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# CONNECTING STATISTICS, PROBABILITY, ALGEBRA AND DISCRETE MATHEMATICS

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Abstract. In this paper, we connect four different branches of Mathematics: Statistics, Probability, Algebra and Discrete Mathematics with the objective of introducing new results on Markov chains and evolution algebras obtained by following a relatively new line of research, already dealt with by several authors. It consists of the use of certain directed graphs to facilitate the study of Markov chains and evolution algebras, as well as to use each of the three objects to make easier the study of the other two. The results obtained can be useful, in turn, to link different scientific disciplines, such as Physics, Engineering and Mathematics, in which evolution algebras are considered very interesting tools.

Keywords: Evolution algebras; directed graphs; Markov chains; mathematical modellings.

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## 1. Introduction

The main goal of this paper is to connect four different branches of Mathematics to each other: Statistics and Operational Research, Probability, Algebra and Graphs Theory. Indeed, we want to endow Markov chains, which are statistics structures with two different structures more, respectively, algebraic and discrete, apart from their already known probabilistic character. Moreover, we also continue the research on a relatively recent introduced topic, the *evolution algebras*, to obtain new results on them, different from those already obtained by Tian (see [17, 18]) and two of the authors in two previous works ([14] and [13]), relying on tools in principle away from these algebras as directed graphs and Markov chains. Regarding the connection between algebraic, discrete and statistics aspects, the reader can check [1, 16], for instance.

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The motivation to deal with evolution algebras is the following. At present, the study of these algebras is very booming, due to the numerous connections between them and many other branches of Mathematics, such as Group Theory, Dynamic Systems and Theory of Knots, among others. Furthermore, they are also related to other sciences, including Biology, since non-Mendelian genetics is precisely which originated them. In fact, Tian already indicated in Chapters 2 and 4 of [17] the relationship between evolution algebras and Markov chains.

For this study we have used in this paper certain objects of Discrete Mathematics, particularly directed graphs, thus linking three branches of the Mathematics which at first would appear to have no connection: evolution algebra Theory, Graph Theory and Markov chains, so that the attainment of new properties of each of them allows to achieve certain advances in the study of the others.

The relationship among these three branches is based on the direct and reverse representation of each Markov chain by an evolution algebra and the one of the latter by a certain graph, so that the study of the properties of each of these objects allows us its subsequent translation to the language of the other two. This research, which could be considered novel, was already somehow used by Tian himself in Chapter 6 of [17].

Recently, several works regarding the use of graphs for studying evolution algebras have appeared in the literature (chronologically, see [14], [13], [8], [3], [4], [7], [6] and [5], for instance). Our intention with this contribution is to show a wide audience new mathematical models which can provide new concepts or new understanding of biological systems which the objective of finding application to multiple biological systems.

The main results of the papers are Theorems 2.11, 2.18, 2.21 and 2.24, which together with Theorems 2.28 and 2.29, allow us to complement those recent works on relationship between Markov chains and evolution algebras.

## 2. Preliminaries

In this section we recall some basic concepts related to evolution algebras, Markov chains and Graph Theory, to be used in this paper. We also deal with the links among all of them. For a more general overview of these three theories, references [17, 18] for the first, [10, 11, 12, 15] for the second and [2, 9] for the third, among others, are available. In this paper, we will consider finite evolution algebras and discrete time homogeneous Markov chains with a finite number of steps.

For extension reasons, in this section we are going to recall only some definitions and results on evolution algebras, since the basic concepts of Markov chains and Graph theories are well known and can be consulted, in any case, in the references indicated above. Note, also, that the paper deals with Markov chains of any state space. Then, as according to [7], the results proved by Tian are true for finite state spaces, it must be considered that all evolution algebras in this manuscript are of finite dimension.

The most basic concepts on evolution algebras were introduced by Tian in [17] and other were obtained by two of the authors in [14, 13].

Evolution algebras, which were firstly introduced by J. P. Tian, and then jointly presented with Vojtechovsky in 2006 [18], and later appeared as a book by Tian in 2008 [17], are those algebras in which the relationships between their generators $V = \{e_1, \dots, e_n\}$ are given by

$$\begin{cases} e_i \cdot e_j = 0, \quad i \neq j, \quad 1 \leq i, j \leq n, \\ e_i^2 = e_i \cdot e_i = \displaystyle\sum_{j=1}^{n} a_{ji}\, e_j, \quad 1 \leq i \leq n. \end{cases}$$

where $a_{ji} \in K$ and $K$ is a field.

