Modulation of Generalized Symmetric Regularized Long-wave Equation: Generalized Nonlinear Schrödinger equation

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Abstract

In this work, the application of "the modified reductive perturbation method" is extended to the generalized symmetric regularized long-wave equation for strongly dispersive case and the contribution of higher order terms in the perturbation expansion is obtained. It is shown that the first order term in the perturbation expansion is governed by the generalized nonlinear Schrödinger quation whereas the second order term is governed by the generalized linear Schrödinger equation with a non-homogeneous term. A travelling wave type of solution to these evolution equations is also given.

Keywords: Generalized symmetric regularized wave equation, Generalized Nonlinear Schrödinger equation, Solitary waves

1. Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for weakly dispersive case one obtains the Korteweg-de Vries (KdV) equation for the lowest order term in the perturbation expansion, the solution of which may be described by solitons (Davidson [1]). To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables (Taniuti [2]). However, in such approach some secular terms appear which can be eliminated by introducing some slow scale variables (Sugimoto and Kakutani [3]) or by a renormalization procedure of the velocity of the KdV soliton (Kodama and Taniuti [4]). Nevertheless, this approach remains some what artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the perturbation parameter (Washimi and Taniuti [5]). On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities made by Kraenkel and Manna [6] for long water waves by use of the multiple time scale expansion but could not obtain explicitly the correction terms to the speed of the wave. In order to remove such secularities, He [7, 8] presented the “modified Lindstedt-Poincare method” and applied it to some strongly nonlinear oscillation problems.

To circumvert these ambiguities, Malfliet and Wieers [9] presented a dressed solitary have approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the long wave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solution to the original
nonlinear equations and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [10], a method so called "the modified reductive perturbation method" had been presented to examine the contributions of higher order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves and solitary waves in a fluid filled elastic tube [11]. In these works, it is shown that the lowest order term in the perturbation expansion is governed by the nonlinear Korteweg-de Vries equation, whereas the higher order terms in the expansion are governed by the degenerate Korteweg-de Vries equation with non-homogeneous term. By employing hyperbolic tangent method a progressive wave type of solution was sought and the possible secularities were removed by selecting the scaling parameter in a special way. The basic idea in this method was the inclusion of higher order dispersive effects through the introduction of the scaling parameter to balance the higher order nonlinearities with dispersion. The negligence of higher order dispersive effects in the classical reductive perturbation method leads to the imbalance of nonlinearity and dispersion, which resulted in some secular terms in the solution of evolution equations. As a matter of fact, the renormalization method presented by Kodama and Taniuti [4] is different but rather involved formulation of the same idea.

In the present work, the application of "the modified reductive perturbation method" is extended to the generalized symmetric regularized long-wave equation for strongly dispersive case and the contribution of higher order terms in the perturbation expansion is obtained. It is shown that the first order term in the perturbation expansion is governed by the generalized nonlinear Schrödinger equation whereas the second order term is governed by the generalized linear Schrödinger equation with a non-homogeneous term. A travelling wave type of solutions to these evolution equations are also given and various order correction terms to the wave amplitude and the wave speed are obtained so as to remove possible secularities in the solution. The present method is seen to be fairly simple and practical as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel and Manna [6].

2. Modified reductive perturbation formalism for GSRLW equation

In this section, the attention is focused on the examination of the generalized symmetric regularized long wave (GSRLW) equation equation [12] given by

\[ u_{tt} - u_{xx} + \frac{1}{2n+1} \left( u^{2n+1} \right)_{xx} - u_{xxx} = 0, \]

(1)

where \( u \) is the velocity in the x direction and \( n > 1 \) is an integer. The equation (1) describes the propagation of ion-plasma acoustic waves in space under weakly nonlinear action.

Motivated with the dispersion relation of the linearized form of equation (1), it is convenient to introduce the following stretched coordinates

\[ \xi = \varepsilon^n (x - \lambda t), \quad \tau = \varepsilon^{2n} gt, \]

(2)

where \( \varepsilon \) is a small parameter measuring the width of wave packet, \( \lambda \) and \( g \) are two scale parameters to be determined from the solution.

