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SOLVABILITY OF A RESONANT FRACTIONAL-ORDER p-LAPLACIAN BOUNDARY VALUE PROBLEM WITH TWO-DIMENSIONAL KERNEL

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Abstract. The goal of this study is to establish the existence of solutions for a fractional-order p-Laplacian boundary value problem with a two-dimensional kernel at resonance case. The non-linearity of this problem forced us to transform it into a semilinear system to use the so called Mawhin's coincidence degree theory. In addition, an example is included to demonstrate the main result.

Keywords: Fractional-order p-Laplacian, Resonance case, Semilinear system, Mawhin's coincidence degree theory.

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1. Introduction

Most natural phenomena have recently been described by some type of boundary value problems (BVPs for short) for differential equations. Examples include the study of physical phenomena, chemistry, engineering, and control of dynamical systems, etc. See ([2], ([10], [15], [19], ([21]). In ([4], [6], [9]), [12]), [16]), [17]), [18]), the authors have investigated resonant problems with linear differential operators and one-dimensional kernels. As we can see the situation becomes more problematic when dealing with non-linear two-dimensional operators like the case of p-Laplace boundary value problems. See ([3], [7], [8], [13]) for more details. Motivated by the works mentioned above, only a few authors have looked into this case to get some existence results based on the assumption that certain algebraic expression is not equal to zero see ([8]). For example, where they assume that

$$C = \begin{vmatrix} Q_1 e^{-t} & Q_2 e^{-t} \\ Q_1 t e^{-t} & Q_2 t e^{-t} \end{vmatrix} \neq 0.$$

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In this work, we shall study the existence of solutions for a kind of problems which is driven by the works stated above, in particular ([8]). When suitable growth conditions are imposed on the nonlinear term, many new difficulties arise, such as the construction of the projector Q, whose formula is quite different from the classical one.
The main work in this paper is concerned with the investigation of the existence of solutions for the following fractional-order p-Laplacian BVP at resonance with integral boundary conditions

$$\begin{cases} (\phi_p(D^{\beta}_{0_+}u(t)))' + g(t)f(t,u(t),D^{\beta}_{0+}u(t)) = 0, \quad t \in [0,T], 0 \leq \beta < 1, \\ \phi_p(D^{\beta}_{0+}u(0)) = \int_0^T g(t)\phi_p(D^{\beta}_{0+}u(t))dt, \\ \phi_p(D^{\beta}_{0+}u(T)) = \int_0^T g(t)\phi_p(D^{\beta}_{0+}u(t))dt, \end{cases} \tag{1.1}$$

where D^{β}_{0+} is the Riemann-Liouville fractional derivative of order β, $g \in L^1[0,T]$ with $g(t) > 0$ and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a g-Caratheodory function, that is, (i) for each $(x,y) \in \mathbb{R}^2$, the mapping $t \to f(t,x,y)$ is Lebesgue measurable, (ii) for a.e. $t \in [0,T]$, the mapping $(x,y) \to f(t,x,y)$ is continuous on $\mathbb{R}^2$ and (iii) for each $r > 0$, there exists $\omega_r(t) : [0,T] \to [0,+\infty)$ satisfying $\int_0^T g(t)\,|\omega_r(t)| < +\infty$ such that, for a.e. $t \in [0,T)$ and every $(x,y) \in [-r,r] \times [-r,r]$, we have

$$|f(t,x,y)| \leq \omega_r(t).$$

Recall also that $\phi_p : \mathbb{R} \to \mathbb{R}$ is an odd continuous, increasing operator and $\phi_p^{-1} = \phi_q\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. In ([5]), the authors investigated the following multi-point boundary value problem for a nonlinear fractional differential equation with a p-Laplacian operator

$$\begin{cases} \left(\phi_p\left(D^{\alpha}_{0^+}x(t)\right)\right)' = f\left(t, x(t), D^{\alpha-1}_{0^+}x(t)\right), \quad t \in [0,1], \\ x(0) = D^{\alpha}_{0^+}x(1) = 0, D^{\alpha-1}_{0^+}x(1) = \sum_{i=1}^{\alpha-m}\beta_i D^{\alpha-1}_{0^+}x\left(\eta_i\right), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$ is the p-Laplacian $(p > 1)$, $D^{\alpha}_{0^+}$ is the standard Riemann–Liouville derivative $(1 < \alpha < 2)$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$, and $\beta_i \in \mathbb{R}_+$, for $i = 1,2,3,\ldots,m-2$.
The problem (1.1) is said to be *at resonance* if $\int_0^T g(t)dt = 1$.
Moreover, because $(\phi_p(D^{\beta}_{0_+}u(t)))'$ is a nonlinear operator, the coincidence degree theory for linear differential operators with resonant boundary value conditions fails to apply to it directly. However, rewriting (1.1) as a semilinear system allows us to apply the continuation theorem to the problem (2.1) and obtain the existence of some solutions.
Our paper consists of four sections. In the first section, we present the general framework of our study. In the second, we recall most of the preliminary notions. In Section 3, we present three lemmas to prove the result existence by applying the so call *Mawhin coincidence degree theory* and an example is given to support our results in the fourth section .

2. Preliminaries about the coincidence degree theory

We begin this section by recalling some definitions concerned with the fractional calculus, and abstract results from the coincidence degree theory. For more details we refer to ([10]), [12]. Let X and Y be two Banach spaces.

Definition 2.1. *([11]) The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

where $\Gamma(\beta)$ represents the gamma function, provided that the right side is pointwisely defined on $(0, +\infty)$.

Definition 2.2. *([11]) The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$D_{0^+}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds,$$

where $n = [\beta] + 1$, provided that the right-hand side is defined pointwise on $(0, +\infty)$. Here $[\beta]$ denotes the integer part of the real number β.

Lemma 2.1. *([11]) Let $\beta > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\beta} u(t) = 0$$

has $u(t) = c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_n t^{\beta-n}, c_i \in \mathbb{R}, i = 1, 2, \ldots, n$ as unique solutions, where n is the smallest integer greater than or equal to β.

Lemma 2.2. *([11]) Given $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\beta > 0$. Then*

$$I_{0^+}^{\beta} D_{0^+}^{\beta} u(t) = u(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_n t^{\beta-n},$$

for some $c_i \in \mathbb{R}, i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to β.