If $E$ is an evolution algebra over a field $K$ with a generator set $V = \{e_i \mid i \in \Lambda\}$, the linear map $L : E \to E \mid L(e_i) = e_i^2 = \sum_j a_{ji} e_j$, for all $i \in \Lambda$ is called the *evolution operator* of $E$, the coefficients $a_{ji}$ are the *structure constants of $E$ relative to $V$* and the matrix $M_V := (a_{ji})$ is said to be the *structure matrix of $E$ relative to $V$*.

A particular subclass of evolution algebras are the graphicable algebras, also introduced by Tian in [17]. An *n-dimensional graphicable algebra* is a commutative, non associative algebra, with a set of generators $V = \{e_1, e_2, \ldots, e_n\}$ endowed with relations

$$\begin{cases} e_i \cdot e_j = 0, \quad i \neq j, \quad 1 \leq i, j \leq n, \\ e_i^2 = \sum_{e_j \subseteq V_i} e_j, \quad 1 \leq i \leq n. \end{cases}$$

where $V_i$ is a subset of $V$.
Thus, it is obvious that a graphicable algebra is an evolution algebra, although the converse is not true in general.

An *evolution subalgebra* of an evolution algebra spanned by $\{e_1, ..., e_n\}$ is a subalgebra that is spanned by $\{e_i : i \in \Lambda\}$, for some subset $\Lambda$ of $\{1, \ldots, n\}$.

Let $E$ be an evolution algebra and $I$ be an evolution subalgebra of $E$. It is said that $I$ is an *evolution ideal* of $E$ if $E \cdot I \subseteq I$. Note that this definition implies that every evolution subalgebra is an evolution ideal because evolution algebras do not have an identity that characterizes them, unlike the Lie, Malcev or Leibniz algebras, for instance.

An evolution algebra $E$ is called *simple* if it has not ideals different from 0 and $E$, and it is called *irreducible* it it has no proper subalgebra.

The following definition is given by Tian in [17]: Let $E$ be an evolution algebra and $\{e_1, e_2, ...e_n\}$ a set of generators. It is said that $e_i$ *appears* in $x \in E$ if the coefficient $\alpha_i \in K$ is different from 0 in the expression $x = \sum_{j=1}^{n} \alpha_j e_j$. If $e_i$ appears in $x$, it is denoted by $e_i \prec x$.

Now, with respect to the association between evolution algebras and graphs (where *graph* might be of any type of them), Tian, in [17], showed how to associate a graph with an evolution algebra. He gave the following

**Definition 2.1.** *Let $G = (V, E)$ be a directed graph, $V$ be the set of vertices and $E$ be the set of edges. It is defined the* associated evolution algebra with $G$ *taking $V = \{e_1, e_2, \ldots, e_n\}$ as the set of generators and $R$ as the set of relations of the algebra*

$$R = \begin{cases} e_i^2 = \sum_{e_k \in \Gamma(e_i)} e_k, \quad 1 \leq i \leq n, \\ e_i \cdot e_j = 0, \quad i \neq j, \quad 1 \leq i, j \leq n. \end{cases}$$

*where $\Gamma(e_i) = \{e_k : (e_i, e_k) \in E\}$ denotes the set of vertices adjacent to $e_i$.*

Conversely, Tian also showed how to associate an evolution algebra with a directed graph: he took the set of generators of the algebra as the set of vertices and as the set of edges those connecting the vertex $e_i$ with the vertices corresponding to generators appearing in the expression of $e_i^2$, for each generator $e_i$.

With respect to the relationship between graphs and Markov chains is already known a representation of the last ones by graphs (see [10], for instance) . Each homogeneous Markov chain with the set of states $\{e_i \mid i \in \Lambda\}$ and transition probabilities $p_{ij} = Pr[X_n = e_j \mid X_{n-1} = e_i]$ can be associated with a weighted directed graph by taking the set of states as the set of vertices, the transitions of one step from a state to other as the set of edges and the transition probabilities as the weight of the edges. The fact that two vertices are non adjacent means that it is not possible the transition in one step between them.

Finally, Tian also gave in [17] the relationship between evolution algebras and Markov chains. He proved that for each homogeneous Markov chain $X$, there is an evolution algebra $M_X$ whose structure constants are transition probabilities, and whose generator set is the state space of the Markov chain. However, he did not the converse procedure, that is, given an evolution algebra, he did not define the Markov chain associated to it.