It will be assumed that the field variable \( u \) is a function of the slow variables \( (\xi, \tau) \) as well as the fast variables \( (x, t) \). Thus, the following operators are valid

\[ \frac{\partial}{\partial t} \rightarrow \varepsilon^n \lambda \frac{\partial}{\partial \xi} + \varepsilon^{2n} g \frac{\partial}{\partial \tau}, \]

\[ \frac{\partial}{\partial x} \rightarrow \varepsilon^a \frac{\partial}{\partial \xi}. \]

(3)

It is further assumed that the field variable \( u \) and the scale parameters \( \lambda \) and \( g \) can be expanded into asymptotic series of \( \varepsilon \) as (Demiray [13])
\[ u = \epsilon u_1 + \epsilon^{n+1} u_2 + \epsilon^{2n+1} u_3 + \epsilon^{3n+1} u_4 + \ldots \]
\[ \lambda = \lambda_0 + \epsilon^{2n} \lambda_1 + \epsilon^{3n} \lambda_2 + \ldots \]
\[ g = 1 + \epsilon^n g_1 + \epsilon^{2n} g_2 + \ldots \]  

Introducing the expansions (3) and (4) into equation (1) and setting the coefficients of like powers of \( \epsilon \) equal to zero the following sets of differential equations are obtained:

\[ O(\epsilon) \text{ equation:} \]
\[ \frac{\partial^3 u_1}{\partial t^3} - \frac{\partial^3 u_2}{\partial x^3} - \frac{\partial^4 u_1}{\partial t^4} = 0. \]  

\[ O(\epsilon^{n+1}) \text{ equation:} \]
\[ \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^4 u_2}{\partial t^4} + 2 \left( \frac{\partial^4 u_1}{\partial x^4} + \lambda \frac{\partial^2 u_1}{\partial t \partial \xi} \right) \\
-2 \left( \frac{\partial^4 u_1}{\partial x \partial t^2 \partial \xi} - \frac{\lambda_0}{\partial x^2 \partial t \partial \xi} \right) = 0. \]  

\[ O(\epsilon^{2n+1}) \text{ equation:} \]
\[ \frac{\partial^4 u_2}{\partial t^4} - \frac{\partial^4 u_3}{\partial x^4} - \frac{\partial^4 u_4}{\partial t^4} + 2 \left( \frac{\partial^4 u_1}{\partial x^4} + \lambda_0 \frac{\partial^2 u_1}{\partial t \partial \xi} \right) \\
+ 2 \lambda_0 \frac{\partial^2 u_1}{\partial t \partial \xi} + \frac{\partial^2 u_1}{\partial t \partial \xi} + \frac{\partial^2 u_2}{\partial t^2 \partial \xi} - \frac{\lambda_0}{\partial x^2 \partial t \partial \xi} \frac{\partial^4 u_2}{\partial x^2 \partial \xi^2} + \frac{\partial^2}{\partial x \partial t} \left( u_1^{2n+1} \right) \\
2 g_1 \frac{\partial^2 u_1}{\partial t \partial \xi} - 2 \lambda \frac{\partial^2 u_1}{\partial \xi^2} - 2 \lambda_0 \frac{\partial^2 u_1}{\partial \xi^2} = 0. \]  

\[ O(\epsilon^{3n+1}) \text{ equation:} \]
\[ \frac{\partial^3 u_3}{\partial t^3} - \frac{\partial^3 u_4}{\partial x^3} - \frac{\partial^3 u_5}{\partial t^3} - 2 \lambda_0 \frac{\partial^3 u_1}{\partial t \partial \xi} - \frac{\partial^3 u_2}{\partial t^2 \partial \xi} - \frac{\partial^2}{\partial t \partial \xi} \left( u_1^{3n+1} \right) \\
-4 \frac{\partial^3 u_1}{\partial x \partial t \partial \xi} + 2 \lambda_0 \frac{\partial^3 u_1}{\partial x^2 \partial \xi^2} - 2 \lambda_0 \frac{\partial^4 u_1}{\partial t \partial \xi^3} + \lambda_0 \frac{\partial^3 u_1}{\partial t \partial \xi}\left( u_1^{2n+1} \right) + \frac{1}{2n+1} \frac{\partial^2}{\partial t \partial \xi} \left( u_1^{2n+1} \right) = 0. \]
2.1 Solution of the field equations

