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AN ITERATIVE METHOD FOR SOLVING TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAYS

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Abstract. This article deals with an iterative method which is a new formulation of Adomian decomposition method for solving time-fractional partial differential equations (TFPDEs) with proportional delays. The fractional derivative taken here is in Caputo sense. Daftardar-Gejji and Jafari (2006) proposed this new technique where the non-linearity is defined by using the new formula of Adomian polynomials and the new iterative formula (NIF) is independent of λ. It does not require any discretization, perturbation, or any restrictive parameters. It is shown that the NIF converges rapidly to the exact solutions. Three test problems have been illustrated in order to confirm the efficiency and validity of NIF.

Keywords: Adomian Polynomials, Partial differential equations, Proportional delay, Fractional Calculus.

AMS Subject Classification: 35R11, 65Mxx, 65Dxx.

1. Introduction

Fractional differential equations involving fractional derivatives are generalizations of classical differential equations of integer order. Fractional partial differential equations (FPDEs) are widely used as models to express many important physical phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics and chemical physics. The literature related to fractional differential equation is very much rich. Due to mathematical complexity the development of analytical solutions are very few and are restricted to solutions of simple FDEs. Due to its demonstrated applications in numerous diverse fields of science and engineering, the widely researched subject of fractional calculus has gained considerable importance and popularity over the past three decades. In recent years, more and more attention has been attracted in the realm of fractional differential equations. The fractional approach has become a strong

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modeling methodology, and it is commonly used in materials and mechanics, anomalous diffusion, wave propagation and turbulence, etc [1], [2], [3], [4], [5].

In general, for nonlinear fractional partial differential equations, there is no method that yields an exact solution. In order to obtain approximate solutions, several powerful methods for solving FPDEs were proposed, such as the homotopy analysis method (HAM) [6], the homotopy perturbation method (HPM) [7], the Adomian decomposition method [8], [9], the meshless method [10], the operational matrix [11] and so on.

In this paper, we consider the following type of TFPDEs with proportional delays, which are of the form

$$ {}_{a}^{C}D_t^{\alpha}u(x,t) = \mathcal{F}\left(x, t, u(p_0x, q_0t), \frac{\partial}{\partial x}u(p_1x, q_1t), ..., \frac{\partial^n}{\partial x^n}u(p_nx, q_nt)\right), \tag{1} $$

with the initial condition,

$$ u^{(k)}(x,0) = g_k(x), \tag{2} $$

where $p_i, q_i \in (0,1)$ for $i, j \in \mathbb{N}$, and $\mathcal{F}$ denotes the nonlinear operator. Very few numerical techniques have been used to solve TFPDEs with delays since the past two decades. Zubik-Kowal[12] used the Chebyshev pseudospectral method for solving linear differential and differential-functional parabolic equations, Zubik-Kowal together with other researchers used the spectral collocation and waveform relaxation methods[13] as well as iterated pseudospectral method[14] for nonlinear delay partial differential equations. By employing the extended two-dimensional differential transform method, Abazari and Ganji[15] obtained approximate solutions of PDE with a proportional delay. Using differential transform method, Abazari and Kilicman[16] obtained analytical solutions of nonlinear integro-differential equations with proportional delay. Tanthanuch[17] provided an implementation of group analysis to the non-homogeneous mucilaginous Burgers equation with proportional delay. The analytical solutions of TFPDE with proportional delay were obtained by Shakeri and Dehghan[18], and Biazar and Ghanbari[19] and Sakar et al.[20] using the homotopy perturbation method. Chena and Wang[21] the well-known technique of variational iteration to solve a neutral functional-differential equation with proportional delays. Polyanin and Zhurov[22] used the technique of functional limitations to find the exact solutions of reaction-diffusion equations of nonlinear delay.

In our work, we have applied an iterative formula for solving TFPDEs. In 2006, Daftardar-Gejji and Jafari [23] proposed this new method for solving linear as well as nonlinear functional equations. This iterative method is formulated from well-known Adomian decomposition method, where the non-linearity is defend by using a new formula of Adomian polynomials and this new formula is independent of λ. The method converges to the exact solution if it exists through successive approximations. As needed by some current methods, the NIM does not involve any restrictive assumptions for nonlinear terms. Applications of this Adomian decomposition method can also be found in the works of Ismael et al.[24], [25].

