ASYMPTOTIC SOLUTIONS OF SEISMIC SURFACE WAVES NEAR CAUSTICS

MASOUD NEGIN

IŞIK UNIVERSITY
2014

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MASOUD NEGIN

Submitted to the Graduate School of Science and Engineering
in partial fulfillment of the requirements for the degree of
Master of Science
in
Mathematics

IŞIK UNIVERSITY
2014

IŞIK UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

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APPROVAL DATE: 6 June 2014

ASYMPTOTIC SOLUTIONS OF SEISMIC
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Abstract

The dispersive behavior of Love waves in an elastic half-space substrate covered by an elastic layer under the effect of inhomogeneous initial stresses has been investigated. Classical linearized theory of elastic waves in initially stressed bodies for small deformations is used and the well-known WKB high-frequency asymptotic technique is applied for the theoretical derivations. The influence of the imperfectness of the contact conditions on Love wave propagation velocity has also been studied through the influence of the interface imperfectness parameter on this velocity. Numerical results on the action of the imperfectness of the contact conditions on the wave dispersion curves, as well as on the influence of the initial stresses in the constituents on the wave propagation velocity for a geophysical example are presented and discussed. Possible asymptotic solutions in the vicinity of caustic points are also considered.

SİSMİK YÜZEY DALGALARIN KOSTİK NOKTALAR
CİVARINDAKİ ASİMPTOTİK ÇÖZÜMLERİ

Özet

Bu tezde, Love dalgalarının dispersiyonu lineer elastik bir tabaka ile örtülü lineer elastik malzemeden oluşmuş yarı düzlemde homojen olmayan ön gerilme etkisi altında incelenmiştir. Denklemlerin elde edilmesinde küçük yerdeyiştirmeler çerçevesinde ön gerilmeli ortanmlarda yayılan elastik dalgaların klasik lineerize edilmiş teorisi kullanılmıştır. Çözüm ise bir asimptotik dizi biçiminde tanınmış WKB yüsek frekans asimptotik yöntemini uygulayarak bulunmuştur. İdeal olmayan temas koşullarının dalga yayılma hızına verdiği etki de ayrıca incelenmiştir. Dispersiyon eğrileri hemde bir geofisik öğrnek için değişik temas koşulları ve ön gerilme deyerleri için sunulmuş ve tartışılmıştır. Kostik noktalarının civarındaki asimptotik çözümlerin olub olmadığı da göz önüne alınmıştır.

Dedicated to the memory of my grandpa

HƏSƏN ƏMOĞLU

Acknowledgements

I would like to thank the Mathematics Department at Işık University for giveing me this opportunity to persue my whole life dream to study mathematics. I would also like to express the deepest appreciation and thanks to my advisor Prof. Elman Hasanoğlu for his continuous support of my study and research, for his patience, motivation and immense knowledge.

I would like to thank all of my friends specially Rza Muxtaroğlu who supported me and encouraged me towards my goal.

Finally, I would like to thank my family, for supporting me spiritually all the way through this long education and throughout my life.

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Chapter 1

Introduction

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. ~Henri Poincare

1.1 Introduction

Seismic surface waves usually form the longest and the most powerful parts of seismic waves and produce harmful and most damaging effects of earthquakes into the buildings. Surface waves also play an important role in geological studies for site characterization, determination of shear wave velocity profiles, damping ratios, fault detection and study of the earthquakes. On the other hand, acoustic surface waves also have enormous applications in material sciences, electronic devises, non-destructive testing and damage detections.

When surface waves propagate in a layered half-space, their velocities depend upon the frequency of the vibrations, thickness, density and elastic properties of the constituents of the media, that is, they are dispersive and the relationships between velocity and frequency are usually plotted in dispersion curves.

An important issue in the study of this type of elastodynamics problems is the study of the effect of initial stresses on the wave propagation characteristics. Initial stresses might occur in composite materials or structural elements during their manufacturing or assembling processes and in the Earth's crust under the action of geostatic and geodynamic forces, for example as a result of difference of temperature, slow process of creep, differential external forces, gravity variations etc. It also important to note that the stress magnitudes in the Earth's crust are not homogeneous throughout the crust and show linear increase with depth. These stresses have a profound influence on the propagation of surface waves.

Second crucial issue in this regard is relates to study of the effect of imperfect bonding conditions between the covering layer and the half-space on a surface wave propagation. In fact, in the depths of the Earth's crust, contacts between layers of rocks are not in perfect bonding conditions and discontinuities in the form of fractures, faults and joints are common geological features. Such irregularities significantly affect the elastic wave propagation characteristics and a main consequence of that, is geometric dispersion of propagated waves. Likewise, in layered composite materials interface defects such as weak-bonding between the constituents which can be caused by interface damages or chemical actions etc. are unavoidable. Two classical boundary conditions are perfect bonded interfaces and full slipping interfaces. In perfect contact condition also known as welded interfaces all the stress and displacement components are continuous across the interface, whereas, in the case of full slipping conditions also known as non-welded interfaces there is a discontinuity in the shear component of the displacement. As mentioned above, this is not the real case and an actual interface conditions between layers is much more complicated in mathematical modeling viewpoint and different investigators spent significant efforts to describe the real physical conditions by different mechanical models.