For the solution of $O(\varepsilon)$ equation the following representation is proposed:

$$u_1 = U(\xi, \tau) e^{i\phi} + c.c., \quad (9)$$

where $U(\xi, \tau)$ is an unknown function whose governing equation will be obtained later, $\phi = kx - \omega t$ is the phasor and c.c. stands for the complex conjugate of the corresponding quantity.

Inserting (9) into (5), the following dispersion relation is obtained

$$\omega^2 (1 + k^2) - k^2 = 0, \quad (10)$$

where $\omega$ is the angular frequency and $k$ is the wave number.

Substitution of (10) into equation (6) yields

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^4 u_2}{\partial x^2 \partial t^2} + 2i\left[ \omega \lambda_0 (1 + k^2) - k (1 - \omega^2) \right] \frac{\partial U}{\partial \xi} e^{i\phi} + c.c. = 0. \quad (11)$$

The form of equation (11) suggests us to seek a solution for $u_2$ as

$$u_2 = U_2^{(1)} e^{i\phi} + c.c. \quad (12)$$

Inserting (12) into (11) and utilizing the dispersion relation (10) one gets

$$\left[ \omega \lambda_0 (1 + k^2) - k (1 - \omega^2) \right] \frac{\partial U}{\partial \xi} = 0, \quad (13)$$

Here it is to be noted that $U_2^{(1)}(\xi, \tau)$ remains as an arbitrary function of its argument and its governing equation will be obtained from the higher order perturbation expansion. In order to have a non-zero solution for $U$ the coefficient of $\frac{\partial U}{\partial \xi}$ in equation (13) must vanish, which yields

$$\omega \lambda_0 (1 + k^2) - k (1 - \omega^2) = 0,$$

or

$$\lambda_0 = \frac{k (1 - \omega^2)}{\omega (1 + k^2)}. \quad (14)$$

Here $\lambda_0 = d\omega / dk$ is the group velocity of the linearized harmonic wave.

To get the solution for equation the solutions (9) and (12) are introduced into (7) and the use of (14) yields the following equation:

$$\frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^4 u_3}{\partial x^2 \partial t^2} + \left\{ -2i\omega (1 + k^2) \frac{\partial U}{\partial \tau} + (\omega^2 + \lambda_0^2 - 1 + \lambda_0^2 \gamma^2 + 4 \lambda_0 \omega k) \frac{\partial^2 U}{\partial \xi^2} \right\} e^{i\phi}$$

$$+ \frac{\omega k}{2n+1} \sum_{l=1}^{l=n+1} (2l-1)^2 \left( \begin{array}{c} 2n+1 \\ n+1-l \end{array} \right) U(2n+1-l) U(2l-1) e^{i(2l-1)\phi} + c.c = 0 \quad (15)$$
where |U|^2 = UU*, U* being the complex conjugate of U and (n + 1 - l) is the Binomial coefficient. The form of equation (15) suggests us that it should have the following form

\[ -2i\omega(1 + k^2)\frac{\partial U}{\partial \tau} + \left[ \lambda_0^2(1 + k^2) + 4\lambda_0\omega k + \omega^2 - 1 \right] \frac{\partial^2 U}{\partial \xi^2} + \frac{\omega k}{(2n+1)} \left( \frac{2n+1}{n} \right) |U|^2n U = 0. \]  

(17)

Re-organizing the terms of the equation (17) one obtains the following generalized nonlinear Schrödinger equation

\[ i\frac{\partial U}{\partial \tau} - \mu_1 \frac{\partial^2 U}{\partial \xi^2} - \mu_2 |U|^2n U = \hat{U}, \]  