This article is structured as follows. We start with some definitions of the fractional calculus which are essential to establish our results. The new iterative method is discussed in Section 3. In Section 4, three numerical results are illustrated to clarify the present method and comparisons are done with the existing results. Section 5 summarizes this paper briefly, followed by references.

2. Preliminaries

Riemann-Liouville Fractional Integral: The Riemann-Liouville fractional integral [1] of order $\alpha > 0$ of a function f is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t > 0, \ \alpha \in \mathbb{R}^+, \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.
Caputo Fractional Derivative: The fractional derivative of $f(t)$ in the Caputo sense is defined by

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, & m-1 < \alpha < m, \ m \in \mathbb{N} \\ \frac{d^m f(t)}{dt^m}, & \alpha = m, \ m \in \mathbb{N} \end{cases} \tag{4}$$

where α denotes the order of the derivative (real or complex). Here, in this work only real and positive α are considered.

3. New Iterative Method

In this section, we discuss an iterative method introduced by Daftardar-Gejji and Jafari [23], which is used for solving the nonlinear functional equations of the form

$$u = f + L(u) + N(u), \tag{5}$$

where f is a known function, L and N are linear and nonlinear operators respectively. The NIM solution for the Eq.(5) has the form

$$u = \sum_{i=0}^{\infty} u_i. \tag{6}$$

The convergence of the series (6) has been discussed in [26].
Since L is linear

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \tag{7}$$

The nonlinear operator N in Eq.(5) is decomposed as below

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \\ &= \sum_{i=0}^{\infty} \hat{A}_i, \end{aligned} \tag{8}$$

where

$$\begin{aligned}
\hat{A}_0 &= N(u_0)\\
\hat{A}_1 &= N(u_0+u_1) - N(u_0)\\
\hat{A}_2 &= N(u_0+u_1+u_2) - N(u_0+u_1)\\
&\vdots\\
\hat{A}_i &= \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\},\ i \geq 1.
\end{aligned}$$

Using Eqs. (6),(7) and (8) in Eq.(5), we get

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} \hat{A}_i. \tag{9}$$

The solution of eq. (1) can be expressed as

$$u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \cdots + u_n + \cdots , \tag{10}$$

where

$$\begin{aligned}
u_0 &= f\\
u_1 &= L(u_0) + \hat{A}_0\\
u_2 &= L(u_1) + \hat{A}_1\\
&\vdots\\
u_n &= L(u_{n-1}) + \hat{A}_{n-1}\\
&\vdots
\end{aligned}$$

Algorithm

$$\begin{aligned}
INPUT :&Read\ M\ (Number\ of\ iterations);\ \alpha\\
&Read\ L(u);\ N(u);\ f\\
Step-1 :&u_{-1} = 0\\
&u_0 = f\\
Step-2 :&For\ (n=0,\ n \leq M,\ n++)\\
&\{\\
Step-3 :&\hat{A}_n = f(u_n) - f(u_{n-1});\\
Step-4 :&\overline{u}_{n+1} = J^{\alpha}(L(u) + \hat{A}_n);\\
Step-5 :&u_{n+1} = \overline{u}_{n+1} + u_n;\\
Step-6 :&u = u_{n+1}\\
&\}\ end\\
OUTPUT :&\ u
\end{aligned}$$

4. Convergence

In this section, we present the convergence and error estimation of the proposed iterative method for solving Eqs.(1) and (2).

Theorem 4.1. *Let $u_n(\xi,\tau)$ and $u(\xi,\tau)$ be defined in Banach space $(C[0,1],\|.\|)$. Then the series solution $\{u(\xi,\tau)\}_{n=0}^{\infty}$ defined by Eq.(10), if $0<k<1$.*

