 defined as: $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (\{a_1, a_2\}, \{d_2\})), (\omega_2, (\{a_3\}, \{d_1\}))\}$. Then, $\tilde{\tilde{\mathcal{F}}}_Q$ can be written as:*
$\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega_1, (\mathcal{F}^1(\omega_1) = \{a_1, a_2\}, \mathcal{F}^2(\omega_1) = \{d_2\})), (\omega_2, (\mathcal{F}^1(\omega_2) = \{a_3\}, \mathcal{F}^2(\omega_2) = \{d_1\}))\}$,
Thus, the two soft sets reduce from $\tilde{\tilde{\mathcal{F}}}_Q$ are: $\mathcal{F}^1_Q = \{(\omega_1, \mathcal{F}^1(\omega_1) = \{a_1, a_2\}), (\omega_2, \mathcal{F}^1(\omega_2) = \{a_3\})\}$, and $\mathcal{F}^2_Q = \{(\omega_1, \mathcal{F}^2(\omega_1) = \{d_2\}), (\omega_2, \mathcal{F}^2(\omega_2) = \{d_1\})\}$.

In the next, we introduce theorem to show that from each $\mathcal{B\check{C}SCS}$ we can obtain two Čech soft closure spaces.

Theorem 4.1. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$, let $\partial_{K_1} : SS(K_1, Q) \longrightarrow SS(K_1, Q)$ given by $\partial_{K_1}(\mathcal{F}_Q) = \mathcal{G}_Q$ for all $\mathcal{F}_Q \in SS(K_1, Q)$; $\mathcal{F}_Q = \{(\omega, \mathcal{F}(\omega)) : \omega \in Q, \mathcal{F}(\omega) \subseteq K_1\}$ and $\mathcal{G}_Q = \{(\omega, \mathcal{G}(\omega)) : \omega \in Q, \mathcal{G}(\omega) \subseteq K_1\}$ where $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{\mathcal{G}}}_Q$ such that $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}(\omega), \emptyset)) : \omega \in Q, \mathcal{F}(\omega) \subseteq K_1\}$ and $\tilde{\tilde{\mathcal{G}}}_Q = \{(\omega, (\mathcal{G}(\omega), M)) : \omega \in Q, \mathcal{G}(\omega) \subseteq K_1, M \subseteq K_2\}$. Then, the mapping ∂_{K_1} is a Čech soft closure operator over K_1. Similarly, let $\partial_{K_2} : SS(K_2, Q) \longrightarrow SS(K_2, Q)$ given by $\partial_{K_2}(H_Q) = U_Q$ for all $H_Q \in SS(K_2, Q)$; $H_Q = \{(\omega, H(\omega)) : \omega \in Q, H(\omega) \subseteq K_2\}$ and $U_Q = \{(\omega, U(\omega)) : \omega \in Q, U(\omega) \subseteq K_2\}$ where $\partial(\tilde{\tilde{H}}_Q) = \tilde{\tilde{U}}_Q$ such that $\tilde{\tilde{H}}_Q = \{(\omega, (\emptyset, H(\omega))) : \omega \in Q, H(\omega) \subseteq K_2\}$ and $\tilde{\tilde{U}}_Q = \{(\omega, (N, U(\omega))) : \omega \in Q, N \subseteq K_1, U(\omega) \subseteq K_2\}$. Then, the mapping ∂_{K_2} is a Čech soft closure operator over K_2.*

Proof. (1) Since ∂ is a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$, then $\partial(\tilde{\tilde{\emptyset}}) = \emptyset$. This implies $\partial_{K_1}(\tilde{\tilde{\emptyset}}_Q) = \tilde{\tilde{\emptyset}}_Q$ and $\partial_{K_2}(\tilde{\tilde{\emptyset}}_Q) = \tilde{\tilde{\emptyset}}_Q$.