(18)

where the coefficients \( \mu_1 \) and \( \mu_2 \) are defined by

\[ \mu_1 = \frac{\omega^2 + \lambda_0^2(1 + k^2) + 4\lambda_0\omega k - 1}{2\omega(1 + k^2)}, \]  

\[ \mu_2 = \frac{k}{2(1 + k^2)} \frac{1}{(2n+1)} \left( \frac{2n+1}{n} \right). \]  

(19)

With the substitution of (16) into equation (15) one obtains

\[ U_3^{(l)} = \sum_{l=1}^{\infty} U_3^{(l)} e^{(2l-1)\varphi} + c.c., \]  

where, due to the dispersion relation (10), \( U_3^{(l)} \) remains an arbitrary function of its argument. Introducing (16) into (15) and considering only the coefficient of \( e^{\varphi} \) one obtains

\[ \begin{align*}
-2i\omega(1 + k^2) &\frac{\partial U_2^{(l)}}{\partial \tau} + \left( \omega^2 + \lambda_0^2k^2 + \lambda_0 + 4\lambda_0\omega k - 1 \right) \frac{\partial^2 U_2^{(l)}}{\partial \xi^2} \\
+ \omega k &\left( \frac{2n}{n} \right) |U_2^{(l)}|^2n + \left( \frac{2n}{n-1} \right) |U_2^{(l)}|^2n-2 U_2^{(l)} U_2^{*(l)} \\
-2i\omega g_1(1 + k^2) &\frac{\partial U}{\partial \tau} + 2i\omega \lambda_1(1 + k^2) \frac{\partial U}{\partial \xi} - \left( 4\omega k + 2\lambda_0 k^2 + 2\lambda_0 \right) \frac{\partial^2 U}{\partial \xi \partial \tau} \\
-2i\lambda_0(\omega + \lambda_0 k) &\frac{\partial^2 U}{\partial \xi^3} - \frac{2n+1}{2n+1} \left( \frac{2n}{n} \right) \frac{\partial}{\partial \xi} \left( |U_2^{(l)}|^2n U \right) = 0.
\end{align*} \]  

(21)

For the solution of \( O(\varepsilon^{3n+1}) \) equation, only the equation associated with the coefficient of \( e^{\varphi} \) is needed. From equation (8) this can be written as

\[ \begin{align*}
-2i\omega(1 + k^2) &\frac{\partial U_2^{(l)}}{\partial \tau} + \left( \omega^2 + \lambda_0^2k^2 + \lambda_0 + 4\lambda_0\omega k - 1 \right) \frac{\partial^2 U_2^{(l)}}{\partial \xi^2} \\
+ \omega k &\left( \frac{2n}{n} \right) |U_2^{(l)}|^2n + \left( \frac{2n}{n-1} \right) |U_2^{(l)}|^2n-2 U_2^{(l)} U_2^{*(l)} \\
-2i\omega g_1(1 + k^2) &\frac{\partial U}{\partial \tau} + 2i\omega \lambda_1(1 + k^2) \frac{\partial U}{\partial \xi} - \left( 4\omega k + 2\lambda_0 k^2 + 2\lambda_0 \right) \frac{\partial^2 U}{\partial \xi \partial \tau} \\
-2i\lambda_0(\omega + \lambda_0 k) &\frac{\partial^2 U}{\partial \xi^3} - \frac{2n+1}{2n+1} \left( \frac{2n}{n} \right) \frac{\partial}{\partial \xi} \left( |U_2^{(l)}|^2n U \right) = 0.
\end{align*} \]  

(21)
Eliminating $\frac{\partial^2 U}{\partial \xi \partial \tau}$ from equation (21) through the use of (18), the following evolution equation is obtained

$$i \frac{\partial U^{(1)}}{\partial \tau} - i \frac{\partial^2 U^{(1)}}{\partial \xi^2} - \mu_2 \left[ n |U|^{2n-2} U^2 U_{x}^{(1)} + (n+1) |U|^{2n} U_t^{(1)} \right] = i S(U).$$