Proof. Assuming that $(C[0,1],\|.\|)$ is the Banach space with norm,

$$\|u(\xi,\tau)\| = \max_{\forall\xi,\tau\in[0,1]} |u(\xi,\tau)|.$$

Let us define $\{S_n\}$ be the sequence of partial sum of Eq.(10) as,

$$\begin{cases} S_0 = u_0(\xi,\tau) \\ S_1 = u_0(\xi,\tau) + u_1(\xi,\tau) \\ S_3 = u_0(\xi,\tau) + u_1(\xi,\tau) + u_2(\xi,\tau) \\ \vdots \\ S_n = u_0(\xi,\tau) + u_1(\xi,\tau) + u_2(\xi,\tau) + \cdots + u_n(\xi,\tau) \end{cases} \tag{11}$$

and we have to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(C[0,1],\|.\|)$. To show this, let

$$\begin{aligned} \|S_{n+1} - S_n\| =& \|u_{n+1}(\xi,\tau)\| \le k\|u_n(\xi,\tau)\| \le k^2\|u_{n-1}(\xi,\tau)\| \le \cdots \\ &\le k^{n+1}\|u_0(\xi,\tau)\|. \end{aligned} \tag{12}$$

Now for every,$n,m\in\mathbb{N}, n\ge m$, by using Eq(12) and triangle inequality successively, we have,

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{m+1} - S_m)\| \\ &\le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \cdots + \|S_{m+1} - S_m\| \\ &\le k^n\|u_0(\xi,\tau)\| + k^{n-1}\|u_0(\xi,\tau)\| + \cdots + k^{m+1}\|u_0(\xi,\tau)\| \\ &= \frac{1-k^{n-m}}{1-k} k^{m+1}\|u_0(\xi,\tau)\| \end{aligned} \tag{13}$$

Since $0<k<1$, we have $1-k^{n-m}<1$; then,

$$\|S_n - S_m\| = \frac{k^{n-m}}{1-k} \max_{\forall\xi,\tau\in[0,1]} \|u_0(\xi,\tau)\|. \tag{14}$$

Since $u_0(\xi,\tau)$ is bounded,

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0 \tag{15}$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[0,1],\|.\|)$, so the series solution defined in Eq(10), converges, which completes the proof. □

5. Illustrative Examples

Example 5.1. [20] *Consider the time-fractional generalized Burgers equation with proportional delay*

$$\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + \frac{\partial}{\partial x} u(x,\frac{t}{2}) u(\frac{x}{2},\frac{t}{2}) + \frac{1}{2} u(x,t), \tag{16}$$

Table 1. Error Comparision of Example 5.1 for $\alpha = 1$

x	t	*HPM*[20]	*VIM*[27]	*NIF*
0.25	*0.25*	2.123×10^{-6}	8.789589×10^{-8}	2.32882×10^{-10}
	0.50	7.0943×10^{-5}	5.838508×10^{-6}	3.17976×10^{-8}
	0.75	5.63483×10^{-4}	6.909595×10^{-5}	5.80491×10^{-7}
	1.00	2.487123×10^{-3}	4.037904×10^{-4}	4.65436×10^{-6}
0.50	*0.25*	4.245×10^{-6}	1.757918×10^{-7}	4.65764×10^{-10}
	0.50	1.41885×10^{-4}	1.167702×10^{-5}	6.35951×10^{-8}
	0.75	1.126970×10^{-3}	1.381919×10^{-4}	1.16098×10^{-6}
	1.00	4.97425×10^{-3}	8.075809×10^{-4}	9.30872×10^{-6}
0.75	*0.25*	6.367×10^{-6}	2.636877×10^{-7}	6.98645×10^{-10}
	0.50	2.1283×10^{-4}	1.751553×10^{-5}	9.53927×10^{-8}
	0.75	1.69045×10^{-3}	2.072879×10^{-4}	1.74147×10^{-6}
	1.00	7.46137×10^{-3}	1.211371×10^{-3}	1.39631×10^{-5}

$0 \le x, t \le 1$ *and* $\alpha \in (0, 1]$, *with* $u(x, 0) = x$. *The exact solution is* xe^t *for* $\alpha = 1$. *Here,*

$$\begin{aligned} L(u) &= \frac{\partial^2}{\partial x^2} u(x,t) + \frac{1}{2} u(x,t) \\ N(u) &= \frac{\partial}{\partial x} u(x, \frac{t}{2}) u(\frac{x}{2}, \frac{t}{2}) \\ f(x) &= u(x,0) = x \end{aligned}$$