Definition 2.3. *([14]) Let $L : \operatorname{dom} L \subset X \to Y$ be a linear operator. Then one says that L is a Fredholm operator provided that*
(i) $\ker L$ is finite dimensional space,
(ii) $\operatorname{Im} L$ is closed and has finite codimension.
The index of L is defined by: ind $L = \dim \ker L - \operatorname{codim} \operatorname{Im} L$.

It follows from definition (2.3) that if L is a Fredholm operator of index zero, then there exist two linear continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Furthermore, the restriction of L on $\operatorname{dom} L \cap \ker P$, $L_P : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$, is invertible. We will denote its inverse by K_P. The generalized inverse of L is denoted by $K_{P,Q} := K_P(I - Q)$.
On the other hand, for every isomorphism $J : \operatorname{Im} Q \to \ker L$, the mapping $JQ + K_{P,Q} : Y \to \operatorname{dom} L$ is an isomorphism. Now let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$.

Definition 2.4. *([14]) Let L be a Fredholm operator of index zero. The operator $N : X \to Y$ is said to be L-compact in Ω if*
(i) the map $QN : \bar{\Omega} \to Z$ is continuous and $QN(\bar{\Omega})$ is bounded in Y and
(ii) $K_{P,Q}N : \bar{\Omega} \to X$ is a compact operator.

Lemma 2.3. *([13]) We will use the following properties of ϕ_p. For $u, v \geq 0$, we have*
(i) $\phi_p(u + v) \leq \phi_p(u) + \phi_p(v)$, if $1 < p < 2$,
(ii) $\phi_p(u + v) \leq 2^{p-2} (\phi_p(u) + \phi_p(v))$, if $p \geq 2$.

Theorem 2.1. *([14]) Let X, Y be two real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be an L-compact mapping on Ω. Assume that the following conditions are satisfied:*
(1) $Lu \neq \rho N u$ for all $(u, \rho) \in [\operatorname{dom} L \setminus \ker L \cap \partial\Omega] \times (0; 1)$,
(2) $QNu \neq 0$ for all $u \in \ker L \cap \partial\Omega$,
(3) $\deg(QN_{|\ker L}, \Omega \cap \ker L, 0) \neq 0$.
Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

Now, we consider the following system :

$$\begin{cases} D^{\beta}_{0^+} u_1(t) = \phi_q\left(u_2(t)\right), \\ u_2'(t) = -g(t) f\left(t, u_1(t), \phi_q u_2(t)\right), \\ u_2(0) = u_2(T) = \int_0^T g(t) u_2(t) dt. \end{cases} \tag{2.1}$$

Where $0 \leq \beta < 1$ and we introduce the spaces

$$X_1 = C^{1-\beta}[0,T] = \left\{u \in C[0,T] \mid \text{ such that } t^{1-\beta} u(t) \in C[0,T]\right\},$$

with the norm

$$\|u\|_{X_1} = \|u\|_{C^{1-\beta}} = \max_{t\in[0,T]} \left| t^{1-\beta} u(t) \right|,$$

and

$$X_2 = C[0,T] = \{u \mid u(t) \text{ is continuous on the interval } [0,T]\},$$

equipped by the norm $\|u\|_{X_2} = \max_{t\in[0,T]} |u(t)|$. Taking the space

$$X = \left\{u = (u_1, u_2)^\top \mid u_1 \in C^{1-\beta}[0,T], u_2 \in C[0,T]\right\},$$

with the norm

$$\|u\|_X = \max\left\{\|u_1\|_{C^{1-\beta}}, \|u_2\|_C\right\},$$

and $Y_1 = C[0,T]$, $Y_2 = L^1[0,T]$ where $\|y\|_{Y_1} = \max_{t\in[0,T]} |y(t)|$, $\|y\|_{Y_2} = \int_0^T |y(t)| dt$. Define the space Y as follows:

$$Y = \left\{y = (y_1, y_2)^\top \mid y_1 \in C[0,T], y_2 \in L^1[0,T]\right\},$$

with the norm

$$\|y\|_Y = \max\left\{\|y_1\|_{C[0,T]}, \|y_2\|_{L^1[0,T]}\right\}.$$

Obviously, $(Y, ||.||_Y)$ is a Banach space and $||.||_{C^{1-\beta}}$ is a norme in $C^{1-\beta}$ because $||u||_{C^{1-\beta}} = ||u||_\infty$.
Now, let's prove that $(X, ||.||_X)$ is also a Banach space. Suppose (u_n) is Cauchy sequence in $C^{1-\beta}$, that's for ever $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m > n > n_0$, we have

$$||u_m - u_n||_{C^{1-\beta}} < \varepsilon,$$

which implies that

$$\sup_{t\in[0,T]} t^{1-\beta} \left|u_m(t) - u_n(t)\right|_{C^{1-\beta}} < \varepsilon,$$

then,

$$\sup_{t\in[0,T]} \left| t^{1-\beta} u_m(t) - t^{1-\beta} u_n(t) \right|_{C^{1-\beta}} < \varepsilon,$$

finally, we obtain

$$||(u_m)_{1-\beta} - (u_n)_{1-\beta}||_\infty < \varepsilon,$$

which shows that $(u_n)_{1-\beta}$ is a Cauchy sequence in $C[0,T]$ (real Banach space), then $(u_n)_{1-\beta}$, is convergent to a function $u \in C[0,T]$. Therefore, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n > n_0$, we have

$$\sup_{t\in[0,T]} \left| t^{1-\beta} u_n(t) - u(t) \right| < \varepsilon,$$

hence

$$\sup_{t\in[0,T]} t^{1-\beta} \left| u_n(t) - \frac{u(t)}{t^{1-\beta}} \right| < \varepsilon,$$