Theory of the seismic waves is based on the theory of elasticity. However, propagation of seismic waves is a complicated process and analytical solutions of the elastodynamics equations in general types of media cannot be solved exactly and either numerical methods such as finite-difference method or approximate solutions must be used. High-frequency asymptotic theory is one of the possible approaches and is a powerful technique introduced to solve these kind of equations approximately. The high-frequency asymptotic methods usually are presented in the form of the, so to say, ray series. For this reason, the ray method is also often called the ray series method, or the Asymptotic Ray Theory (ART). Indeed, in asymptotic ray theory the solution of the elastodynamics equation is approximated in the form of a ray series. However, the classical asymptotic ray theory fails near singularities and it is valid only for regions where the ray field is regular. The points of the ray, at which the ray Jacobian vanishes, are called caustic points. Theoretically, caustics correspond to points at which the amplitude of the signal is infinite. This can be viewed as limitations to the classical asymptotic ray theory, nevertheless, it provide valuable insight into the physical phenomena.

In the present study, dispersive behavior of surface waves in the half-space substrate covered by a layer, which are assumed to be linear and elastic, under the effect of inhomogeneous initial stresses with imperfect interface conditions is investigated. Classical linear elasticity theory with small initial deformations is applied and the Wentzel–Kramers–Brillouin (WKB) high-frequency asymptotic technique is adopted for the theoretical derivations. We are particularly interested in asymptotic solutions in the vicinity of caustic points.

1.2 Literature Review

Study of the elastodynamics problems became necessary in the second half of the 20^{th} century. Since then, elastic waves have been studied elaborately and a large amount of literature can be found in the standard books such as Ewing *et al* (1957); Biot (1965); Achenbach (1973); Eringen and Suhubi (1975). After then a large amount of investigations have been done and published in many journals. More recently, a detail study of elastic wave propagation from seismological point of view also have been made by Pujol (2003) and Chapman (2004).

Amount of works related to the study of wave propagation in initially stresses bodies are enormous. Here we will present a few of those numerous studies in line with the purpose of our study. Dey *et al* (1986) studied the possibility of propagation of

Rayleigh waves in an initially stressed incompressible half-space under a rigid layer and showed that such waves cannot propagate in an isotropic medium without tensile initial stress. Ahmed (2000) investigated the effect of initial stress on the propagation of Rayleigh waves in a granular medium under incremental thermal stresses. Liu *et al* (2001) studied the propagation behavior of Love waves in a layered piezoelectric structure with an initial stress and investigated the effect of the initial stress on the phase velocity of the propagation. Qian *et al* (2004, 2009) studied the effect of inhomogeneous initial stresses on dispersion of Love waves in a piezoelectric layered structure by solving the coupled electromechanical field equations. Du *et al* (2008) analytically investigated the dispersive and attenuated characteristics of Love waves under the effect of initial stress in piezoelectric layered structures loaded with viscous liquid. Qian *et al* (2010) investigated the effect of initial stress on the propagation behavior of Love waves in a piezoelectric half-space of polarized ceramics carrying a functionally graded material layer for both electrical open case and electrical short case, respectively. Chattaraj *et al* (2011) studied the propagation of torsional surface wave in fluid saturated poro-elastic layer lying over nonhomogeneous elastic half-space under compressive initial stress. Gupta *et al* (2011, 2012) studied the influence of irregularity, initial stress and porosity on the propagation of torsional surface waves in an initially stressed anisotropic poro-elastic layer over a semi-infinite heterogeneous half-space with linearly varying rigidity and density. Kakar and Kakar (2013) also investigated the effects of the gravity, rotation and magnetic field on the propagation of Rayleigh waves in a pre-stressed inhomogeneous, orthotropic elastic solid medium.

The influence of interface conditions on elastic wave propagation also have been investigated using several theoretical methods in the literature depending on the character of the problem to be considered. Martin (1992) provided a brief review of different imperfect interface models in the literature and formulated the problem mathematically. He applied a simple linear modification to the perfect interface continuity conditions to model various intermediate imperfect conditions. In a particular case, this model is performed as a shear-spring type imperfect interface model, according to which, only shear displacements have discontinuity across this interface and the jump of this discontinuity is connected linearly with the corresponding shear stress. An application and review of the related investigations on the influence of the shear-spring type imperfectness have been considered in many papers. To summarize some, Pecorari (2001) investigated the scattering problem of a Rayleigh wave by surface-breaking cracks with partial contact interfaces. Leungvichcharoen and Wijeyewickrema (2003) discussed the effect of an imperfect interface on harmonic extensional wave propagation in a pre-stressed, symmetric layered composite by employing shear spring type resistance model to simulate the imperfect interface. Melkumyan and Mai (2008) studied the effects of imperfect bonding in piezoelectric/piezomagnetic composites and showed that imperfection of the interface bonding has significant impact on the existence of interface waves and on their velocities of propagation. Kumara and Singh (2009) considered the propagation of plane waves at an imperfectly bonded interface of two orthotropic generalized thermoelastic rotating half-spaces with different elastic and thermal properties. Liu *et al* (2010) analyzed *SH* surface waves in a piezoelectric elastic layer and an elastic half-space structure with imperfect bonding. Zhou *et al* (2012) also tried to simulate the imperfect interface conditions by using linear spring model to study bulk wave propagation in laminated piezomagnetic and piezoelectric plates with initial stresses. Vishwakarma *et al* (2014) considered different types of imperfect interface to study the propagation of a torsional surface wave in a homogeneous crustal layer over an initially stressed mantle with varying rigidities, density and initial stresses. This completes our consideration of the investigations related to wave propagation in initially stresses bodies with perfect and/or imperfect interface conditions.