## 3. Associating evolution algebras, Markov chains and graphs

In this section we show novel results obtained on the study of the relationships among evolution algebras, graphs and Markov chains. All of them complete the study made by Tian in Chapter 4 of [17].

3.1. **Some results on Markov evolution subalgebras.** In this section we show some properties and characterizations on Markov evolution subalgebras.

**Definition 3.1** (Tian, Chapter 4 of [17])**.** *Let $X$ be a homogeneous Markov chain with the set of states $\{e_i \mid i \in \Lambda\}$ and transition probabilities $p_{ij} = Pr[X_n = e_j \mid X_{n-1} = e_i]$; The* evolution algebra $M_X$ corresponding to $X$ *has $\{e_i \mid i \in \Lambda\}$ as set of generators and the following expressions as the laws of the algebra*

$$R = \begin{cases} e_i^2 = \sum_{k\in\Lambda} p_{ik} e_k, & i \in \Lambda, \\ e_i \cdot e_j = 0, & i \neq j, \quad i,j \in \Lambda. \end{cases}$$

*where $0 \leq p_{ik} \leq 1$ and $\sum_{k\in\Lambda} p_{ik} = 1$.*

Observe that the transition probabilities matrix $(p_{ij})$ of the Markov chain $X$ defines the structure matrix of $M_X$ related to $\{e_i \mid i \in \Lambda\}$.

We give now a similar definition, but in the other sense, which will allow us to set the converse result.

**Definition 3.2.** *Let $E$ be an evolution algebra with a set of generators $\{e_i \mid i \in \Lambda\}$ and laws given by the products*

$$R = \begin{cases} e_i^2 = \sum_{k\in\Lambda} a_{ik} e_k, & i \in \Lambda, \\ e_i \cdot e_j = 0, & i \neq j, \quad i,j \in \Lambda. \end{cases}$$

*where $0 \leq a_{ik} \leq 1$ and $\sum_{k\in\Lambda} a_{ik} = 1$. The structure obtained taking the set of generators as the set of states and the structure constants involved in the products of $E$ as transition probabilities is called the Markov chain associated with $E$.*

Note that the name given to that structure is consistent, as it is proved in the following

**Theorem 3.1.** *The structure obtained in the previous definition starting from the evolution algebra $E$ is a Markov chain.*

*Proof.* According to the previous definition, it is immediate to check that the structure obtained is a Markov chain, due to the one-to-one correspondence between $E$ and a discrete time Markov chain $X$, with state space the generators and transition probabilities given by the structure constants (note that each state of the Markov chain $X$ is identified with a generator of S). □

**Definition 3.3** (Tian, Introduction of Chapter 4 of [17])**.** *An evolution algebra which is associated with a Markov chain is called* Markov evolution algebra.

Now, by construction, we will associate a weighted directed graph to a Markov evolution algebra as follows. We will call it *Markov graph.*

**Definition 3.4.** *The weighted directed graph associated with a Markov evolution algebra is $G = (V, E)$, with $V$ the set of generators of the algebra, $(e_i, e_j) \in E$ if and only if $e_j \prec e_i^2$ and the weight of the edge $(e_i, e_j)$ is precisely $p_{ij}$ given by the structure matrix of the algebra related to $V$. This weighted directed graph is called* Markov graph.

Note that the Markov graph coincides with the weighted directed graph corresponding to the Markov chain which is associated with the algebra.

**Example 3.1.** Let us consider a 4-dimensional evolution algebra defined by

$$\begin{array}{rlllll} e_1 \cdot e_1 = & 0.5\,e_1 & + \; 0.2\,e_2 & & + & 0.3\,e_4, \\ e_2 \cdot e_2 = & 0.1\,e_1 & & + \; 0.9\,e_3, & & \\ e_3 \cdot e_3 = & & & 0.4\,e_3 & + & 0.6\,e_4, \\ e_4 \cdot e_4 = & & 0.15\,e_2 & & + & 0.85\,e_4. \end{array}$$

By Definition 3.3, it is straightforwardly checked that it is a Markov evolution algebra with set of states $\{e_1, e_2, e_3, e_4\}$, and its transition probabilities matrix is the matrix $P$ given in Figure 1. Moreover, the weighted directed graph associated to such an algebra is also shown in that figure.

$$P = \begin{pmatrix} 0.5 & 0.2 & 0 & 0.3 \\ 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0.15 & 0 & 0.85 \end{pmatrix}$$

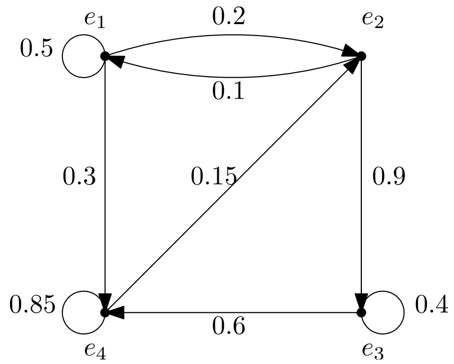


FIGURE 1. The directed graph and its transition probabilities matrix.