(2) For $\mathcal{F}_Q \in SS(K_1, Q)$ and $H_Q \in SS(K_2, Q)$. We must prove $\mathcal{F}_Q \sqsubseteq \partial_{K_1}(\mathcal{F}_Q)$ and $H_Q \sqsubseteq \partial_{K_2}(H_Q)$. Let $\mathcal{F}_Q = \{(\omega, \mathcal{F}(\omega)) : \omega \in Q, \mathcal{F}(\omega) \subseteq K_1\}$. Then, $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}(\omega), \emptyset)) : \omega \in Q, \mathcal{F}(\omega) \subseteq K_1\} \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q)$. Hence, $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q)$. Now, since $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \{(\omega, (\partial_{K_1}(\mathcal{F}_Q), M)) : \omega \in Q, \text{ for some } M \subseteq K_2\}$. That means $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q) = \{(\omega, (\partial_{K_1}(\mathcal{F}_Q), M)) : \omega \in Q, \text{ for some } M \subseteq K_2\}$. It follows, $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \{(\omega, (\partial_{K_1}(\mathcal{F}_Q), M)) : \omega \in Q, \text{ for some } M \subseteq K_2\}$. Which gives $\mathcal{F}_Q \sqsubseteq \partial_{K_1}(\mathcal{F}_Q)$ for all $\mathcal{F}_Q \in SS(K_1, Q)$. Similarly, $\tilde{\tilde{H}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{H}}_Q), where \tilde{\tilde{H}}_Q = \{(\omega, (\emptyset, H(\omega))) : \omega \in Q, H(\omega) \subseteq K_2\}$ which gives $H_Q \sqsubseteq \partial_{K_2}(H_Q)$.

(3) Let $\mathcal{G}_Q, \mathcal{F}_Q \in SS(K_1, Q)$ and $H_Q, U_Q \in SS(K_2, Q)$ such that $\mathcal{F}_Q \sqsubseteq \mathcal{G}_Q$ and $H_Q \sqsubseteq U_Q$. Then, $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{\mathcal{G}}}_Q$ and $\tilde{\tilde{H}}_Q \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{U}}_Q$, where $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}(\omega), \emptyset)) : \omega \in Q, \mathcal{F}(\omega) \subseteq K_1\}$, $\tilde{\tilde{\mathcal{G}}}_Q = \{(\omega, (\mathcal{G}(\omega), M)) : \omega \in Q, \mathcal{G}(\omega) \subseteq K_1, M \subseteq K_2\}$, $\tilde{\tilde{H}}_Q = \{(\omega, (\emptyset, H(\omega))) : \omega \in Q, H(\omega) \subseteq K_2\}$ and $\tilde{\tilde{U}}_Q = \{(\omega, (N, U(\omega))) : \omega \in Q, N \subseteq K_1, U(\omega) \subseteq K_2\}$. Therefore, $\partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)$ and $\partial(\tilde{\tilde{H}}_Q) \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{U}}_Q)$ implies $\partial_{K_1}(\mathcal{F}_Q) \sqsubseteq \partial_{K_1}(\mathcal{G}_Q)$ and $\partial_{K_2}(H_Q) \sqsubseteq \partial_{K_2}(U_Q)$. Now, let $\mathcal{G}_Q, F_Q \in SS(K_1, Q)$ and $H_Q, U_Q \in SS(K_2, Q)$. Since $\partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{\mathcal{G}}}_Q) = \partial(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q)$, then $\partial_{K_1}(\mathcal{F}_Q) \sqcup \partial_{K_1}(\mathcal{G}_Q) = \partial_{K_1}(\mathcal{F}_Q \sqcup \mathcal{G}_Q)$. Also, since $\partial(\tilde{\tilde{H}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{U}}_Q) = \partial(\tilde{\tilde{H}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{U}}_Q)$, then $\partial_{K_2}(H_Q) \sqcup \partial_{K_2}(U_Q) = \partial_{K_2}(H_Q \sqcup U_Q)$. Hence, $\partial_{K_1}, \partial_{K_2}$ are Čech soft closure operators.

□

In the next proposition, we show that from any two Čech soft closure spaces we can obtain a $\mathcal{B}\check{\mathcal{C}}\mathcal{SCS}$.