which means that (u_n) converges to the function $v \in C^{1-\beta}$, defined by $v(t) = \frac{u(t)}{t^{1-\beta}}, 0 < t \leq T$, which shows that X_1 is Banach space.
On the other hande $X_2 = C[0,T]$ with the norm $\|u\|_{X_2} = \max_{t\in[0,T]} |u(t)|$ is Banach space, then $X = X_1 \times X_2$, with the norm $\|u\|_X = \max\left\{\|u_1\|_{C^{1-\beta}}, \|u_2\|_C\right\}$, is Banach space.
It is clear that, $(u_1(\cdot), u_2(\cdot))^\top$ is a solution of the problem (2.1), if and only if $u_1(\cdot)$ is a solution of the problem (1.1). Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lu(t) := \begin{pmatrix} (Lu)_1(t) \\ (Lu)_2(t) \end{pmatrix} = \begin{pmatrix} D^{\beta}_{0+} u_1(t) \\ u_2'(t) \end{pmatrix}, \quad \forall t \in [0,T], \tag{2.2}$$

where

$$\operatorname{dom} L = \left\{ u \in X, (D^{\beta}_{0+} u_1, u_2') \in Y, u_2(0) = u_2(T) = \int_0^T g(t) u_2(t) dt \right\},$$

$(Lu)_1(.)$ is the first component and $(Lu)_2(.)$ is the second. Let the operator $N : X \to Y$ be defined by

$$Nu(t) := \begin{pmatrix} (Nu)_1(t) \\ (Nu)_2(t) \end{pmatrix} = \begin{pmatrix} \phi_q (u_2(t)) \\ -g(t) f(t, u_1(t), \phi_q u_2(t)) \end{pmatrix}, \quad \forall t \in [0,T]. \tag{2.3}$$

It is easy to see that problem (2.1) can be converted to the operator equation

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

Throughout this paper we will use the following notations: $D_1, D_2 : Y_2 \to Y_2$ are two linear operators defined by the following relations

$$D_1 y_2 := \int_0^T y_2(s) ds \text{ and } D_2 y_2 := \int_0^T g(t) \int_0^t y_2(s) ds dt.$$

Where $Y_2 = L^1[0,T]$, and for all $\beta \in [0,1)$ denote by For all $\beta \in [0,1)$ denote by

$$\Delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21},$$

where

$$\delta_{11} = T,\ \delta_{22} = \frac{1}{\beta} \int_0^T t^{\beta} g(t) dt,\ \delta_{12} = \frac{T^{\beta}}{\beta} \text{ and } \delta_{21} = \int_0^T t g(t) dt,$$

and the operators $R_1, R_2 : Y_2 \to Y_2$ as

$$\begin{cases} R_1 y_2 := \frac{1}{\Delta} \left(\delta_{22} D_1 y_2 - \delta_{12} D_2 y_2 \right), \\ R_2 y_2 := \frac{1}{\Delta} \left(\delta_{11} D_2 y_2 - \delta_{21} D_1 y_2 \right). \end{cases} \tag{2.4}$$

Proposition 2.1. *(Proposition, p-219,* [1]*) If the continuous function $f \geq 0$ in $[a,b]$ then:*

$$\int_a^b f(x) dx = 0 \Longrightarrow \forall x \in [a,b]; f(x) = 0.$$

Proof. Indeed, if $f(x_0) > 0, x_0 \in [a,b]$, then by continuity, there exist an interval $[\alpha,\beta] \subset [a,b]$, where $f > \frac{f(x_0)}{2}$, which imply

$$\int_a^b f dx \geq \int_\alpha^\beta f dx \geq \frac{f(x_0)(\beta-\alpha)}{2},$$

a contradiction. □

Remark 2.1. *In view of the precedent proposition* $\Delta = \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \neq 0$.
Indeed, for each $0 \leq \beta < 1$, *the function* F *defined by* $F(t) = Tt^\beta - T^\beta t$ *is positive continuous, so* $F(t)g(t) \geq 0$ *(because* $g(t) > 0$*), and as* $g\left(\frac{T}{2}\right) F\left(\frac{T}{2}\right) \neq 0$, *then the function* Fg *is not identically zero in* $[0,T]$ *which prove that*

$$\Delta = \frac{1}{\beta}\int_0^T F(t)g(t)dt \neq 0.$$

3. Existence result

3.1. **Some auxiliary lemmas.** In this part, we needed three lemmas to prove the existence of solutions of our problem by applying Mawhin's coincidence degree theory.

Lemma 3.1. *We have the following results:*

$$\operatorname{Ker} L = \left\{c_1(t^{\beta-1},0)^\top + c_2(0,1)^\top, \forall t \in [0,T], c_1, c_2 \in \mathbb{R}\right\}, \tag{3.1}$$

$$\operatorname{Im} L = \left\{y = (y_1,y_2)^\top \in Y : D_1 y_2 = D_2 y_2 = 0\right\}. \tag{3.2}$$

Proof. On one hand, for each $u = (u_1,u_2)^\top \in \ker L$, we have $Lu(t) = 0$ for all $t \in [0;1]$, so

$$\begin{cases} D_{0+}^\beta u_1(t) = 0 \\ u_2'(t) = 0 \end{cases} \Rightarrow \begin{cases} u_1(t) = c_1 t^{\beta-1} \\ u_2(t) = c_2 \end{cases}$$

then

$$\operatorname{Ker} L = \left\{c_1(t^{\beta-1},0)^\top + c_2(0,1)^\top, \forall t \in [0,T], c_1, c_2 \in \mathbb{R}\right\}.$$

If $y = (y_1,y_2)^\top \in \operatorname{Im} L$, then there exists $u = (u_1,u_2)^\top \in \operatorname{dom} L$ such that $y = Lu$, i.e., $y_1(t) = D_{0+}^\beta u_1(t), y_2(t) = u_2'(t)$, which yields

$$u_2(t) = c_2 + \int_0^t y_2(s)ds, \quad c_2 \in \mathbb{R},$$

with consideration of the boundary conditions $u_2(0) = u_2(T) = \int_0^T g(t)u_2(t)dt$, we conclude that

$$c_2 = c_2 + \int_0^T y_2(s)ds = c_2 + \int_0^T g(t)\int_0^t y_2(s)ds,$$

i.e.

$$\int_0^T y_2(s)ds = \int_0^T g(t)\int_0^t y_2(s)ds = 0.$$

Then,

$$D_1 y_2 = D_2 y_2 = 0, \tag{3.3}$$

thus

$$\operatorname{Im} L \subset \left\{y = (y_1,y_2)^\top \in Y : D_1 y_2 = D_2 y_2 = 0\right\}. \tag{3.4}$$