However, it is clear that there are evolution algebras which are not Markov. As a particular case, some graphicable algebras also have associated Markov chains. Those algebras, graphicable and Markov, are those whose laws are

$$\begin{cases} e_i^2 = e_{k(i)}, & \text{for } i, k(i) \in \Lambda, \\ e_i \cdot e_j = 0, & i \neq j, \quad i, j \in \Lambda. \end{cases}$$

It can be found in [14] the following characterization of the evolution operator of graphicable algebras

**Theorem 3.2.** ([14]) *Let $G$ be a simple graph with $V(G) = \{x_1, x_2, ..., x_n\}$ and let $L$ be the evolution operator of graphicable algebra $A(G)$. If we express $L^n(e_i) = n_{i1}e_1 + n_{i2}e_2 + ... + n_{ir}e_r$, then $n_{ij}$ coincides with the number of walks with length $n$ between the vertices corresponding to generators $e_i$ and $e_j$.*

The natural question is now which would be the translation of this result in the language of Markov chains.

We now introduce the following novel concept

**Definition 3.5.** *Given a weighted directed graph $G$, the* Markov weight *of a walk in $G$ is the product of the weights of the edges of the walk.*

This definition allows us to state the following result

**Theorem 3.3.** *Let $G$ be a directed graph with $V(G) = \{x_1, x_2, ..., x_n\}$, and let $L$ be the evolution operator of the evolution algebra $A(G)$. If we express $L^n(e_i) = n_{i1}e_1 + n_{i2}e_2 + ... + n_{ir}e_r$, then $n_{ij}$ coincides with the sum of the Markov weights of the walks of length $n$ from the vertex corresponding to the generator $e_i$ to the vertex corresponding to the generator $e_j$.*

Previous definition and theorem make possible to improve some results by Tian. Indeed, Tian showed a result in [17] which links the concepts of closed subset of states and evolution subalgebras. It is the following

**Theorem 3.4** (Tian, Theorem 17, Chapter 4 of [17])**.** *Let $C$ be a closed subset of the set of states $S = \{e_i \mid i \in \Lambda\}$ of a Markov chain $X$. $C$ is closed in the sense of probability if and only if $C$ generates an evolution subalgebra of the evolution algebra $M_X$.*

Recall that a set of states $S_1 \subset S$ of a Markov chain is *closed* if $p_{ij}^{(m)} = 0$, for all $i \in S_1$, $j \notin S_1$ and $m \geq 1$. If a closed subset contains only one state, then that state is *absorbent*.

Then, from this definition, this result can be formulated in an alternative way, as follows

**Theorem 3.5.** *Let $C$ be a subset of the set of generators $\{e_i | i \in \Lambda\}$ of a Markov evolution algebra. $C$ generates an evolution subalgebra of the Markov evolution algebra if and only if*

$$n_{ij} = 0, \text{ for } e_i \in C, e_j \notin C, n \geq 1$$

*where $n_{ij}$ represents the elements of the evolution operator $L^n$.*

Tian gave a proof in [17] of the following result (which can be also proved by 'reductio ad absurdum', taking into consideration that every evolution subalgebra is an evolution ideal):

**Theorem 3.6** (Tian, Theorem 18, Chapter 4 of [17])**.** *A Markov chain $X$ is irreducible if and only if its corresponding evolution algebra $M_X$ is simple.*

Now, as a consequence, we can reformulate that assert as follows

**Theorem 3.7.** *A Markov evolution algebra is simple if and only if there exists an integer $n$ verifying that $n_{ij}$, the element of the representation matrix of the operator $L^n$, is positive for each pair of generators $e_i$ and $e_j$.*

**Remark 3.1.** *This result allows us to give an alternative way to decide whether a Markov evolution algebra is simple, without the need of finding proper ideals of the algebra. Let us see it in the following example*

**Example 3.2.** Let $M_X$ be the Markov evolution algebra with generators $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and laws $e_1^2 = 0.3\, e_2 + 0.7\, e_6; e_2^2 = e_3; e_3^2 = 0.8\, e_1 + 0.2\, e_4; e_4^2 = e_6; e_5^2 = e_4; e_6^2 = e_5$. Its associated Markov chain has states $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the transition probabilities matrix is

$$P = \begin{pmatrix} 0 & 0.3 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Observe that $P = L$, $n_{ij} = 0$, when $i = 3, 4, 5$ and $j = 1, 2, 3$ in L and thus for any power of $L$.