Proposition 4.1. *If (K_1, ∂_1, Q) and (K_2, ∂_2, Q) are two Čech soft closure spaces, then $(K_1, K_2, \partial_{K_1K_2}, Q)$ where $\partial_{K_1K_2} : SS(K_1, K_2, Q) \longrightarrow SS(K_1, K_2, Q)$ is given by $\partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q)$ $= \{(\omega, (\partial_1(\mathcal{F}^1_Q)(\omega), \partial_2(\mathcal{F}^2_Q)(\omega))) : \omega \in Q, \partial_1(\mathcal{F}^1_Q)(\omega) \subseteq K_1, \partial_2(\mathcal{F}^2_Q)(\omega) \subseteq K_2\}$. Where $\mathcal{F}^1_Q$ and $\mathcal{F}^2_Q$ are soft sets which reduced from the $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{F}}}_Q$ (as we explain in Notation 4.1).*

Proof. (1) $\partial_{K_1K_2}(\tilde{\tilde{\emptyset}}) = \{(\omega, (\partial_1(\tilde{\emptyset}_Q)(\omega), \partial_2(\tilde{\emptyset}_Q)(\omega))) : \omega \in Q\} = \{(\omega, (\emptyset, \emptyset)) : \omega \in Q\} = \tilde{\tilde{\emptyset}}$ since ∂_1 and ∂_2 are Čech soft closure operators.

(2) Let $\mathcal{F}_Q \in SS(K_1, Q)$ and $\mathcal{G}_Q \in SS(K_2, Q)$. Then, $\mathcal{F}_Q \sqsubseteq \partial_1(\mathcal{F}_Q)$ and $\mathcal{G}_Q \sqsubseteq \partial_2(\mathcal{G}_Q)$. This implies $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}(\omega), \mathcal{G}(\omega))) : \omega \in Q\} \sqsubseteq \{(\omega, (\partial_1(\mathcal{F}_Q)(\omega), \partial_2(\mathcal{G}_Q)(\omega))) : \omega \in Q\} = \partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q)$.

(3) Let $\tilde{\tilde{\mathcal{F}}}_Q, \tilde{\tilde{\mathcal{G}}}_Q \in SS(K_1, K_2, Q)$. Then, from Notation 4.1, $\mathcal{F}^1_Q, G^1_Q \in SS(K_1, Q)$ and $\mathcal{F}^2_Q, G^2_Q \in SS(K_2, Q)$. So, $\partial_1(\mathcal{F}^1_Q) \sqcup \partial_1(\mathcal{G}^1_Q) = \partial_1(\mathcal{F}^1_Q \sqcup \mathcal{G}^1_Q)$ and $\partial_2(\mathcal{F}^2_Q) \sqcup \partial_2(\mathcal{G}^2_Q) = \partial_2(\mathcal{F}^2_Q \sqcup \mathcal{G}^2_Q)$. Now,

$$\begin{aligned}
\partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q)\tilde{\tilde{\sqcup}}\partial_{K_1K_2}(\tilde{\tilde{\mathcal{G}}}_Q) &= \{(\omega, (\partial_1(\mathcal{F}^1_Q)(\omega), \partial_2(\mathcal{F}^2_Q)(\omega))) : \omega \in Q\}\tilde{\tilde{\sqcup}} \\
&\quad \{(\omega, (\partial_1(\mathcal{G}^1_Q)(\omega), \partial_2(\mathcal{G}^2_Q)(\omega))) : \omega \in Q\} \\
&= \{(\omega, (\partial_1(\mathcal{F}^1_Q)(\omega) \cup \partial_1(\mathcal{G}^1_Q)(\omega), \partial_2(\mathcal{F}^2_Q)(\omega) \cup \partial_2(\mathcal{G}^2_Q)(\omega))) : \\
&\quad \omega \in Q\} \\
&= \{(\omega, (\partial_1(\mathcal{F}^1_Q \sqcup \mathcal{G}^1_Q)(\omega), \partial_2(\mathcal{F}^2_Q \sqcup \mathcal{G}^2_Q)(\omega))) : \omega \in Q\} \\
&= \partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q\tilde{\tilde{\sqcup}}\tilde{\tilde{\mathcal{G}}}_Q).
\end{aligned}$$

Thus, $\partial_{K_1K_2}$ is a binary Čech soft closure operator.