Now, suppose that $y=(y_1,y_2)^\top \in Y$, and satisfies 3.3. Let

$$\begin{cases} u_1(t)=I^{\beta}_{0+}y_1(t), \\ u_2(t)=I^{1}_{0+}y_2(t). \end{cases}$$

Since $\int_0^T g(t)dt=1$, we get

$$u_2(0)=u_2(T)=\int_0^T g(t)u_2(t)dt,$$

then $u=(u_1,u_2)^\top \in \operatorname{dom} L$ and $Lu=y$ i.e. $y\in \operatorname{Im} L$. Hence,

$$\left\{ y=(y_1,y_2)^\top \in Y : D_1y_2=D_2y_2=0 \right\} \subset \operatorname{Im} L, \tag{3.5}$$

From 3.4 and 3.5, we conclude that

$$\operatorname{Im} L=\left\{ y=(y_1,y_2)^\top \in Y : D_1y_2=D_2y_2=0 \right\}.$$

The proof is completed. □

Lemma 3.2. *Under the assumption $\int_0^T g(t)dt=1$, the following conditions hold:*
(i) $L:\operatorname{dom} L\subset \Omega \to X$ is a Fredholm operator of index zero. Furthermore, the linear continuous projectors $P:X\to X$ and $Q:Y\to Y$ satisfy

$$Pu(t):=\begin{pmatrix} (Pu)_1(t) \\ (Pu)_2(t) \end{pmatrix}=\begin{pmatrix} T^{1-\beta}u_1(T)\,t^{\beta-1} \\ \int_0^T g(s)u_2(s)ds \end{pmatrix}, \quad \forall t\in[0,T],$$

where the first and the second component of P are independent of each other, and

$$Qy(t):=\begin{pmatrix} (Qy)_1(t) \\ (Qy)_2(t) \end{pmatrix}=\begin{pmatrix} 0 \\ R_1y_2+R_2y_2t^{\beta-1} \end{pmatrix}, \quad \forall t\in[0,T],$$

where R_1,R_2 are defined in 2.4.

(ii) The inverse $K_P:\operatorname{Im} L\to \operatorname{dom} L\cap \ker P$ of L_P can be written as

$$K_Py:=\begin{pmatrix} (K_Py)_1 \\ (K_Py)_2 \end{pmatrix}=\begin{pmatrix} I^{\beta}_{0+}y_1 \\ I^{1}_{0+}y_2 \end{pmatrix},$$

and satisfy

$$\|K_Py\|_X \le L\,\|y\|_Y\,,$$

where $L=\max\{\frac{T}{\Gamma(\beta+1)},1\}$.

Proof. (i) For all $u\in X$, and $t\in[0,T]$, we have

$$\begin{aligned} P^2u(t)&=P(Pu)(t)=P\begin{pmatrix} (Pu)_1(t) \\ (Pu)_2(t) \end{pmatrix}=P\begin{pmatrix} T^{1-\beta}u_1(T)\,t^{\beta-1} \\ \int_0^T g(s)u_2(s)ds \end{pmatrix} \\ &=\begin{pmatrix} T^{1-\beta}(Pu)_1(T)\,t^{\beta-1} \\ \int_0^T g(s)(Pu)_2(s)ds \end{pmatrix}=\begin{pmatrix} T^{1-\beta}(T^{1-\beta}u_1(T)\,T^{\beta-1})t^{\beta-1} \\ (Pu)_2(t)\int_0^T g(s)ds \end{pmatrix} \\ &=Pu(t); \end{aligned}$$

because $\int_0^T g(t)dt = 1$.
For all $y \in Y$, and $t \in [0,T]$, we get

$$\begin{aligned}
R_1\left(R_1 y_2\right) &= \frac{1}{\Delta}\left[\delta_{22} D_1\left(R_1 y_2\right) - \delta_{12} D_2\left(R_1 y_2\right)\right] \\
&= \frac{1}{\Delta}\left[\delta_{22}\delta_{11} - \delta_{12}\delta_{21}\right] R_1 y_2 \\
&= R_1 y_2,
\end{aligned}$$

and similarly we can derive that

$$R_1\left(R_2 y_2 t^{\beta-1}\right) = 0,$$

$$R_2\left(R_1 y_2\right) = 0,$$

and

$$R_2\left(R_2 y_2 t^{\beta-1}\right) = R_2 y_2.$$

So, for $y = (y_1, y_2)^\top \in Y$, it follows from the four relations above that

$$\begin{aligned}
Q^2 y_2 = Q(Qy_2) &= R_1[R_1 y_2 + R_2 y_2 t^{\beta-1}] + R_2[R_1 y_2 + R_2 y_2 t^{\beta-1}]t^{\beta-1} \\
&= R_1\left(R_1 y_2\right) + R_1\left(R_2 y_2 t^{\beta-1}\right) + R_2\left(R_1 y_2\right) + R_2\left(R_2 y_2 t^{\beta-1}\right) \\
&= R_1 y_2 + R_2 y_2 t^{\beta-1} = Qy_2.
\end{aligned}$$

Thus, we get

$$Q^2 y(t) = Q(Qy)(t) = Q\begin{pmatrix} (Qy)_1 \\ (Qy)_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Q(Qy_2) \end{pmatrix} = Qy, \quad \forall t \in [0,T].$$