Then, according to Theorem 3.5, $\{e_4, e_5, e_6\}$ generates an evolution subalgebra (equivalently, it is a proper closed subset in the sense of probability). Therefore, Theorem 3.7 implies that $M_X$ is not a simple evolution algebra and its associate Markov chain is not irreducible.

**Remark 3.2.** *Not every subgraph of a weighted directed graph associated with a Markov evolution algebra generates an evolution subalgebra of the evolution algebra. For it, the subgraph must meet the following requirement, which is to be* closed*, a new concept that we now introduce.*

**Definition 3.6.** *Let $G = (V, E)$ be a directed graph, not necessarily weighted. We say that the subgraph $\langle V' \rangle$ induced by $V' \subset V$ is* closed *if $(e_i, e_j) \notin E$, for all $e_i \in V'$ and $e_j \in V \setminus V'$.*

This new concept allows us to deduce the following

**Theorem 3.8.** *Let $G'$ be the weighted directed graph associated to a Markov evolution algebra $M'$ and let $G$ be the weighted directed graph associated to another Markov evolution algebra $M$. Then, $G'$ is a closed induced subgraph of $G$ if and only if $M'$ generates an evolution subalgebra of the evolution algebra $M$.*

This concept also allows us to give a new characteristic of simple Markov evolution algebras expressed by means of its associated graph.

**Theorem 3.9.** *Let $G = (V, E)$ be the directed graph associated to a Markov evolution algebra $M$. Then, $G$ has no closed induced subgraph if and only if the evolution algebra $M$ is simple.*

3.2. **Classification of the generators of a Markov evolution algebra.** Observe that so far there have been several ways to see if a Markov evolution algebra is simple or not. Now, given a Markov evolution algebra, we wish to find its simple evolution subalgebras. To do this, we turn to the classification of individual generators of a Markov evolution algebra.

Tian proved in [17] that the concepts algebraically recurrent generator and recurrent state in the sense of probability are equivalent, and that the same occurs with transient generator and transient state in that sense. In addition, Tian proved in Chapter 4 of [17] the following result

**Theorem 3.10** (Tian, Lemma 10 in Chapter 4 of [17])**.** *A generator $e_i$ is algebraically persistent if and only if all generators $e_j$ which occurs in $< e_i >$, $e_i$ also occurs in $< e_j >$.*

This result can be increased and reformulated according to our research as follows

**Theorem 3.11.** *A generator $e_i$ of the evolution algebra $M$ is transient if and only if there exists one $e_j$ appearing in $\langle e_i \rangle$, such that $e_i$ does not appear in $\langle e_j \rangle$. A generator $e_i$ is algebraically recurrent in $M$ if and only if for each generator $e_j$ which appears in $\langle e_i \rangle$, $e_i$ also appears in $\langle e_j \rangle$.*

Now, we move on to introduce the concept of *period* of a generator $e_i$ of the evolution algebra $M$ in the same way as it is defined the period of a state $e_i$ in a Markov chain $X$.

**Definition 3.7.** *The* period *of a generator $e_j$ of an evolution algebra $M$ is defined as the greatest common divisor of the set of integers $n$ for which $n_{jj} > 0$, where $n_{jj}$ is the $j-th$*

*diagonal element of the representation matrix of the operator $L^n$. A generator with period 1 is called* aperiodic.

As a consequence, if $G$ is the weighted directed graph associated with a Markov evolution algebra, the period of a generator $e_j$ is the greatest common divisor of the lengths of the walks from $e_j$ to $e_j$.

**Example 3.3.** Given the following Markov evolution algebra

$$e_1^2 = e_2, \quad e_2^2 = 0.17\, e_1 + 0.83\, e_3, \quad e_3^2 = e_2,$$

the weighted directed graph is shown in Figure 2.