□

Lemma 4.1. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B}\check{\mathcal{C}}\mathcal{S}\mathcal{C}\mathcal{S}$. Then, $\{(\omega, (\partial_{K_1}(\mathcal{F}^1)(\omega), \partial_{K_2}(\mathcal{F}^2_Q)(\omega))) : \omega \in Q\}\tilde{\tilde{\sqsubseteq}}\partial(\tilde{\tilde{\mathcal{F}}}_Q)$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ and $\mathcal{F}^1_Q, \mathcal{F}^2_Q$ are the associated soft sets of $\tilde{\tilde{\mathcal{F}}}_Q$.*

Proof. Let $\tilde{\tilde{\mathcal{F}}}_Q$ be a $\mathfrak{B}\mathfrak{S}$-set. From Notation 4.1, $\tilde{\tilde{\mathcal{F}}}_Q$ can be represented as the form $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}^1(\omega), \mathcal{F}^2(\omega))) : \omega \in Q, \mathcal{F}^1(\omega) \in P(K_1), \mathcal{F}^2(\omega) \in P(K_2)\}$. Then, $\partial_{K_1} : SS(K_1, Q) \longrightarrow SS(K_1, Q)$ and $\partial_{K_2} : SS(K_2, Q) \longrightarrow SS(K_2, Q)$. Now,

$$\begin{aligned}
\{(\omega, (\partial_{K_1}(\mathcal{F}^1_Q)(\omega), \partial_{K_2}(\mathcal{F}^2_Q)(\omega))) : \omega \in Q\} &= \{(\omega, \partial_{K_1}(\mathcal{F}^1_Q)(\omega), \emptyset)) : \omega \in Q\} \cup \\
&\quad \{(\omega, (\emptyset, \partial_{K_2}(\mathcal{F}^2_Q)(\omega))) : \omega \in Q\} \\
&\subseteq \{(\omega, (\partial_{K_1}(\mathcal{F}^1_Q)(\omega), M)) : \omega \in Q, M \subseteq K_2\} \\
&\quad \cup\{(\omega, (N, \partial_{K_2}(\mathcal{F}^2_Q)(\omega))) : \omega \in Q, N \subseteq K_1\} \\
&= \partial(\{(\omega, (\mathcal{F}^1(\omega), \emptyset)) : \omega \in Q\}) \cup \\
&\quad \partial(\{(\omega, (\emptyset, \mathcal{F}^2(\omega))) : \omega \in Q\}) \\
&= \partial(\tilde{\tilde{\mathcal{F}}}_Q).
\end{aligned}$$

□

Remark 4.1. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B}\check{\mathcal{C}}\mathcal{S}\mathcal{C}\mathcal{S}$. Then ∂ is coarser than $\partial_{K_1K_2}$.*

Proof. We must show for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ we have $\partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q)\tilde{\tilde{\sqsubseteq}}\partial(\tilde{\tilde{\mathcal{F}}}_Q)$. From Notation 4.1, any $\mathfrak{B}\mathfrak{S}$-set $\tilde{\tilde{\mathcal{F}}}_Q$ can be represented as
$\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, \tilde{\tilde{\mathcal{F}}}(\omega) = (F^1(\omega), F^2(\omega))) : \omega \in Q, F^1(\omega) \in P(K_1), F^2(\omega) \in P(K_2)\}$.
$\partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q) = \{(\omega, (\partial_{K_1}(F^1_Q)(\omega), \partial_{K_2}(F^2_Q)(\omega))) : \omega \in Q\}\tilde{\tilde{\sqsubseteq}}\partial(\tilde{\tilde{\mathcal{F}}}_Q)$ by Lemma 4.1. Hence, the result. □

Proposition 4.2. *Let (K_1, ∂_1, Q) and (K_2, ∂_2, Q) be two Čech soft closure spaces and $\partial_{K_1K_2}$ as in Proposition 4.1. If $\mathcal{F}_Q \in SS(K_1, Q)$ is ∂_1-closed soft set and $\mathcal{G}_Q \in SS(K_2, Q)$ is ∂_2-closed soft set, then $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}(\omega), \mathcal{G}(\omega))) : \omega \in Q\}$ is $\partial_{K_1K_2}$-closed $\mathfrak{B}\mathfrak{S}$-set.*