(22)

Here the function $S(U)$ is defined by

$$S(U) = -g_1 \frac{\partial U}{\partial \tau} + \lambda_2 \frac{\partial U}{\partial \xi} + \mu_3 \frac{\partial}{\partial \xi} \left( |U|^{2n} U \right) + \mu_4 \frac{\partial^2 U}{\partial \xi^2},$$

(23)

where the coefficients $\mu_3$ and $\mu_4$ are defined by

$$\mu_3 = \left[ - \left( \frac{\omega + \lambda_0 k}{2n+1} \right) \left( \frac{2n+1}{n} \right) + 2 \left( 2 \omega k + \lambda_0 k^2 + \lambda_0 \right) \mu_2 \right] / 2 \omega \left( 1 + k^2 \right),$$

$$\mu_4 = \left[ -2 \lambda_0 \left( \omega + \lambda_0 k \right) + 2 \left( 2 \omega k + \lambda_0 k^2 + \lambda_0 \right) \mu_1 \right] / 2 \omega \left( 1 + k^2 \right).$$

(24)

The evolution equation (23) is the degenerate (linearized) form of the generalized nonlinear Schrödinger equation with non-homogeneous term $S(U)$. If proceeded further, a hierarchy of evolution equations similar to the equation (22) can be obtained for the higher order perturbation expansion.

### 2.2 Progressive wave solution

As is well known, the form of the progressive wave solution for the nonlinear Schrödinger equation depends on the sign of the product of the coefficients $\mu_1 \mu_2$. Therefore, it might be useful first to give the explicit form of the coefficients $\mu_1$ and $\mu_2$ in terms of the wave number $k$. From the dispersion relation and the definition of the group velocity one can write

$$\omega = k \left( 1 + k^2 \right)^{-1/2}, \quad \lambda_0 = i \left( 1 + k^2 \right)^{-3/2}.$$  

(25)

Introducing these expressions into the expressions of the coefficients $\mu_1$ and $\mu_2$ one has

$$\mu_1 = \frac{3}{2} \frac{k}{(1 + k^2)^{3/2}},$$

$$\mu_2 = \frac{k}{2(1 + k^2)} \left( \frac{1}{2} \right) \left( \frac{2n+1}{n} \right).$$

(26)

Physically, the wave number $k$ is a positive quantity, so the product $\mu_1 \mu_2$ is positive.

Having determined the sign of $\mu_1 \mu_2$, in this sub-section, a progressive wave solution of the following form is proposed

$$U = f(\zeta) \exp[i(K \xi - \Omega \tau)],$$

$$U_x^{(1)} = h(\zeta) \exp[i(K \xi - \Omega \tau)].$$

(27)

Where $f(\zeta)$ is a real function, $h(\zeta)$ is a complex function, $K$ and $\Omega$ are some constants. Introducing (27) into (18) one has

$$\mu_1 f' - (\Omega + \mu_1 K^2) f + \mu_2 f^{2n+1} = 0.$$  

(28)

Here a prime denotes the differentiation of the corresponding quantity with respect to $\zeta$. 


Since the coefficients $\mu_1$ and $\mu_2$ satisfy the inequality $\mu_1^2 > 0$, the solution for $f(\xi)$ may be given by

$$f(\xi) = a \sec h^{1/n} \beta \, \xi,$$  

(29)

where $a$ is the amplitude of the solitary wave, $\Omega$ and $\beta$ are defined by

Introducing the solution given in (29) and (30) into equation (22), the governing equation for $\mu_1$ may be given by

$$\mu_1 \left\{ h'' + \frac{\beta^2}{n^2} \left[(n+1)^2 \sec h^2 \beta \xi - 1\right] h + \frac{n+1}{n} \beta^2 \sec h^2 \beta \xi h^* \right\}$$

$$= \left[ 3\mu_4 K f'' + (\Omega g_1 + K \lambda_1 - \mu_4 K^3) f + \mu_3 K f^{2n+1} \right]$$

$$+ i \left[ -\mu_4 f'' + (2\mu_1 K g_1 - \lambda_1 + 3K^2 \mu_1) f' - (2n+1)\mu_3 K f^{2n} f' \right]$$