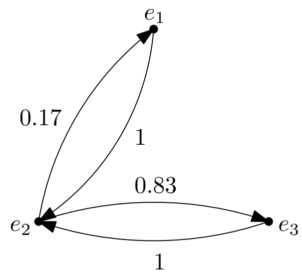


FIGURE 2. The Markov graph of Example 3.3

Now, observe that the generator $e_1$ has period 2 because the simple walks $e_1e_2e_1$ and $e_1e_2e_3e_2e_1$ from $e_1$ to $e_1$ have lengths 2 and 4, respectively. Similarly, the period of $e_2$ is the greatest common divisor $\{2, 2\} = 2$ and the period of $e_3$ is the greatest common divisor $\{2, 4\} = 2$.

Note that in this example all generators have the same period. It is so because in an irreducible Markov chain all states are either periodic with the same period or aperiodic. Then, the following result follows

**Theorem 3.12.** *In a simple evolution algebra all generators are either periodic with the same period or all states are aperiodic.*

For instance, according to Theorem 3.12, the evolution algebra of the example 3.3 is a simple evolution algebra because its associated Markov chain is irreducible.

In previous results we have dealt with the classification of individual generators. We will now deal with the classification of subsets of generators. To do this, we recall that Tian already proved the following result

**Theorem 3.13** (Tian, Corollary 12, Chapter 4 of [17])**.** *The state $e_k$ is an absorbent state in a Markov chain $X$ if and only if $e_k$ is an idempotent element in the evolution algebra $M_X$.*

Note that this fact is easy to observe since $e_k$ is absorbent in a Markov chain $X$ if and only if $p_{kk} = 1$. So, in the algebra $M_X$ it is verified $e_k \cdot e_k = e_k$.

Besides, it is easy to note that the absorbent states are closed subsets of the set of states of a Markov chain and thus, by Theorem 3.4, they generate an evolution subalgebra of the algebra associated to the chain. It also explains the following result by Tian

**Theorem 3.14** (Tian, Remark 4 in Chapter 4 of [17])**.** *Every idempotent element in a Markov evolution algebra $M$ generates an evolution subalgebra of $M$.*

Continuing with our novel contributions in this study, other concepts which can be introduced in evolution algebras inspired by their analogous in Markov chain are the following

**Definition 3.8.** *Let $M$ be a Markov evolution algebra. The generator $e_j$ is said to be* accessible *from the generator $e_i$ if there exists an integer $n$ such that $n_{ij} > 0$, where $n_{ij}$ is the element of the row $i$ and column $j$ of the representation matrix of the operator $L^n$. Two generators $s_i$ and $s_j$* communicate *if $s_j$ is accessible from $s_i$ and $s_i$ is accessible from $s_j$. A generator $e_i$ is said to be a* generator of return *if $a_{ii} > 0$, where $a_{ii}$ is the element of the column $i$ and row $i$ of the representation matrix of the operator $L$. The set of all generators communicated to generator $e_i$ constitutes a* class *denoted by $C(e_i)$.*

Then, in the same way as in Markov chains, the generators of an evolution algebra can be partitioned into connected classes. Classes may or may not be closed, as it occurs in Markov chains. If the generator $e_i$ is recurrent, its connected class is closed, that is, by Theorem 3.4, the connected class to which it belongs generates an evolution subalgebra of the Markov evolution algebra. Only transient generators can belong to non closed connected classes.

Besides, if the generator $e_i$ is recurrent and $e_j$ is accesible from the generator $e_i$, then the generator $e_j$ must be connected to $e_i$ and must also be recurrent. Thus the recurrent generators only connect with other recurrent generators, so the set of recurrent generators must be closed and therefore they generate an evolution subalgebra of the Markov evolution algebra.

Then, if the generator $e_i$ is recurrent, the class $C(e_i)$ is an irreducible closed set and contains only recurrent generators. Thus by Theorems 3.4 and 3.6 if $e_i$ is a recurrent generator, then $C(e_i)$ generates an evolution subalgebra of the simple Markov evolution algebra.

These new concepts involve the following new results in evolution algebras, which are related with Corollary 9 in Chapter 3 of [17]. The first of them is

**Theorem 3.15.** *If all generators of an evolution algebra belong to the same connected class, then the algebra is simple.*

whereas the second one is

**Theorem 3.16.** *The set of generators of an evolution algebra can be partitioned into two subsets. The first one contains only transient generators and the second subset only recurrent generators. Moreover, these last ones generate an evolution subalgebra of the Markov evolution algebra and this second subset may also be partitioned into connected and irreducible classes which generate simple evolution subalgebras.*

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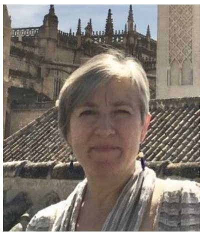

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