We now consider a brief review of the investigations related to the asymptotic solution of high-frequency wave propagation problems in layered half-spaces as these investigations are also relevant to our study. Seismic ray theory presents an approximate high-frequency solution to the elastodynamics equation. Seismic ray theory has been described elaborately in several text books, for example, Babich and Buldyrev (1991), Cerveny (2001), Chapman (2004). Again, number of studies which applied the high-frequency asymptotic approximation to wave propagation problems are enormous and here we will present a few of recent studies in the direction of our study purpose. Li *et al* (2004) studied the propagation behaviors of Love waves in inhomogeneous medium using WKB method and obtained the dispersion relations of Love waves for different gradient variation of material constants. Jin *et al* (2005) also applied the WKB method to solve the Rayleigh surface wave propagation in a homogeneous isotropic elastic structures with curved surfaces of arbitrary form. Liu *et al* (2007) on the basis of WKB method derived the dispersion equations for Love wave propagation in layered graded composites structures using the shear spring model for the rigid, imperfect, and slip interface cases. Cao *et al* (2008) employed the WKB technique for the asymptotic solutions of propagation of Rayleigh surface waves in a transversely isotropic graded piezoelectric half-space when material properties varying continuously along depth direction. Liu *et al* (2009) obtained the asymptotic solutions of Love waves by applying the WKB method and solved the fourth order differential equation with variable coefficients to investigate the effects of gradient variations of the piezoelectric and dielectric constants. Qian *et al* (2009, 2010) investigated the existence and propagation behavior of transverse surface waves in a layered structure concerning a gradient metal layer by WKB method and obtained the dispersion equation for such structures. Balogun and Achenbach (2013) studied the surface waves generated by a time-harmonic line load on an isotropic linearly elastic half-space whose elastic moduli and mass density vary with the depth direction.

1.3 Scope and Objectives

The scope of this thesis is a contribution to understanding of the dispersive behavior of surface waves in a half-space substrate covered by a layer under the effect of linearly varying initial stresses. Classical linear elasticity theory with small initial deformations will be applied and the WKB high-frequency asymptotic technique is applied for the theoretical derivations. We are particularly interested in asymptotic solutions in the vicinity of caustic points.

First, the dispersion of surface waves in the half-space which is assumed to be linear and elastic and covered also by linear elastic layer with perfect contact interfaces will be investigated considering pre-stressing in the layer and in the half-space. Then, the same analysis is repeated this time considering the effect of imperfect interface conditions.

1.4 Organization of thesis

The thesis is divided into five chapters. The present chapter provides general introduction to the subject. Chapter 2 presents a brief review of the wave propagation theory in elastic media. Derivation of the elastodynamics equation of motion is described concisely and different types of surface waves in an elastic half-space covered by a layer is presented. Chapter 3 describes the method of WKB for high-frequency approximation of the elastic waves. Basic concepts and brief derivation of the asymptotic ray series for a simple second order differential equation is presented and the implementation of the method to obtain approximate solution for surface wave propagation is discussed. Chapter 4 is dedicated to the formulation of the problem and some numerical analysis are performed. And finally some conclusion and discussions about the results and some suggestions for the future works is presented in chapter 5.

Chapter 2

Elastic Waves

2.1 Wave Equation

In the linear theory of elasticity it is assumed that the deformations are very small and the relation between stress and strain is linear. Although this may not give an exact description of the physical problem, it provides a very useful solution which is reasonable as long as those assumptions are valid (Rose J.L. 1999). Consider Cauchy's equations of motion:

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \tag{2.1}$$

To derive the wave equation, we assume that a continuum is isotropic and homogeneous. Thus, the corresponding stress-strain relations can be written as follows (Hooke's law):

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{2.2}$$

where λ and μ are Lame's constants and the strain-displacement relations are:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

where σ is the Cauchy's stress tensor, ε is the strain tensor, ρ is the material density, δ_{ij} is the Kronecker delta function, u represent the displacement in the spatial coordinate of the system and f represent the external forces on the material particles which are assumed to be zero in our formulations. Note that the second order derivative of u with respect to time is represented by $\ddot{u}$. Combining the equations (2.1), (2.2) and (2.3) in terms of the displacement u yields the equation of motion in elastodynamics for isotropic homogeneous materials:

$$(\lambda+\mu)u_{j,ji}+\mu\, u_{i,jj}=\rho\ddot{u}_i. \tag{2.4}$$

Note that we use the standard Einstein's index notation in all the above equations. We can also state equations (2.4) concisely using vector calculus notation. The first summation term in the left-hand side of these equations is the divergence of u, namely, $\nabla\cdot\mathbf{u}$, while the second summation term is Laplace's operator, namely, ∇^2. Therefore, we can rewrite equations (2.4) as:

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\,\nabla^2\mathbf{u}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}. \tag{2.5}$$

This is the equations of motion, i.e. equation (2.4), in vector representation. It can be shown using Helmholtz decomposition that the displacement field u decomposes into two independent vector fields for plane-wave assumptions (Slawinski 2007). These two fields represent two different kinds of waves and are solutions to the equations:

$$\nabla^2\varphi=\frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2}, \tag{2.6}$$

$$\nabla^2\psi=\frac{1}{\beta^2}\frac{\partial^2\psi}{\partial t^2}, \tag{2.7}$$

for the longitudinal and the shear waves, respectively. Here φ and ψ are scalar and vector potentials of the field, and