Now,

$$u(x,t) = x + J^\alpha L(u) + J^\alpha N(u),$$

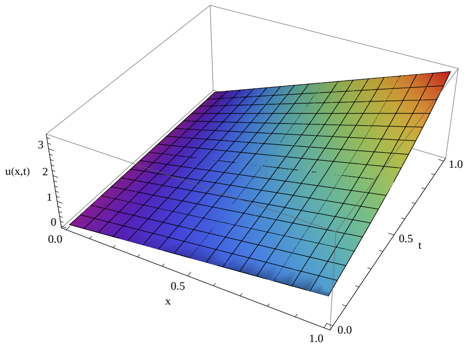
where J^α *is the fractional integral operator defined in eq.* (3). *From Section 3,* $u_0, u_1, u_2, ...$ *and* $\hat{A}_0, \hat{A}_1, \hat{A}_2, ...$ *can be calculated for* $\alpha = 1$ *as*

$$\begin{aligned} u_0 &= x \\ \hat{A}_0 &= \frac{1}{2} x \\ u_1 &= tx \\ \hat{A}_1 &= \frac{1}{8} t(4+t)x \\ u_2 &= \frac{1}{24} t^2 (12+t)x \\ \hat{A}_2 &= \frac{1}{73728} t^2 (9216 + 4992t + 768t^2 + 48t^3 + t^4)x \\ &\cdots \end{aligned}$$

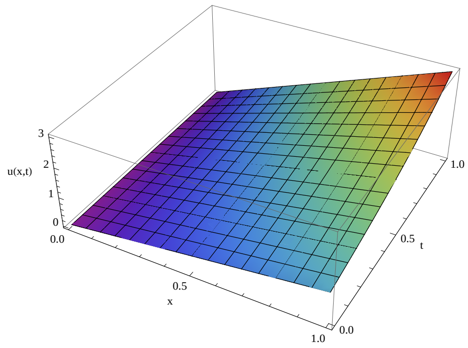
Therefore,

$$u(x,t) = x \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{972841 t^7}{5284823040} + \frac{7183013 t^8}{338228674560} + \cdots \right). \tag{17}$$

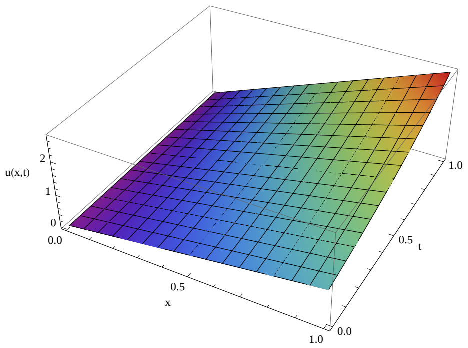
The analytical solution of Example-5.1 for $\alpha = 1$ *is calculated in eq.*(17) *and is very accurate with the exact solution. The solution by present method has been compared with the same by HPM*[20] *and VIM*[27] *and cited in Table-1. The Figures-1(A,B,C) show the behavior of approximate solution* $u(x,t)$ *for different values of* α. *The approximate*



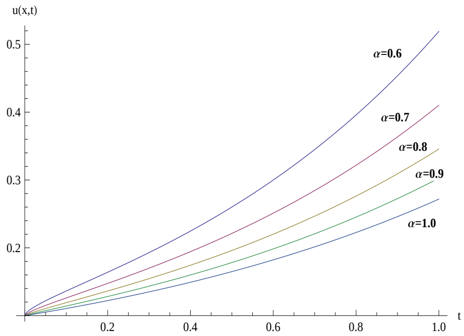
(A) $\alpha = 0.8$



(B) $\alpha = 0.9$



(C) $\alpha = 1.0$



(D) Approximate Solutions at $x = 0.1$

FIGURE 1. Behavior of NIF solution of Example 5.1

solutions of Example-5.1 for different values of α *when* $x = 0.1$ *have been shown in Figure-1(D).*

Example 5.2. [20] *Consider the fractional partial differential equation with proportional delay*

$$\mathcal{D}_t^{\alpha} u(x,t) = u(x, \frac{t}{2}) \frac{\partial^2}{\partial x^2} u(\frac{x}{2}, \frac{t}{2}) - u(x,t), \tag{18}$$