We have also

$$\begin{aligned}
\|(Pu)_1\|_{C^{1-\beta}} &= \max_{t\in[0,T]}\left|t^{1-\beta} T^{1-\beta} u_1(T)\, t^{\beta-1}\right| = \left|T^{1-\beta} u_1(T)\right| \\
&\le \max_{t\in[0,T]}\left|t^{1-\beta} u_1(t)\right| = \|u_1\|_{C^{1-\beta}}, \\
\|(Pu)_2\|_\infty &= \max_{t\in[0,T]}\left|\int_0^T g(s) u_2(s) ds\right| \\
&\le \int_0^T g(s)\left|u_2(s)\right| ds = \|u_2\|_\infty,
\end{aligned}$$

then

$$\begin{aligned}
\|Pu\|_X &= \max\{\|(Pu)_1\|_{C^{1-\beta}}, \|(Pu)_2\|_\infty\} \\
&\le \max\{\|u_1\|_{C^{1-\beta}}, \|u_2\|_\infty\} = \|u\|_X.
\end{aligned}$$

and we have also

$$\|(Qy)_1\|_\infty = \max_{t\in[0,T]}\left|(Qy)_1(t)\right| = 0 \le \|y_1\|_\infty,$$

$$\|(Qy)_2\|_{L^1[0,T]} = \int_0^T \left|(Qy)_2(s)\right| ds \le C\,\|y_2\|_{L^1[0,T]},$$

where $C = \frac{|\delta_{11}\delta_{22}| + 2|\delta_{12}\delta_{11}| + |\delta_{12}\delta_{21}|}{|\Delta|}$, then

$$\|Qy\|_Y = \max\{\|(Qy)_1\|_\infty, \|(Qy)_2\|_{L^1[0,T]}\} \le \max\{\|y_1\|_\infty, C\,\|y_2\|_{L^1[0,T]}\} \le C\,\|y\|_Y. \tag{3.6}$$

Now we prove that $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, each $y = (y_1, y_2)^\top \in Y$, can be write as $y = ((I-Q)y + Qy) = (y_1, (I-Q)y_2 + Qy_2))^\top$ where $y - Qy \in \operatorname{Im} L = \operatorname{Ker} Q$ and $Qy \in \operatorname{Im} Q$, thus we have $Y = \operatorname{Im} L + \operatorname{Im} Q$. Let $y \in \operatorname{Im} L \bigcap \operatorname{Im} Q$ so $y(t) = (y_1, y_2)^\top = (0, a + bt^{\beta-1})^\top$ where $a, b \in \mathbb{R}$, and since $y \in \operatorname{Im} L$, then $D_1 y_2 = \int_0^T (a + bs^{\beta-1}) ds = 0$ and $D_2 y_2 = \int_0^T g(t) \int_0^T (a + bs^{\beta-1}) ds dt = 0$, we derive $a = b = 0$, thus $\operatorname{Im} L \bigcap \operatorname{Im} Q = \{0\}$, which implies that $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, and as

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2.$$

So L is a Fredholm operator of index zero.
Furthemore, for all u in X, we can write $u = (u - Pu) + Pu$ and since $\operatorname{Im} P = \operatorname{Ker} L$, $P^2 u = Pu$ then, $X = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we can get that $\operatorname{Ker} P \cap \operatorname{Ker} L = \{0\}$ which prove that

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

(ii) From the definition of K_P, for $y \in \operatorname{Im} L$, we have

$$LK_P y = \begin{pmatrix} D^{\beta}_{0+} I^{\beta}_{0+} y_1 \\ \frac{d}{dt} I^{1}_{0+} y_2 \end{pmatrix} = y.$$

For $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, and by using Lemma 2.2 we get

$$K_P L u = \begin{pmatrix} u_1 + c_1 t^{\beta-1} \\ u_2 + c_2 \end{pmatrix},$$

where c_1, c_2 are real constants. Since $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, it is easily to show that $c_1 = c_2 = 0$. So K_P is the inverse of L_P. We have also

$$\begin{aligned} \|K_P y\|_X &= \{\|(K_P y)_1\|_{C^{1-\beta}}, \|(K_P y)_2\|_\infty\} \\ &= \max\{\left\|I^{\beta}_{0+} y_1\right\|_{C^{1-\beta}}, \left\|I^{1}_{0+} y_2\right\|_\infty\} \\ &\le \max\{\frac{T}{\Gamma(\beta+1)} \|y_1\|_{Y_1}, \|y_2\|_{Y_2}\} \\ &\le L \|y\|_Y, \end{aligned}$$

where $L = \max\{\frac{T}{\Gamma(\beta+1)}, 1\}$, which completes the proof. □

Lemma 3.3. *Let $\Omega \subset X$ be open and bounded subset with* $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$. *If f is g-Caratheodory, then N is L-compact on $\bar{\Omega}$.*

Proof. Let $\Omega = B(0, r)$, then for $u \in \bar{\Omega}, \|u\| \le r$. Since f is a g-Caratheodory function then, there exists $\omega_r : [0, T] \to [0, +\infty)$ satisfying $\int_0^T g(t) |\omega_r(t)| < +\infty$,
for a.e $t \in [0, T]$,

$$\left| f\left(t, u(t), D^{\beta}_{0+} u(t)\right) \right| \le \omega_r(t),$$

then

$$\|QNu\|_Y \le C \|Nu\|_Y \le C_1,$$

where $C_1 = C(r^{q-1} + \|\omega_r\|_{L^1[0,T]})$. We will use the following two steps to prove that $K_P(I-Q)N(\bar{\Omega})$ is compact.
Step 1: Let $u \in \bar{\Omega}$, then

$$\|K_P(I-Q)Nu\|_X \le L \|(I-Q)Nu\|_Y \le L \left(\|Nu\|_Y + \|QNu\|_Y\right) \le C_2,$$

where $C_2 = L(C_1 + r^{q-1} + \|\omega_r\|_{L^1[0,T]})$.
Step 2: Let $u \in \bar{\Omega}$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ then

$$\begin{aligned}
& \left| t_2^{1-\beta}(K_P(I-Q)Nu)_1(t_2) - t_1^{1-\beta}(K_P(I-Q)Nu)_1(t_1) \right| \\
= & \left| \frac{t_2^{1-\beta}}{\Gamma(\beta)} \int_0^{t_2} (t_2-s)^{\beta-1} ((I-Q)Nu)_1(s)ds \right. \\
- & \left. \frac{t_1^{1-\beta}}{\Gamma(\beta)} \int_0^{t_1} (t_1-s)^{\beta-1} ((I-Q)Nu)_1(s)ds \right| \\
\leq & \left| \frac{t_2^{1-\beta}}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} ((I-Q)Nu)_1(s)ds \right| \\
& + \left| \frac{t_2^{1-\beta} - t_1^{1-\beta}}{\Gamma(\beta)} \int_0^{t_1} (t_2-s)^{\beta-1} \ ((I-Q)Nu)_1(s)ds \right| \\
& + \left| \frac{t_1^{1-\beta}}{\Gamma(\beta)} \int_0^{t_1} \left((t_2-s)^{\beta-1} - (t_1-s)^{\beta-1} \right) ((I-Q)Nu)_1(s)ds \right| \\
\leq & \ \frac{(C+1)(r^{q-1} + \|\omega_r\|_{L^1[0,T]})}{\Gamma(\beta+1)} |t_2 - t_1| \, \|\omega_r\|_{L^1[0,T]} \ \to 0
\end{aligned}$$