(31)

From the form of equation (31), the function $h(\xi)$ should have the form

$$h(\xi) = h_1(\xi) + ih_2(\xi).$$

Here $h_1(\xi)$ and $h_2(\xi)$ satisfy the following differential equations

$$= 3\mu_4 K f'' + (\Omega g_1 + K \lambda_1 - \mu_4 K^3) f + \mu_3 K f^{2n+1},$$

(32)

$$\mu_1 \left\{ h_1'' + \frac{\beta^2}{n^2} \left[(n+1)\sec h^2 \beta \xi - 1\right] h_1 \right\}$$

$$= -\mu_4 f'' + (2\mu_1 K g_1 - \lambda_1 + 3K^2 \mu_1) f'(2n+1)\mu_3 K f^{2n} f'.$$

(33)

Introducing the explicit expression of $f$ into equations (32) and (33) the followings are obtained:

$$h_1'' + \frac{\beta^2}{n^2} \left[(n+1)(2n+1)\sec h^2 \beta \xi - 1\right] h_1 = \left( \alpha_1 + \alpha_2 \sec h^2 \beta \xi \right) \sec h^2 \beta \xi$$

(34)

$$h_2'' + \frac{\beta^2}{n^2} \left[(n+1)\sec h^2 \beta \xi - 1\right] h_2 = \left( \gamma_1 + \gamma_2 \sec h^2 \beta \xi \right) \sec h^{1/n} \beta \xi \tanh \beta \tau$$

(35)

where the coefficients $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are defined by

$$\alpha_1 = a K \left[ 3\mu_4 K \frac{\beta^2}{n^2} + (\Omega g_1 + K \lambda_1 - \mu_4 K^3) \right] / \mu_1$$

$$\alpha_2 = a K \left[ -3\mu_4 \frac{\beta^2}{n^2} (n+1) + \mu_3 a^{2n} \right] / \mu_1$$
\[ \gamma_1 = \frac{a \beta}{n} \left[ -3 \mu_4 \frac{\beta^2}{n^2} - (2 \mu_4 K g_1 - \lambda_1 + 3 \mu_4 K^2) \right] / \mu_4, \]

\[ \gamma_2 = \frac{a \beta}{n} \left[ \mu_4 \frac{\beta^2}{n^2} - (n+1)(2n+1) - (2n+1)3 \mu_4 a^2 n \right] / \mu_4. \]

(36)

In equation (34) the term \( \alpha_1 = \sec h^{1/n} \beta \xi \) on the right hand side causes the secularity in the solution for this equation. Similarly, in equation (35) the term \( \gamma_1 = \sec h^{1/n} \beta \xi \tanh \beta \xi \) causes the secularity in the solution. In order to remove the secularities in the solution the coefficients \( \alpha_1 \) and \( \gamma_1 \) must vanish, which yields

\[ 3 \mu_4 K \frac{\beta^2}{n^2} + (\Omega g_1 + K \lambda_1 - \mu_4 K^2) = 0, \]

The remaining parts of the equations (33) and (34) become

\[ h_i' + \frac{\beta^2}{n^2} \left[ (n+1)(2n+1) \sec h^2 \beta \xi - 1 \right] h_i = \frac{a_1}{\mu_1} \sec h^{1/n^2} \beta \xi \]

(39)

\[ h_2' + \frac{\beta^2}{n^2} \left[ (n+1) \sec h^2 \beta \xi - 1 \right] h_2 = \frac{\gamma_2}{\mu_1} \sec h^{1/n^2} \beta \xi \tanh \beta \xi. \]

(40)