$0 \leq x, t \leq 1$ *and* $\alpha \in (0, 1]$, *with* $u(x, 0) = x^2$. *The exact solution is* $x^2 e^t$ *for* $\alpha = 1$. *Here,*

$$\begin{aligned} L(u) &= -u(x,t) \\ N(u) &= u(x, \frac{t}{2}) \frac{\partial^2}{\partial x^2} u(\frac{x}{2}, \frac{t}{2}) \\ f(x) &= u(x, 0) = x^2 \end{aligned}$$

Now,

$$u(x,t) = x^2 + J^{\alpha} L(u) + J^{\alpha} N(u),$$

TABLE 2. Error Comparision of Example 5.2 for $\alpha = 1$

x	t	*HPM*[20]	*VIM*[27]	*NIF*
0.25	0.25	5.3×10^{-7}	2.197397×10^{-8}	1.39018×10^{-11}
	0.50	1.7735×10^{-5}	1.459627×10^{-6}	3.49631×10^{-9}
	0.75	1.40870×10^{-4}	1.727399×10^{-5}	8.76459×10^{-8}
	1.00	6.21780×10^{-4}	1.009476×10^{-4}	8.51906×10^{-7}
0.50	0.25	2.123×10^{-6}	8.789589×10^{-8}	5.5607×10^{-11}
	0.50	7.0943×10^{-5}	5.838508×10^{-6}	1.39852×10^{-8}
	0.75	5.63483×10^{-4}	6.909595×10^{-5}	3.50583×10^{-7}
	1.00	2.487123×10^{-3}	4.037904×10^{-4}	3.40762×10^{-6}
0.75	0.25	4.776×10^{-6}	1.977658×10^{-7}	1.25116×10^{-10}
	0.50	1.59620×10^{-4}	1.313664×10^{-5}	3.14668×10^{-8}
	0.75	1.267830×10^{-3}	1.554659×10^{-4}	7.88813×10^{-7}
	1.00	5.596030×10^{-3}	9.085285×10^{-4}	7.66715×10^{-6}

where J^α *is the fractional integral operator defined in eq.* (3)*. From Section 3,* $u_0, u_1, u_2, ...$ *and* $\hat{A}_0, \hat{A}_1, \hat{A}_2, ...$ *can be calculated for* $\alpha = 1$ *as*

$$\begin{aligned}
u_0 &= x^2 \\
\hat{A}_0 &= 2x^2 \\
u_1 &= tx^2 \\
\hat{A}_1 &= \frac{1}{2}t(4+t)x^2 \\
u_2 &= \frac{1}{6}t^2(3+t)x^2 \\
\hat{A}_2 &= \frac{1}{1152}t^2(576+384t+84t^2+12t^3+t^4)x^2 \\
&\cdots
\end{aligned}$$