as $t_1 \to t_2$ uniformly. Similarly we can derive that

$$\begin{aligned}
& |(K_P(I-Q)Nu)_2(t_2) - (K_P(I-Q)Nu)_2(t_1)| \\
= & \left| \int_0^{t_2} ((I-Q)Nu)_2(s)ds - \int_0^{t_1} ((I-Q)Nu)_2(s)ds \right| \\
= & \left| \int_0^{t_1} ((I-Q)Nu)_2(s)ds + \int_{t_1}^{t_2} ((I-Q)Nu)_2(s)ds \right. \\
- & \left. \int_0^{t_1} ((I-Q)Nu)_2(s)ds \right| \\
= & \left| \int_{t_1}^{t_2} ((I-Q)Nu)_2(s)ds \right| \leq (t_2 - t_1) \left| ((I-Q)Nu)_2 \right| \\
\leq & \ (t_2 - t_1)(C+1)(r^{q-1} + \|\omega_r\|_{L^1[0,T]}) \ \to 0
\end{aligned}$$

as $t_1 \to t_2$ uniformly. Thus, $K_P(I-Q)Nu(\bar{\Omega})$ is compact, therefore, the nonlinear operator N is L -compact on $\bar{\Omega}$. □

3.2. **Existence theorem for the fractional-order p-Laplacian boundary value problem.**

Theorem 3.1. *Assume the following condition holds.*
(H_1) *There exist functions $a_1(t) > 0, a_2(t) > 0, a_3(t) > 0$, in $L^1[0,T]$ such that*

$$g(t) \left| f(t,u,v) \right| \leq a_1(t) + a_2(t) |t^{1-\beta} u|^{p-1} + a_3(t) |v|^{p-1},$$

for all $(u,v) \in \mathbb{R}^2$ and $t \in [0,T]$, where $g(t) \in L^1[0,T], g(t) > 0$.
(H_2) *There exists a constant $A > 0$ such that if $|u| > A$ or $|v| > A$, then either*

$$uD_1(N(u,v)^\top)_2 + vD_2(N(u,v)^\top)_2 > 0, \tag{3.7}$$

or

$$uD_1(N(u,v)^\top)_2 + vD_2(N(u,v)^\top)_2 < 0. \tag{3.8}$$

Then problem 2.1 has at least one solution, provided that

$$\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0;T]} + \|a_3\|_{L^1[0;T]} < 1, \ \textit{if}\ 1<p<2, \tag{3.9}$$

$$\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0;T]} + \|a_3\|_{L^1[0;T]} < 1, \ \textit{if}\ p \geq 2. \tag{3.10}$$

Proof. **Step 1**. Consider the set

$$\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lu = \rho Nu, \rho \in (0,1)\}.$$

For $u \in \Omega_1$, and $\rho \neq 0$, we get $Nu \in \operatorname{Im} L = \operatorname{Ker} Q$, hence

$$\int_0^T g(t) f\left(t, u_1(t), \phi_q(u_2(t))\right) dt = \int_0^T g(t) \int_0^t g(s) f\left(s, u_1(s), \phi_q(u_2(s))\right) ds dt = 0.$$

From the integral mean value theorem there exist $t_0 \in (0,T)$, such that

$$f\left(t_0, u_1(t_0), \phi_q(u_2(t_0))\right) = 0,$$

according to condition (H_2), we get $|u_2\left(t_0\right)| \leq A^{p-1}$. Since

$$u_2(t) = u_2(t_0) + \int_{t_0}^{t_1} u_2'\left(s\right) ds.$$

We have

$$\|u_2\|_C \leq A^{p-1} + \left\|u_2'\right\|_{L^1[0,T]}, \tag{3.11}$$

by Lemma 2.2 we can write, $t^{1-\beta}u_1(t) = t^{1-\beta} I^\beta_{0^+} D^\beta_{0^+} u_1(t) + c_1$, then

$$\left|t^{1-\beta}u_1(t)\right| \leq \frac{T}{\Gamma(\beta+1)}\left\|D^\beta_{0^+}u_1\right\|_C + |c_1|, \forall t \in [0,T],$$

which gives

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)}\left\|D^\beta_{0^+}u_1\right\|_C + |c_1|. \tag{3.12}$$

Now, $Lu = \rho Nu$ is equivalent to

$$\begin{cases} D^\beta_{0^+}u_1(t) = \rho\phi_q\left(u_2(t)\right), \\ u_2'(t) = -\rho g(t) f\left(t, u_1(t), \phi_q\left(u_2(t)\right)\right). \end{cases} \tag{3.13}$$

Using (3.13), we get $\left\|D^\beta_{0^+}u_1\right\|_C \leq \|u_2\|_C^{\frac{1}{p-1}}$. Substitute this inequality in (3.12) we conclude that

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)}\|u_2\|_C^{\frac{1}{p-1}} + |c_1|. \tag{3.14}$$