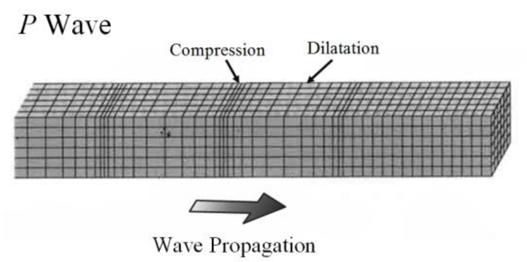
$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}, \tag{2.8}$$

$$\beta=\sqrt{\frac{\mu}{\rho}}, \tag{2.9}$$

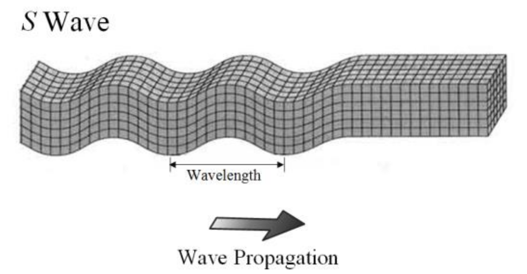
are propagation velocities of the longitudinal and the shear waves, respectively. Indeed, it follows from these equations that two types of elastic waves can propagate in a homogeneous isotropic medium, namely longitudinal and shear or transverse waves.

2.2 Body Waves

In general, there are two principle types of elastic body waves: *Longitudinal waves* and *Transverse waves*. Longitudinal waves are called in seismology *P* or primary waves, because they represent the first arriving waves on seismograms. Transverse or shear waves, however, are called *S* or secondary waves, because they appear in seismograms after the primary waves. In longitudinal waves, the particle motion is in the direction of wave propagation, while in shear waves, the particle motion is normal to the direction of propagation (Figure 2.1). The velocities of longitudinal waves, α, and of transverse waves, β, in a homogeneous, isotropic medium are given by the equations (2.8) and (2.9) which is derived in the previous section, respectively.



(a)



(b)

Figure 2.1 Body Waves: (a) *P* wave; (b) *S* wave.

2.3 Surface Waves

In addition to the body waves that propagate in all directions into the medium, there exists another type of waves which for the most part restricted to the free boundary or surface of the medium and for that reason are called *surface waves*. These waves propagate along the surface of the body and their amplitudes rapidly decrease as the distance from the boundary goes to infinity. There are two the most important types of surface waves, *Rayleigh waves* and *Love waves*, which take their names after the scientists who studied them for the first time. Rayleigh waves essentially occur by the interactions of compressional waves (*P* waves) and vertically polarized shear waves (*S* waves) with the free boundaries of the medium, however, Love waves take place in a system consisting of a layer over a half-space or in general in layered structures when the velocity of *S* wave increase with depth. An important characteristic of surface waves is that they are dispersive, i.e. their propagation velocity is frequency dependent. In the following sections of this chapter we will consider this feature of surface waves more precisely.

2.3.1 Rayleigh Waves

As mentioned above, Rayleigh waves travel only along the free surface of an elastic bodies. The particle motion is elliptic and retrograde with respect to the direction of propagation (Figure 2.2). The components of displacement contain a vertical component and a horizontal component and the amplitude of the particle motion in these waves decreases exponentially with depth. Rayleigh waves in an elastic half-space are non-dispersive, i.e. phase and group velocities of propagation are equal and do not depend on the frequency of the waves. However, when surface waves propagate in a layered half-space, their velocities depend on the frequency of the vibrations, thickness of the layer, density and elastic properties of the constituents of the layers and the half-space. Therefore, they are dispersive and the relationships between velocity and frequency (or wavelength) with parameters of the elastic properties are plotted as dispersion curves. In this case they are called generalized

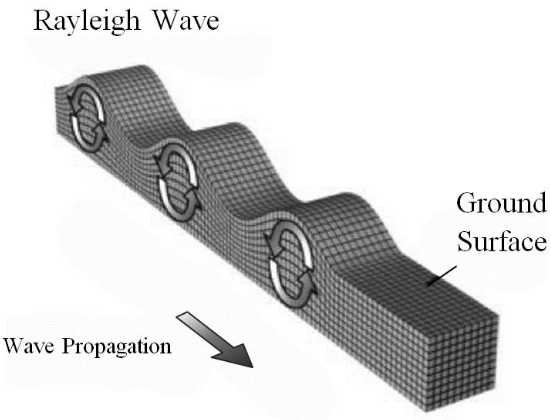


Figure 2.2 Rayleigh Wave.

Rayleigh waves. It was established by Tolstoy and Usdin (1953) that the dispersion equation of the generalized Rayleigh waves has infinitely many modes unlike ordinary Rayleigh waves, which can propagate only in one mode.

Consider an elastic layer over an elastic half-space as shown in Figure 2.3. Wave propagation is two dimensional, thus the displacement components along x and z directions *u* and *w* are non-zero, while displacement component *v* along *y* direction is set to be zero, that is

$$\begin{cases} u_m = u_m\left(x,z,t\right) \\ v_m = 0 \\ w_m = w_m\left(x,z,t\right) \end{cases}, \qquad m = 1,2 \tag{2.10}$$

and Rayleigh wave propagate in the positive direction of *x* axis. Note that the values related to the layer and the half-space are denoted by lower index $m = 1,2$, respectively. According to Helmholtz decomposition rule of the displacement vector, these displacement fields can be described by the potentials for the longitudinal waves, φ, and transverse waves, ψ, in the following form:

$$u_m = \frac{\partial \varphi_m}{\partial x} - \frac{\partial \psi_m}{\partial z}, \tag{2.11}$$

$$w_m = \frac{\partial \varphi_m}{\partial z} + \frac{\partial \psi_m}{\partial x}, \tag{2.12}$$