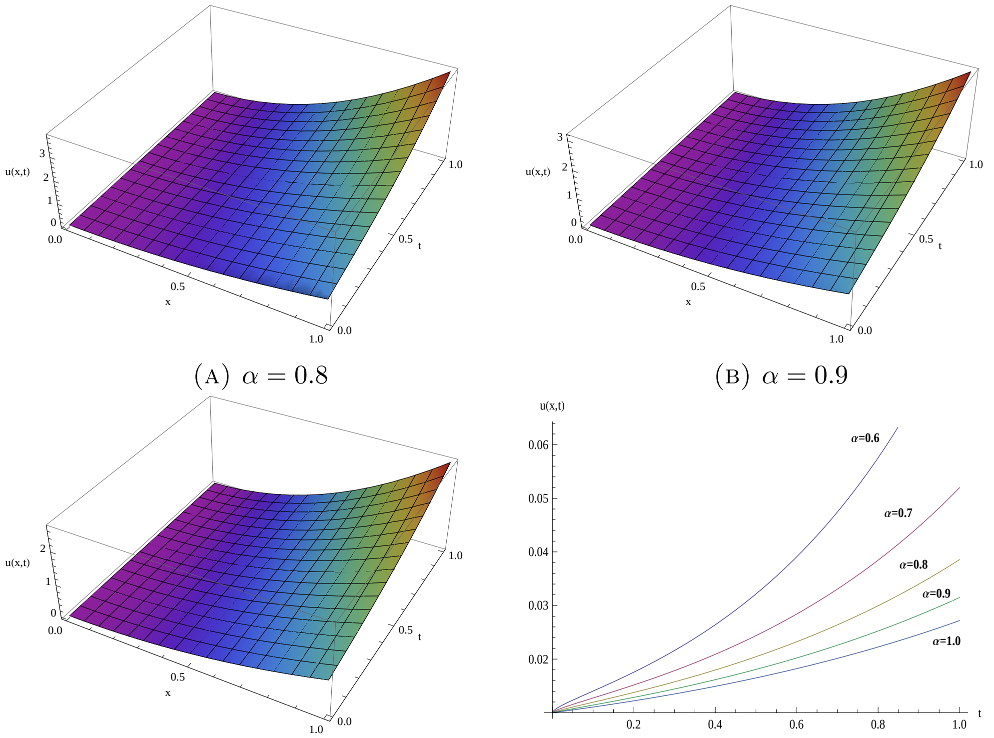
Therefore,

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \frac{6619t^8}{660602880} + \cdots\right). \tag{19}$$

The analytical solution of Example 5.2 for $\alpha = 1$ *is calculated in eq.*(19) *and is very accurate with the exact solution. The solution by present method has been compared with the same by HPM*[20] *and VIM*[27] *and cited in Table-2. The Figures-2(A),2(B),2(C) show the behavior of approximate solutions* $u(x,t)$ *for different values of* α*. The approximate solutions of Example 5.2 for different values of* α *when* $x = 0.1$ *have been shown in Figure-3(D).*

Example 5.3. [20] *Consider the fractional partial differential equation with proportional delay*

$$\mathcal{D}_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2}u(\frac{x}{2},\frac{t}{2})\frac{\partial}{\partial x}u(\frac{x}{2},\frac{t}{2}) - \frac{1}{8}\frac{\partial}{\partial x}u(x,t) - u(x,t), \tag{20}$$



(A) $\alpha = 0.8$ (B) $\alpha = 0.9$

(C) $\alpha = 1.0$ (D) Approximate Solutions at $x = 0.1$

FIGURE 2. Behavior of NIF solution of Example 5.2

$0 \le x, t \le 1$ and $\alpha \in (0, 1]$, with $u(x, 0) = x^2$. The exact solution is xe^{-t} for $\alpha = 1$. Here,

$$L(u) = -\frac{1}{8}\frac{\partial}{\partial x}u(x,t) - u(x,t)$$
$$N(u) = \frac{\partial^2}{\partial x^2}u(\frac{x}{2},\frac{t}{2})\frac{\partial}{\partial x}u(\frac{x}{2},\frac{t}{2})$$
$$f(x) = u(x,0) = x^2$$

Now,

$$u(x,t) = x^2 + J^\alpha L(u) + J^\alpha N(u),$$

where J^α is the fractional integral operator defined in eq. (3). From Section 3, $u_0, u_1, u_2, ...$ and $\hat{A}_0, \hat{A}_1, \hat{A}_2, ...$ can be calculated for $\alpha = 1$ as

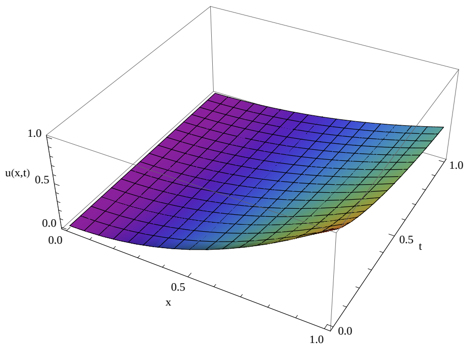
$$u_0 = x^2$$
$$\hat{A}_0 = \frac{x}{4}$$
$$u_1 = -tx^2$$
$$\hat{A}_1 = \frac{1}{16}(-4+t)tx$$
$$u_2 = \frac{1}{48}t^2x(t+24x)$$
$$\hat{A}_2 = \frac{1}{12288}t^2[t^3 + t(8-384x) + 768x + t^2(-4+48x)]$$
$$\dots$$