By the second equation of (3.13), and (H_1), we get

$$\begin{aligned}
\left\|u_2'\right\|_{L^1[0,T]} &= \|g(t)f\left(t,u_1(t),\phi_q\left(u_2(t)\right)\right)\|_{L^1[0,T]} \\
&= \int_0^T g(s)\left|f\left(s,u_1(s),\phi_q\left(u_2(s)\right)\right)\right| ds \\
&\leq \int_0^T \left[a_1(s)+a_2(s)|t^{1-\beta}u|^{p-1}+a_3(s)|v|^{p-1}\right] ds \\
&\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\left|t^{1-\beta}u_1\right|^{p-1}+\|a_3\|_{L^1[0,T]}\left\|\phi_q(u_2)\right\|_C^{p-1} \\
&= \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\|u_1\|_{X_1}^{p-1}+\|a_3\|_{L^1[0,T]}\left\|u_2\right\|_C,
\end{aligned}$$

where the functions $a_1(t)>0, a_2(t)>0, a_3(t)>0$, in $L^1[0,T]$.
If $1<p<2$, then from the above inequalities and Lemma 2.3 we obtain

$$\begin{aligned}
\left\|u_2'\right\|_{L^1[0,T]} &\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\left\|u_2\right\|_{L^1[0,T]}+|c_1|^{p-1}\right) \\
&+\|a_3\|_{L^1[0,T]}\left\|u_2\right\|_C \\
&\leq \|a_1\|_{L^1[0,T]}+\|a_2\|_{L^1[0,T]}\,|c_1|^{p-1}+A^{p-1}\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}\right. \\
&\left.+\|a_3\|_{L^1[0,T]}\right)+\left(\frac{T^{p-1}}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right)\left\|u_2'\right\|_{L^1[0,T]}.
\end{aligned}$$

Similarly, if $p\geq 2$, then

$$\begin{aligned}
\left\|u_2'\right\|_{L^1[0,T]} \leq \|a_1\|_{L^1[0,T]} &+\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\left\|a_2\right\|_{L^1[0,T]}+|c_1|^{p-1} \\
&+A^{p-1}\left(\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\left\|a_2\right\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right) \\
&+\left(\frac{2^{p-2}T^{p-1}}{\Gamma(\beta+1)^{p-1}}\left\|a_2\right\|_{L^1[0,T]}+\|a_3\|_{L^1[0,T]}\right)\left\|u_2'\right\|_{L^1[0,T]}.
\end{aligned}$$

Where A is positive constant, using (3.9) or (3.10), we have

$$\left\|u_2'\right\|_{L^1[0;T]} \leq K_0.$$

by (3.11), we get

$$\left\|u_2\right\|_C \leq A^{p-1}+K_0=K_1.$$

and by (3.14)

$$\|u_1\|_{X_1} \leq \frac{T}{\Gamma(\beta+1)}\left\|u_2\right\|_C^{\frac{1}{p-1}}+|c_1| \leq |c_1|+\frac{T}{\Gamma(\beta+1)}K_1^{\frac{1}{p-1}}=K_2.$$

As a consequence, we obtain

$$\|u\|_X=\max\left\{\|u_1\|_{X_1},\|u_2\|_{X_2}\right\}=\max\left\{K_1,K_2\right\}=k$$

then the set Ω_1 is bounded.
Step 2. Let

$$\Omega_2=\{u\in \operatorname{Ker} L \mid QNu=0\}.$$

For $u \in \Omega_2$, with $u = (u_1, u_2)^\top = (c_1 t^{\beta-1}, c_2)^\top, \forall (c_1, c_2) \in \mathbb{R}^2$. We have $D_1 (Nu)_2 = D_2 (Nu)_2 = 0$, from (H_2) there exist $t_1 \in [0, T]$, such that $\left|c_1 t^{\beta-1}\right| \leq A$ and $|c_2| \leq A$, then we get

$$\|u\|_X = \max_{t\in[0,T]} \left\{ \|u_1\|_{X_1}, \|u_2\|_{X_2} \right\} \leq \max_{t\in[0,T]} \left\{ At^{1-\beta}, A \right\} = B.$$

Thus the set Ω_2 is bounded.

Step 3.

Define the isomorphism $J : \ker L \to \operatorname{Im} Q$ by

$$J \begin{pmatrix} c_1 t^{\beta-1} \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\Delta} \left[\delta_{22} c_1 - \delta_{12} c_2 + (\delta_{11} c_2 - \delta_{21} c_1) t^{\beta-1} \right] \end{pmatrix}, \quad \forall t \in [0, T]$$

for $(c_1, c_2) \in \mathbb{R}^2$. Let

$$\Omega_3 = \{ u \in \ker L, \rho J u + (1-\rho) Q N u = 0, \text{ for some } \rho \in [0, 1] \}.$$

By definition, $u \in \Omega_3$ means that $u = (u_1, u_2)^\top = (c_1 t^{\beta-1}, c_2)^\top$ and $\rho J u + (1-\rho) Q N u = 0$ with $a, b \in \mathbb{R}$. If $\rho = 0$, then $QNu = 0$. By Step 2 we get $\|u\|_X \leq B$. For $\rho = 1$, we obtain $Ju(t) = J(c_1 t^{\beta-1}, c_2)^\top = (0, 0)^\top$, then

$$\begin{cases} \delta_{22} c_1 - \delta_{12} c_2 = 0 \\ \delta_{11} c_2 - \delta_{21} c_1 = 0. \end{cases}$$

Since $\Delta \neq 0$, then $c_1 = c_2 = 0$.

If $0 < \rho < 1$, from $-\rho J u = (1-\rho) Q N u$, we obtain

$$\begin{aligned} &\frac{-\rho}{\Delta} \left[\delta_{22} c_1 - \delta_{12} c_2 + (\delta_{11} c_2 - \delta_{21} c_1) t^{\beta-1} \right] \\ &= \frac{1-\rho}{\Delta} \Big[\delta_{22} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2 - \delta_{12} D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2 + (\delta_{11} D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2 \\ &\quad - \delta_{21} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2) t^{\beta-1} \Big], \end{aligned}$$

which implies that

$$\begin{aligned} -\rho \delta_{22} c_1 + \rho \delta_{12} c_2 &= (1-\rho) \left[\delta_{22} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2 - \delta_{12} D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2 \right] \\ -\rho \delta_{11} c_2 + \rho \delta_{21} c_1 &= (1-\rho) \left[\delta_{11} D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2 - \delta_{21} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2 \right]. \end{aligned}$$

Since the determinant $\Delta \neq 0$, then by simple calculations, we obtain

$$\rho c_1 = -(1-\rho) D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2,$$

and

$$\rho c_2 = -(1-\rho) D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2.$$

Then

$$\begin{cases} \rho c_1^2 t^{\beta-1} = -(1-\rho) c_1 t^{\beta-1} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2 \\ \rho c_2^2 = -(1-\rho) c_2 D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2. \end{cases}$$

By hypothesis (H_2) and from (3.6), we get

$$\rho \left(c_1^2 t^{\beta-1} + c_2^2 \right) = -(1-\rho) \left[c_1 t^{\beta-1} D_1 (N(c_1 t^{\beta-1}, c_2)^\top)_2 + c_2 D_2 (N(c_1 t^{\beta-1}, c_2)^\top)_2 \right] < 0$$

and this is a contradiction. So, the set Ω_3 is bounded.