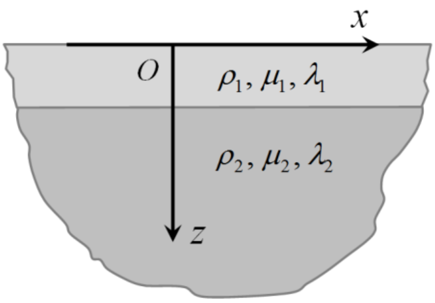


Figure 2.3 System of a layer over a half-space.

where the potentials must satisfy the wave equations:

$$\nabla^2 \varphi_m = \frac{1}{\alpha_m^2} \frac{\partial^2 \varphi_m}{\partial t^2}, \tag{2.13}$$

$$\nabla^2 \psi_m = \frac{1}{\beta_m^2} \frac{\partial^2 \psi_m}{\partial t^2}. \tag{2.14}$$

We are looking for the harmonic solutions of equations (2.13) and (2.14), which represent harmonic wave propagation along *x* direction. Thus,

$$\varphi_m = \Phi_m(z)\, e^{ik(x-ct)}, \tag{2.15}$$

$$\psi_m = \Psi_m(z)\, e^{ik(x-ct)}, \tag{2.16}$$

where k is the wavenumber and c is the phase velocity of wave propagation. By inserting displacements (2.15) and (2.16) into the waves equations (2.13) and (2.14), we obtain the following general solutions to the corresponding differential equations considering the decay conditions for the half-space:

$$\varphi_1 = \left(A\cos kpz + B\sin kpz\right) e^{ik(x-ct)}, \tag{2.17}$$

$$\psi_1 = \left(C\cos kqz + D\sin kqz\right) e^{ik(x-ct)}, \tag{2.18}$$

$$\varphi_2 = Ee^{-krz} e^{ik(x-ct)}, \tag{2.19}$$

$$\psi_2 = Fe^{-ksz} e^{ik(x-ct)}, \tag{2.20}$$

where

$$p = \sqrt{\frac{c^2}{\alpha_1^2} - 1}, \qquad q = \sqrt{\frac{c^2}{\beta_1^2} - 1}, \tag{2.21}$$

$$r = \sqrt{1 - \frac{c^2}{\alpha_2^2}}, \qquad s = \sqrt{1 - \frac{c^2}{\beta_2^2}}\,. \tag{2.22}$$

2.3.2 Love Waves

While Rayleigh waves exist at free surface of a body, Love waves require some kind of a wave guide like an elastic layer over a half-space to propagate. The particles motion in Love waves are only horizontal and perpendicular to the direction of propagation (Figure 2.4). These wave are also dispersive because their velocity obviously depends on the frequency of the propagation. Moreover, several modes of propagation exist, because of the periodic nature of their dispersion function. We will consider this type of waves again in chapter 4 where we mathematically derive their equation of motion and their dispersion relation profoundly. It is worthy to mention once more that, like Rayleigh waves, Love waves also have significant importance in many engineering and scientific disciplines. For example, it is used by earthquake engineers for understanding the causes and amount of damages to the buildings and by seismologist for studying the subsurface structure and properties of the Earth's crustal layer. Love waves propagating in the piezoelectric materials are also extensively used in electronic devices such as sensors and transducers.

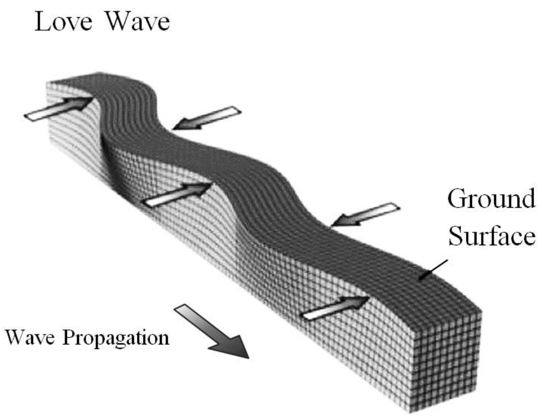


Figure 2.4   Love Wave.

Consider again an elastic layer over an elastic half-space as shown in Figure 2.3. Now, we will consider only the displacement v along y direction which is perpendicular to the direction of wave propagation, while the displacement components along x and z directions, u and w both are zero, thus $v = v(x, z, t)$, $u = w = 0$. In this case, the equations of motion for *SH* type waves is written as:

$$\nabla^2 v_1 = \frac{1}{\beta_1^2} \frac{\partial^2 v_1}{\partial t^2}, \tag{2.23}$$

$$\nabla^2 v_2 = \frac{1}{\beta_2^2} \frac{\partial^2 v_2}{\partial t^2}, \tag{2.24}$$

where

$$\beta_1^2 = \frac{\mu_1}{\rho_1}, \qquad \beta_2^2 = \frac{\mu_2}{\rho_2}, \tag{2.25}$$

v_1 and v_2 are displacement components in the layer and the half-space, respectively. The resulting harmonic wave propagation along x direction for equations (2.23) and (2.24) can be written as:

$$v_1(x, z, t) = V_1(z)\, e^{ik(x-ct)}, \tag{2.26}$$

$$v_2(x, z, t) = V_2(z)\, e^{ik(x-ct)}, \tag{2.27}$$

where k is the wavenumber, c the phase velocity of wave propagation, $V_1(z)$ and $V_2(z)$ are two undetermined functions with respect to z coordinate only. By inserting displacements (2.26) and (2.27) into the waves equations (2.23) and (2.24) we obtain the following second order ordinary differential equations for undetermined functions $V_1(z)$ and $V_2(z)$:

$$V_1''(z) + k^2 \left(\frac{c^2}{\beta_1^2} - 1 \right) V_1(z) = 0, \tag{2.28}$$

$$V_2''(z) + k^2 \left(\frac{c^2}{\beta_2^2} - 1 \right) V_2(z) = 0, \tag{2.29}$$