Proof. Let $\mathcal{F}_Q$ be a ∂_1-closed soft set. Then, $\partial_1(\mathcal{F}_Q) = \mathcal{F}_Q$ and let $\mathcal{G}_Q$ be a ∂_2-closed soft set. Then, $\partial_2(\mathcal{G}_Q) = \mathcal{G}_Q$. Since $\partial_{K_1K_2}(\tilde{\tilde{\mathcal{F}}}_Q) = \{(\omega, \partial_1(\mathcal{F}_Q)(\omega), \partial_2(\mathcal{G}_Q)(\omega)) : \omega \in Q\} = \{(\omega, (\mathcal{F}(\omega), \mathcal{G}(\omega))) : \omega \in Q\} = \tilde{\tilde{\mathcal{F}}}_Q$. Therefore, $\tilde{\tilde{\mathcal{F}}}_Q$ is $\partial_{K_1K_2}$-closed $\mathfrak{BS}$-set. □

Definition 4.1. *Let (V, C, R) be a soft closure space and (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$. Then, the binary soft mapping $f_p : (V, C, R) \longrightarrow (K_1, K_2, \partial, Q)$ is called a $C - \partial$ binary soft morphism if $f_p(C(\mathcal{F}_Q))\tilde{\tilde{\sqsubseteq}}\partial(f_p(\mathcal{F}_Q))$ for all $\mathcal{F}_Q \in SS(V, R)$.*

Definition 4.2. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$. A $\mathfrak{BS}$-set $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$ is said to be ∂-dense $\mathfrak{BS}$-set, if $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{Q}}$.*

Proposition 4.3. *Let (K_1, K_2, ∂, Q) be a $\mathcal{B\check{C}SCS}$, $\mathcal{F}_Q \in SS(K_1, Q)$ is ∂_{K_1}-dense $\mathfrak{BS}$-set and $\mathcal{G}_Q \in SS(K_2, Q)$ is ∂_{K_2}-dense $\mathfrak{BS}$-set. Then, $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, \mathcal{F}(\omega), \mathcal{G}(\omega)) : \omega \in Q\}$ is ∂-dense $\mathfrak{BS}$-set.*

Proof. Since $\mathcal{F}_Q$ is ∂_{K_1}-dense $\mathfrak{BS}$-set, this implies $\partial_{K_1}(\mathcal{F}_Q) = \tilde{K}_1 = \{(\omega, K_1) : \omega \in Q\}$, and since $\mathcal{G}_Q$ is ∂_{K_2}-dense $\mathfrak{BS}$-set, this implies $\partial_{K_2}(\mathcal{G}_Q) = \tilde{K}_2 = \{(\omega, K_2) : \omega \in Q\}$. Now, $\tilde{\tilde{Q}} = \{(\omega, K_1, K_2) : \omega \in Q\} = \{(\omega, (\partial_{K_1}(\mathcal{F}_Q)(\omega), \partial_{K_2}(\mathcal{G}_Q)(\omega))) : \omega \in Q\} \tilde{\tilde{\sqsubseteq}}\partial(\tilde{\tilde{\mathcal{F}}}_Q)$. Therefore, $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \tilde{\tilde{Q}}$. Thus, $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂-dense $\mathfrak{BS}$-set. □

Corollary 4.1. *Let (K_1, ∂_1, Q) and (K_2, ∂_2, Q) be two Čech soft closure spaces. Then, $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂-dense if and only if $\mathcal{F}^1_Q$ is ∂_1-dense and $\mathcal{F}^2_Q$ is ∂_2-dense.*

Proof. Let $\tilde{\tilde{\mathcal{F}}}_Q$ be a $\mathfrak{BS}$-set over K_1 and K_2. Then, $\tilde{\tilde{\mathcal{F}}}_Q = \{(\omega, (\mathcal{F}^1(\omega), \mathcal{F}^2(\omega))) : \omega \in Q\}$. Suppose that $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂-dense. This implies

$$\begin{aligned}
\partial(\tilde{\tilde{\mathcal{F}}}_Q) \quad &= \quad \tilde{\tilde{Q}} \\
&\iff \{(\omega, (\mathcal{F}^1(\omega), \mathcal{F}^2(\omega))) : \omega \in Q\} = \{(\omega, (K_1, K_2)) : \omega \in Q\} \\
&\iff \{(\omega, (\partial_1(\mathcal{F}^1_Q)(\omega), \partial_2(\mathcal{F}^2_Q)(\omega))) : \omega \in Q\} = \{(\omega, (K_1, K_2)) : \omega \in Q\} \\
&\iff \partial_1(\mathcal{F}^1_Q) = \tilde{K}_1 \text{ and } \partial_2(\mathcal{F}^2_Q) = \tilde{K}_2 \\
&\iff \mathcal{F}^1_Q \text{ is } \partial_1 - dense \text{ and } \mathcal{F}^2_Q \text{ is } \partial_2 - dense.
\end{aligned}$$