Following Demiray [14], the particular solution to these equations are given by

\[ h_i = \frac{1}{2} \left( -3 \frac{\mu_4}{\mu_1} + \frac{\mu_1}{\mu_2} \right) \frac{K a}{n} \sec h^{1/n} \beta \xi \]

(41)

Hence in terms of the real physical quantities, the solution becomes

\[ u = e a \sec h^{1/n} \beta \xi \left\{ 1 + e^{n} \frac{1}{2} \left( -3 \frac{\mu_4}{\mu_1} + \frac{\mu_1}{\mu_2} \right) \frac{K a}{n} + ie^{n} \frac{2n+1}{2n^2} \left( \frac{\mu_4}{\mu_1} - \frac{\mu_1}{\mu_2} \right) \beta \tanh \beta \xi \right\} e^{i \phi} + c.c... \]

(43)

where

\[ \xi = e^{n} (x - \lambda_1 t) + e^{2n} 2 \mu_4 K t + e^{3n} \mu_4 \left( \frac{\beta^2}{n^2} - 3 K^2 \right) t + ... \]
\[ \phi = K \varepsilon^n (x - \lambda_0 t) + \varepsilon^{2n} \Omega t + \varepsilon^{3n} \mu_4 K \left( \frac{3 \beta^2}{n^2} - K^2 \right) t + ... \]  

(44)

Here it is seen that the second order correction terms to the wave speed of enveloping and harmonic waves are, respectively, given by

\[ \mu_4 (\beta^2 / n^2 - 3K^2) \]  

and \[ \mu_4 K(3 \beta^2 / n^2 - K^2). \]

From the general formulation, the evolution equations for various order of nonlinearities can be obtained for symmetric regularized long-wave equation. For instance, when \( n = 1 \), i.e., cubic nonlinearity, the evolution equations become

\[ i \frac{\partial U}{\partial \tau} - \mu_2 \frac{\partial^3 U}{\partial \xi^3} - \mu_2 |U|^2 U = U, \] 

(45)

\[ \frac{\partial U^{(1)}}{\partial \tau} - \mu_1 \frac{\partial^2 U^{(1)}}{\partial \xi^2} - \mu_2 U^2 U^{(1)} + 2 |U|^2 U^{(1)} = i S(U) \] 

(46)

where

\[ S(U) = -g_1 \frac{\partial U}{\partial \tau} + \lambda_1 \frac{\partial U}{\partial \xi} + \mu_3 \frac{\partial (|U|^2)}{\partial \xi} + \mu_4 \frac{\partial^3 U}{\partial \xi^3} \] 

(47)

Here the coefficients \( \mu_1, \mu_2, \mu_3 \) and \( \mu_4 \) are defined by

\[ \mu_1 = \frac{\omega^2 + \lambda_0^2 (1 + k^2)}{2 \omega (1 + k^2)} \]

\[ \mu_2 = \frac{k}{2(1 + k^2)} \]

\[ \mu_3 = \left[ - \langle \omega + \lambda_0 k \rangle + 2(2 \omega k + \lambda_0 k^2 + \lambda_0) \mu_2 \right] / 2 \omega (1 + k^2) \]

\[ \mu_4 = \left[ -2 \langle \omega + \lambda_0 k \rangle + 2(2 \omega k + \lambda_0 k^2 + \lambda_0) \mu_4 \right] / 2 \omega (1 + k^2). \] 

(48)

This result is exactly the same with those of given given in [15]. By using this general formulation one can obtain the evolution equations and associated progressive wave solutions for various order nonlinearities.

**Conclusion**

By utilizing the modified reductive perturbation method developed by Demiray [10], the higher order perturbation expansion for the amplitude modulation of the generalized symmetric regularized long-wave equation is studied and the related evolution equations and various correction terms for the amplitude and the wave speed are obtained so as to remove some secular terms that might occur in the solution. The results indicate that, with this method, it is possible to obtain a progressive wave solution to evolution equations of any order perturbation expansion without causing the secularities. However, it is not possible to obtain similar results through the use of the conventional reductive perturbation method [4, 6].

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References