The general solutions of the second order differential equations (2.28) and (2.29) can be expressed as follows:

$$V_1(z) = Ae^{ikpz} + Be^{-ikpz}, \tag{2.30}$$

$$V_2(z) = Ce^{ikqz} + De^{-ikqz}, \tag{2.31}$$

where

$$p = \sqrt{\frac{c^2}{\beta_1^2} - 1}, \qquad q = \sqrt{\frac{c^2}{\beta_2^2} - 1}, \tag{2.32}$$

then,

$$v_1(x,z,t) = \left(Ae^{ikpz} + Be^{-ikpz} \right) e^{ik(x-ct)}, \tag{2.33}$$

$$v_2(x,z,t) = \left(Ce^{ikqz} + De^{-ikqz} \right) e^{ik(x-ct)}, \tag{2.34}$$

and *A*, *B*, *C*, *D* are arbitrary constants which must be determined from boundary conditions.

2.4 Dispersion of Waves

If propagation velocity of waves depends on frequency we say that the waves are dispersive. In a dispersive medium a wave changes its shape during propagation, because its spectral components propagate with different velocities. This may cause some technical problems in the measurements of the velocity of propagation or in their transmission. However this process can be used to study the properties of the medium which the waves have propagated through.

There are two types of wave dispersion: material dispersion and geometrical dispersion. The material dispersion as its name implies is due to the change in the material properties of the structure. For example, this type of dispersion is familiar from optics, since the velocity of light in media depends on its frequency. The geometrical dispersion, on the other hand, is because of the interference of waves. This type of waves dispersion occur when the waves propagate in layered structures, or along the surface of a medium. As mentioned in the previous sections we shall study this type of dispersion in our investigations.

2.5 Boundary Conditions

In general quantitative evaluation of physical properties of a considered problem requires adequate knowledge and understanding of its nature mathematically. Several theoretical methods are available in the literature for studying the influence of interfaces on elastic wave propagation depending on the character of the problem to be considered. The most frequently used method by many investigators is the *displacement discontinuity method*.

Two classical boundary conditions, that are, perfect bonded interfaces and full slipping ones idealize real physical contact between two layers. In perfect contact condition also known as *welded interfaces* all the stress and displacement components are continuous across the interface, whereas, in the case of full slipping conditions also known as *non-welded interfaces* there is a discontinuity in the shear component of the displacement (Rokhlin and Wang 1991).

Now we consider the formulation of the imperfect contact conditions on the interface plane between the covering layer and the half-space. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities of the displacements and forces across the mentioned interface. A review of the mathematical modeling of the various types of incomplete contact conditions for elastodynamics problems has been detailed in a paper by Martin (1992). It follows from this paper that for most models the discontinuity of the displacement $\mathbf{u}^{+}$ and force $\mathbf{f}^{+}$ vectors on one side of the interface are assumed to be linearly related to the displacement $\mathbf{u}^{-}$ and force $\mathbf{f}^{-}$ vectors on the other side of the interface. This statement, as in the paper by Rokhlin and Wang (1991), can be presented as follows:

$$[\mathbf{f}] = \mathbf{C}\mathbf{u}^{-} + \mathbf{D}\mathbf{f}^{-}, \quad [\mathbf{u}] = \mathbf{G}\mathbf{u}^{-} + \mathbf{F}\mathbf{f}^{-}, \tag{2.35}$$

where **C**, **D**, **G** and **F** are three-dimensional (3×3) matrices and the square brackets indicate a jump in the corresponding quantity across the interface. Consequently, if the interface is at say $z = h$, then:

$$\left[\mathbf{u}\right]=\mathbf{u}\big|_{z=h^{+}}-\mathbf{u}\big|_{z=h^{-}}, \quad \left[\mathbf{f}\right]=\mathbf{f}\big|_{z=h^{+}}-\mathbf{f}\big|_{z=h^{-}}. \tag{2.36}$$

It follows from (2.35) that we can write incomplete contact conditions for various particular cases by selection of the matrices **C**, **D**, **G** and **F**. One such selection was made in the paper by Jones and Whitter (1967), according to which, it was assumed that $\mathbf{C}=\mathbf{D}=\mathbf{G}=\mathbf{0}$. In this case the following can be obtained from (2.35):

$$\left[\mathbf{f}\right]=\mathbf{0}, \quad \left[\mathbf{u}\right]=\mathbf{F}\mathbf{f}^{-}, \tag{2.37}$$