Finally, if we assume that (3.7) holds, then by the same method, we can prove the boundedness of the set $\{u \in \ker L : -\rho J u + (1-\rho) Q N u = 0$, for some $\rho \in [0.1]\}$. Next, we prove that all conditions of Theorem (2.1) are satisfied: Let Ω to be an open bounded subset of X such that $\cup_{i=1}^{i=3} \bar{\Omega}_i \subset \Omega$. From Lemma 3.2, we known that L is a Fredholm operator of index zero. By Lemma 3.3, N is L-compact on $\bar{\Omega}$. Since $\Omega_i, (i = 1, 2, 3)$ are bounded sets

and $\Omega_i \subset \Omega$, we have
(1) $Lu \neq \rho Nu$ for all $(u,\rho) \in [(dom\ L \setminus \operatorname{Ker}(L)) \cap \partial\Omega] \times (0,1)$,
(2) $QNu \neq 0$ for all $u \in \ker L \cap \partial\Omega$.
Finaly we prove that condition (3) of Theorem 2.1 is satisfied. Let

$$H(u,\rho) = \pm\rho Ju + (1-\rho)QNu.$$

As $\bar{\Omega}_3 \subset \Omega$, for all $u \in \operatorname{Ker}(L) \cap \partial\Omega$ and $\rho \in [0,1]$, we obtain that $H(u,\rho) \neq 0$. So, by the homotopy property of the degree, we conclude that

$$\begin{aligned} \deg\left(QN_{|}\ker L,\ \ker L \cap \Omega, 0\right) &= \deg(H(.,0),\ \ker L \cap \Omega, 0) \\ &= \deg(H(.,1),\ \ker L \cap \Omega, 0) \\ &= \deg(\pm J,\ \ker L \cap \Omega, 0) \neq 0, \end{aligned}$$

which implies that $Lu = Nu$ has at least a solution in $dom\ L \cap \bar{\Omega}$. □

4. A numerical example

Consider the following fractional differential equation

$$(\phi_3(D_{0+}^{\frac{1}{2}}u(t)))' + \cos t(\tfrac{\sin t}{36\pi}\sin^2 u - \tfrac{5\cos 3t}{36\pi}\phi_3(D_{0+}^{\frac{1}{2}}u(t)) - \tfrac{\pi+2}{72\pi}) = 0, \quad t \in [0, \tfrac{\pi}{2}], \tag{4.1}$$

With

$$\begin{cases} \phi_3(D_{0+}^{\frac{1}{2}}u(0)) = \int_0^{\frac{\pi}{2}} \cos t\phi_3(D_{0+}^{\frac{1}{2}}u(t))dt, \\ \phi_3(D_{0+}^{\frac{1}{2}}u(\frac{\pi}{2})) = \int_0^{\frac{\pi}{2}} \cos t\phi_3(D_{0+}^{\frac{1}{2}}u(t))dt, \end{cases} \tag{4.2}$$

where $\beta = \frac{1}{2}, p = 3, q = \frac{3}{2}, T = \frac{\pi}{2}, g(t) = \cos t, \int_0^{\frac{\pi}{2}} \cos t dt = 1$.
Here $f(t,u,v) = \frac{\sin t}{36\pi}\sin^2 u - \frac{5\cos 3t}{36\pi}v^2 - \frac{\pi+2}{72\pi}$, then

$$\cos t|f(t,u,v)| \leq \frac{\pi+2}{72\pi} + \frac{1}{36\pi}|t^{\frac{1}{2}}u|^2 + \frac{5}{36\pi}|v|^2.$$

So we may take

$$a_1(t) = \frac{\pi+2}{72\pi}, a_2(t) = \frac{1}{36\pi}, a_3(t) = \frac{5}{36\pi},$$

we have $\|a_1\|_{L^1[0,\frac{\pi}{2}]} = \frac{\pi+2}{144}; \|a_2\|_{L^1[0,\frac{\pi}{2}]} = \frac{1}{72}; \|a_3\|_{L^1[0,\frac{\pi}{2}]} = \frac{5}{72}$, and $\Delta = \delta_{11}\delta_{22} - \delta_{21}\delta_{12} = 0.81 \neq 0$ and we have also

$$\frac{2^{p-2}T^p}{\Gamma(\beta+1)^{p-1}}\|a_2\|_{L^1[0,\frac{\pi}{2}]} + T\|a_3\|_{L^1[0,\frac{\pi}{2}]} = 0.13 < 1.$$

Let $A = \frac{\pi}{2}$, if $|v| > A$, then we get

$$\begin{aligned} & uD_1(N(u,v)^{\top})_2 + vD_2(N(u,v)^{\top})_2 \\ &= u\left(\frac{\sin^2 u}{36\pi} - \frac{\sin^2 u}{72} + \frac{\pi+2}{72\pi}\right) + v\left(\frac{5}{216\pi}v + (\frac{\pi^2}{8} - 1)(\frac{\pi+2}{72\pi} - \frac{\sin^2 u}{36\pi})\right) \\ &> \frac{1}{72}v\left(\frac{5}{3\pi}v + (\frac{\pi^2}{8} - 1)\right) > 0. \end{aligned}$$

Hence, all conditions of Theorem 3.1 hold, which implies that the problem 4.1 - 4.2 has at least one solution in X.

5. Conclusions

In this study, we have proved the conditions of existence of solutions of a fractional-order p-Laplacian boundary value problem at resonance case where the differential operator is nonlinear and has a kernel dimension equal to two, The proof of our result is based on the so called *Mawhin's coincidence degree theory* which can only used after transforming the nonlinear problem into a semilinear system.

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