□

5. Operation on binary Čech soft closure operators

Definition 5.1. *Let ∂_1 and ∂_2 be two binaries Čech soft closure operators over K_1 and K_2. Then, $(\partial_1 \cup \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q) = \partial_1(\tilde{\tilde{\mathcal{F}}}_Q)\tilde{\tilde{\sqcup}}\partial_2(\tilde{\tilde{\mathcal{F}}}_Q)$ and $(\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q) = \partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q))$.*

Proposition 5.1. *If ∂_1 and ∂_2 binaries Čech soft closure operators over K_1 and K_2 and $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$. Then, $(\partial_1 \cup \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q)\tilde{\tilde{\sqsubseteq}}(\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q)$.*

Proof. Let $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$. Since $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)$, then $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_1(\partial_2((\tilde{\tilde{\mathcal{F}}}_Q))$. Also, $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q))$. Hence,

$$\begin{aligned}
(\partial_1 \cup \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q) &= \partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q) \\
&\tilde{\tilde{\sqsubseteq}} \partial_1(\partial_2((\tilde{\tilde{\mathcal{F}}}_Q)) \tilde{\tilde{\sqcup}} \partial_1(\partial_2((\tilde{\tilde{\mathcal{F}}}_Q)) \\
&\tilde{\tilde{\sqsubseteq}} \partial_1(\partial_2((\tilde{\tilde{\mathcal{F}}}_Q)) \\
&= (\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q).
\end{aligned}$$

□

Proposition 5.2. *Let $(K_1, K_2, \partial_1, Q)$ and $(K_1, K_2, \partial_2, Q)$ be two $\mathcal{B}\check{\mathcal{C}}\mathcal{S}\mathcal{C}\mathcal{S}$'s. Then, $\tau_{\partial_1 \circ \partial_2} = \tau_{\partial_1 \circ \partial_2} = \tau_{\partial_1} \cap \tau_{\partial_2} = \tau_{\partial_1 \cup \partial_2}$.*

Proof. First, we prove $\tau_{\partial_1 \circ \partial_2} = \tau_{\partial_1} \cap \tau_{\partial_2}$. Let $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1 \circ \partial_2}$. Then, $(\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ that means $\partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$. Since $\tilde{\tilde{\mathcal{F}}}_Q^{'} \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})$, then we get $\partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})$. By the second condition of the property of ∂_1, we have $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) \tilde{\tilde{\sqsubseteq}} \partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}))$ which implies $\partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})$. That means $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})$ is ∂_1-closed $\mathfrak{B}\mathfrak{S}$-set. Now, since $\partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})$ and since $\partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$, then we have $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$. That means $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂_2-open $\mathfrak{B}\mathfrak{S}$-set. Therefore, $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_2}$. Now, $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = (\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$. Hence, $\tilde{\tilde{\mathcal{F}}}_Q$ is ∂_1-open $\mathfrak{B}\mathfrak{S}$-set. It follows, $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1}$. This yields, $\tau_{\partial_1 \circ \partial_2} \subseteq \tau_{\partial_1} \cap \tau_{\partial_2}$.