TABLE 3. Error Comparision of Example 5.3 for $\alpha = 1$

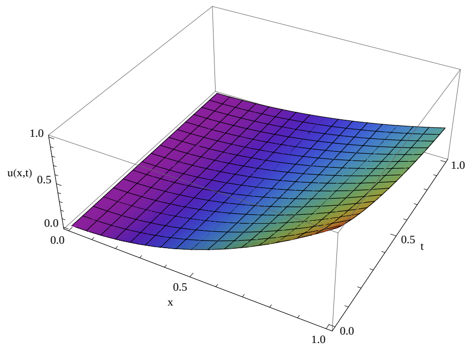
x	t	*HPM*[20]	*VIM*[27]	*NIF*
0.25	*0.25*	4.88×10^{-7}	2.045889×10^{-8}	6.13524×10^{-9}
	0.50	1.50109×10^{-5}	1.265190×10^{-6}	7.72137×10^{-7}
	0.75	1.096588×10^{-4}	1.393738×10^{-5}	1.29728×10^{-5}
	1.00	4.450349×10^{-4}	7.579841×10^{-5}	9.55782×10^{-5}
0.50	*0.25*	1.9520×10^{-6}	8.183556×10^{-8}	1.13262×10^{-8}
	0.50	6.00430×10^{-5}	5.060761×10^{-6}	1.42056×10^{-6}
	0.75	4.38636×10^{-4}	5.574951×10^{-5}	2.37896×10^{-5}
	1.00	1.7801398×10^{-3}	3.031936×10^{-4}	1.74731×10^{-4}
0.75	*0.25*	4.3940×10^{-6}	1.841300×10^{-7}	1.7985×10^{-8}
	0.50	1.350980×10^{-4}	1.138671×10^{-5}	2.25128×10^{-6}
	0.75	9.869290×10^{-4}	1.254364×10^{-4}	3.76306×10^{-5}
	1.00	4.005314×10^{-3}	6.821857×10^{-4}	2.75897×10^{-4}

Therefore,

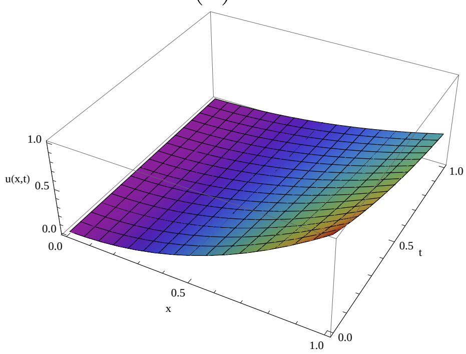
$$u(x,t) = x^2\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \frac{t^6}{720} + \cdots\right) + \cdots . \tag{21}$$



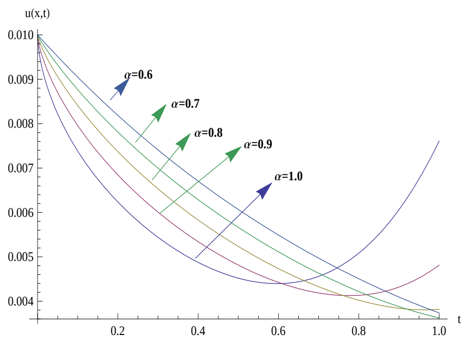
(A) $\alpha = 0.8$



(B) $\alpha = 0.9$



(C) $\alpha = 1.0$



(D) Approximate Solutions at $x = 0.1$

FIGURE 3. Behavior of NIF solution of Example 5.3

The analytical solution of Example 5.3 for $\alpha = 1$ is calculated in eq.(21) *and is very accurate with the exact solution. The solution by present method has been compared with the same by HPM*[20] *and VIM*[27] *and cited in Table-3. The Figures-3(A),3(B),3(C) show* *the behavior of approximate solutions $u(x,t)$ for different values of α. The approximate solutions of Example 5.3 for different values of α when $x = 0.1$ have been shown in Figure-3(D).*

6. Conclusion

In this work, the new iterative formula (NIF) was successfully implemented to obtain the approximate solution of TFPDEs. The suggested approximate series solutions are achieved without any discretization, perturbation, or restrictive circumstances that converge very quickly. In this technique, the non-linearity is defined by using a new formula of Adomian polynomials which is independent of λ. So, this proposed method takes less number of computations than Adomian decomposition method and other analytical methods. From the tables and figures it manifests that the proposed method gives better result than the other methods. Illustrating examples confirm the efficiency and applicability of the proposed method.

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