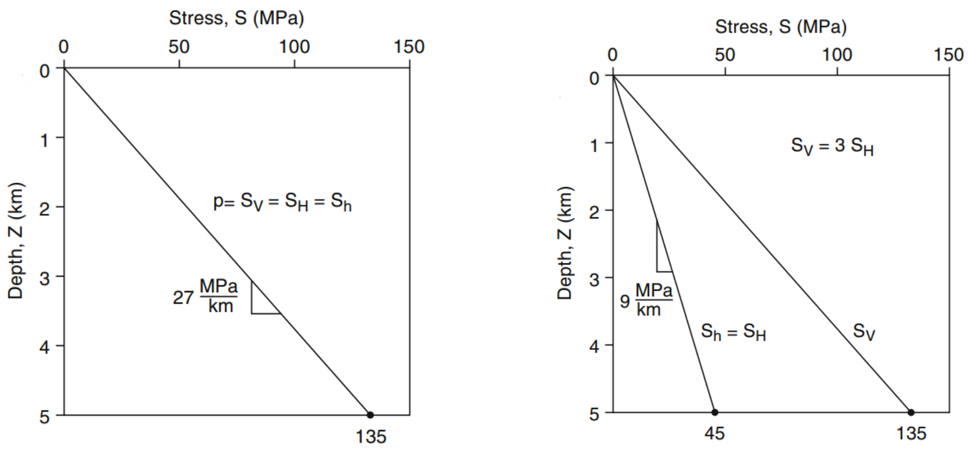
where **F** is a constant diagonal matrix. The model (2.37) simplifies significantly the solution procedure of the corresponding problems and is adequate in many real cases. Therefore, this model is called a shear-spring type resistance model and has been used in many investigations carried out within the framework of classical elastodynamics. According to this statement, we also use the model (2.37) for the mathematical formulation of the imperfectness of the contact conditions in our investigations.

2.6 Elastic Waves in Bodies with Initial Stresses

Problems related to the nonlinear effects of the elastic waves arise in almost all fields of modern engineering branches such as civil, mechanical, aircraft and geophysical engineering and many others. One of the most common sources of nonlinearity in such problems associated to the initial stress state in the constituents of the medium. However, problems related to study of the wave propagation in initially stressed bodies cannot be solves within the framework of the classical linear theory of elasticity. Although many attempts have been done going back to the 19th century, but it did not achieve noticeable progress until the second half of the 20th century that the general nonlinear theory of elastic waves intensively developed. Therefore, at the present time, most studies of these types of elastodynamics problems are made in the framework of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWIB). The general concepts and equations of the TLTEWIB have been presented elaborately in many classical texts such as Biot (1965), Eringen and Suhubi (1975) and others. It should be noted that, the equations of the TLTEWIB are obtained from exact relations of the nonlinear theory of elasticity in initially stressed bodies by linearization of the equations with respect to small deformation.

2.7 Crustal Stress Model

The Earth is not a rigid body at rest and the Earth's crust moves continuously caused by plate tectonics. In addition, earthquake ruptures also create discontinuous movements in different regions of the crust. Theses geological processes generate different types of stress fields in the crustal regions of the Earth. Moreover, anisotropy and heterogeneity of rock mass also can create structural stresses. In the context of rock mechanics, stresses within the Earth's crust are usually described in terms of three principal stress components as the vertical stress generally due to the weight of the above rock masses (S_V), the minimum horizontal stress (S_h) and the maximum horizontal tectonic stress (S_H). In rock mechanics also the state of stress in which these three principal stresses are equal, (i.e. $S_V = S_H = S_h$), is referred to as *lithostatic* stress. As shown in Figure 2.5a, in the Earth's crust the increase of lithostatic stress magnitudes with depth is linear with a slope of about $27\,\text{MPa/km}$.



(a) (b)

Figure 2.5 Stress models: (a) lithostatic, and (b) biaxial state of stress in the Earth's crust (Zang and Stephansson, 2010).

For rock having Poisson ratio of $\upsilon = 0.25$, due to the biaxial state of stress model the horizontal stresses are each equal to one-third of the vertical stress (Figure 2.5b). Consequently, as stresses increase with depths by $27\,\mathrm{MPa/km}$ in the lithostatic case (Figure 2.5a), the slope of horizontal stress with depth is only $9\,\mathrm{MPa/km}$ in the biaxial case (Figure 2.5b). Note that, this law is valid at least in the top third of the continental crust beneath Europe down to 9.1 km and approaching temperatures of 265°C close to the brittle ductile transition (Zang and Stephansson 2010).

Chapter 3

Asymptotic Ray Theory

3.1 Introduction

The propagation of seismic waves generally in inhomogeneous and anisotropic structure of the Earth is very complicated processes. In such a complex structure, analytical solutions of the elastodynamics equations cannot be obtained exactly and standard modeling techniques cannot be used. Common approaches to face such types of problems are:

- Numerical solutions of the elastodynamics equations based on the methods such as finite-difference or finite-element methods.
- Asymptotic solutions of the elastodynamics equations using approximate high-frequency analysis. The most well-known representative of these techniques is the Wentzel-Kramers-Brillouin (WKB) method which is based on the *Asymptotic Ray Theory*. The solution is usually given in the form of the so-called ray series or asymptotic series. In fact, this is another way of using an infinite series to describe a functions at a point. It is also interesting to note that most of the time even only the leading term of such a series make desired accuracy for the problem (zero-order ray solution).

Asymptotic ray theory (ART) for elastodynamics problems has been introduced for the first time by Babich (1956) and Karal and Keller (1959). The theory has been described elaborately in several text books, such as, Nayfeh (1981), Bender (1999), Holms (1995) and Awrejcewicz and Krysko (2006). Here we only give some basic definitions and ideas behind asymptotic analysis and consider a brief review of the principle concepts related to the asymptotic solution of high-frequency wave propagation problems.

3.2 Asymptotic Approximation

Definitions:

1. We say:

$$f(x) = O\left(g(x)\right) \text{ as } x \to x_0, \tag{3.1}$$

if there is a constant *A* such that:

$$|f(x)| \leq A|g(x)|,$$

for all *x* sufficiently near x_0.