Conversely, let $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1} \cap \tau_{\partial_2}$. Then, $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1}$ and $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_2}$ which means $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ and $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ respectively. On the other hand, $(\partial_1 \circ \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \partial_1(\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = \partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'} = \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \partial_2(\partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'})) = (\partial_2 \circ \partial_1)(\tilde{\tilde{\mathcal{F}}}_Q^{'})$. It follows, $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1 \circ \partial_2}$ and $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_2 \circ \partial_1}$. Hence, $\tau_{\partial_1 \circ \partial_2} = \tau_{\partial_1} \cap \tau_{\partial_2}$ and $\tau_{\partial_2 \circ \partial_1} = \tau_{\partial_1} \cap \tau_{\partial_2}$. Now, to prove $\tau_{\partial_1} \cap \tau_{\partial_2} = \tau_{\partial_1 \cup \partial_2}$. Suppose $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1 \cup \partial_2}$. Then, $(\partial_1 \cup \partial_2)(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ this implies $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'}) \tilde{\tilde{\sqcup}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ if and only if $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ and $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q^{'}) = \tilde{\tilde{\mathcal{F}}}_Q^{'}$ that means $\tilde{\tilde{\mathcal{F}}}_Q \in \tau_{\partial_1} \cap \tau_{\partial_2}$. □

Remark 5.1. *Let ∂_1 and ∂_2 be two binary Čech soft closure operators. Then, $\partial_1 \cap \partial_2$ need not to be binary Čech soft closure operator.*

Proof. Let $\partial = \partial_1 \cap \partial_2$, i.e., $\partial(\tilde{\tilde{\mathcal{F}}}_Q) = \partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)$ for all $\tilde{\tilde{\mathcal{F}}}_Q \in SS(K_1, K_2, Q)$.

(1) $\partial(\tilde{\tilde{\emptyset}}) = \tilde{\tilde{\emptyset}}$ since $\partial_1(\tilde{\tilde{\emptyset}}) = \tilde{\tilde{\emptyset}}$ and $\partial_2(\tilde{\tilde{\emptyset}}) = \tilde{\tilde{\emptyset}}$.

(2) Since $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial_1(\tilde{\tilde{\mathcal{F}}}_Q) and \tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)$, then $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q)$.

(3) Let $\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqsubseteq}} \tilde{\tilde{\mathcal{G}}}_Q$. Then, $\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_1(\tilde{\tilde{\mathcal{G}}}_Q)$ and $\partial_2(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)$. It follows,
$\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial_1(\tilde{\tilde{\mathcal{G}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)$ implies $\partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqsubseteq}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)$. Thus, ∂ is a binary soft closure operator.
But,

(4)

$$\begin{aligned}
\partial(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q) &= \partial_1(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q) \\
&= \{\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial_1(\tilde{\tilde{\mathcal{G}}}_Q)\} \tilde{\tilde{\sqcap}} \{\partial_2(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)\} \\
&= \{\{\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial_1(\tilde{\tilde{\mathcal{G}}}_Q)\} \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)\} \tilde{\tilde{\sqcup}} \{\{\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial_1(\tilde{\tilde{\mathcal{G}}}_Q)\} \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)\} \\
&= \{\{\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)\} \tilde{\tilde{\sqcup}} \{\partial_1(\tilde{\tilde{\mathcal{G}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{F}}}_Q)\}\} \tilde{\tilde{\sqcup}} \\
&\quad \{\{\partial_1(\tilde{\tilde{\mathcal{G}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)\} \tilde{\tilde{\sqcup}} \{\partial_1(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcap}} \partial_2(\tilde{\tilde{\mathcal{G}}}_Q)\}\} \\
&\tilde{\tilde{\sqsupseteq}} \partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)
\end{aligned}$$

Hence, $\partial(\tilde{\tilde{\mathcal{F}}}_Q \tilde{\tilde{\sqcup}} \tilde{\tilde{\mathcal{G}}}_Q)$ need not be equal $\partial(\tilde{\tilde{\mathcal{F}}}_Q) \tilde{\tilde{\sqcup}} \partial(\tilde{\tilde{\mathcal{G}}}_Q)$.

□

6. Conclusions

This study introduces and investigates the concept of binary Čech soft closure space, which is defined over two initial universe sets with fixed parameter sets. Čech soft closure space is extended and generalized in this space. Closed (open) binary soft sets, binary soft interior, and dense binary soft sets are defined and studied as one of the most basic concepts in this space. Relationships between binary Čech soft closure space and Čech soft closure space are deduced. Examples and counterexamples are presented to illustrate some of our results. Some operations on binary Čech soft closure operators are defined.

Acknowledgement. The author would like to thank the referee(s) for their helpful suggestions, which improved this paper.

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