If one function is much smaller than another we write:

$$f(x) = o\left(g(x)\right) \text{ as } x \to x_0, \tag{3.2}$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

2. Given two functions $f(x)$ and $g(x)$, we say that g is an asymptotic approximation to f as $x \to x_0$ whenever the relative error between f and g goes to zero as $x \to x_0$:

$$\lim_{x \to x_0} f(x) - g(x) = 0,$$

where f and g are defined on some interval containing x_0. Or equivalently:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1. \tag{3.3}$$

We denote this equivalency by $f \sim g$ as $x \to x_0$. Usually g is called the *gauge* function.

3. A set of gauge functions $\{g_n(x)\}, n = 1, 2, ...$ is an asymptotic sequence as $x \to x_0$ if:

$$g_{n+1}(x) = o\left(g_n(x)\right).$$

4. We say that $f(x)$ has an asymptotic expansion with respect to the asymptotic sequence $\{g_n(x)\}$ if there are constants a_k such that for each *n*,

$$f(x)=\sum_{k=0}^{n}a_k\, g_k(x)+o\big(g_k(x)\big) \text{ as } x\to x_0,$$

or

$$f(x)\sim\sum_{k=0}^{n}a_k\, g_k(x) \text{ as } x\to x_0. \tag{3.4}$$

5. The power series $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is said to be asymptotic to function $f(x)$ as $x\to x_0$ if:

$$\lim_{x\to x_0}\left(f(x)-\sum_{n=0}^{\infty}a_n(x-x_0)^n\right)=0,$$

then we write:

$$f(x)\sim\sum_{n=0}^{\infty}a_n(x-x_0)^n. \tag{3.5}$$

We can simply find the coefficients of the asymptotic series from definition (3.5), starting from computing a_0, then a_1 and so on. Note that, asymptotic series are divergent most of the time. It definitely makes no sense to sum terms of an asymptotic series up to some high number. In fact, we usually take just the first few terms of an asymptotic series or even the leading term only.

3.3 WKB Method

As mentioned earlier, wave propagation in inhomogeneous and anisotropic media is a complex physical phenomena, whose analytic solution is difficult or may be impossible. However, at high-frequency excitation of the medium the process can be accurately described by a relatively simple asymptotic expression. Practically, high-frequency asymptotic approximation is valid if no significant changes in the elastic parameters of the medium occur within a wavelength of excitation. The asymptotic representation that we describe in this section is the well-known WKB expansion. The exponential approximation known as a WKB approximation, named after Wentzel, Kramers, and Brillouin who popularized the theory. Consider wave equation:

$$\nabla^2 v = n^2 \frac{\partial^2 v}{\partial t^2}, \tag{3.6}$$

where ∇^2 is Laplacian operator. We are interested in the time-harmonic response of the equation (3.6) and so let:

$$v(\boldsymbol{x},t) = u(\boldsymbol{x})\, e^{-i\omega t}. \tag{3.7}$$

Substituting this into the equation (3.6) yields what is known as Helmholtz equation or the reduced wave equation, given as:

$$\nabla^2 u + \omega^2 n^2 u = 0. \tag{3.8}$$

We are looking for the solution of equation (3.8) in the following form:

$$u = Ae^{i\omega S} \tag{3.9}$$

where A is the amplitude and S is the eikonal, both depend on the coordinates. According to WKB procedure, in order to find the asymptotic solution of (3.8) for large values of ω we substitute equation (3.9) into equation (3.8) and equate the coefficients of power of ω and ω^2 to zero, we got equations for S and A as:

$$\left(\nabla S\right)^2 = n^2, \tag{3.10}$$

$$2\nabla S \cdot \nabla A + A\,\Delta S = 0. \tag{3.11}$$

Equation (3.10) is known as the *eikonal* equation in optics. This is a first-order nonlinear partial differential equation which plays a very important role in optics and in wave propagation in elastic media. Physically, this equation determines a surface of constant phase as *wave fronts*. The solution of this nonlinear equation can be obtained for example using the classical characteristics or ray method for the first-order nonlinear partial differential equations (e.g. Courant and Hilbert 1989). Advantage of the characteristics method is that the partial differential equation is replaced by a set of ordinary differential equations, which are in general, easier to solve. It should be mentioned that, it is customary to write characteristics equations in terms of Hamiltonians, so, in Hamiltonian formulation we can obtain ray equations as follows:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i},$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, 3. \tag{3.12}$$

Equation (3.11) is called the *transport* equation. In fact, the transport equation describes the amplitude along the wave front. Although this is a linear ordinary differential equation its character might be complicated, however, it turns out that the transport equation can be solved in terms of rays as well.

Theoretically WKB method can be applied to multi-dimensional wave equation with partial derivatives, however, here we only consider one dimensional wave equation in more details because it describes our model for surface wave equation which we will use in the next chapter.

We are interested to approximate the solution of differential equations of the form:

$$y'' + \omega^2 q(x) y = 0, \tag{3.13}$$

where ω^2 is a large parameter. Closed form solution of the equation (3.13) only exists for some very simple forms of the function $q(x)$. However, when ω^2 is very large then the equation (3.13) would have the solutions of the form:

$$y = A(x) e^{i\omega S(x)}, \tag{3.14}$$

where A and S are undetermined functions of *x*. Assuming,

$$A = A_0 + \frac{1}{\omega} A_1 + \frac{1}{\omega^2} A_2 + \cdots,$$

we will have:

$$y' = \left(A_0' + \frac{1}{\omega} A_1' + i\omega A_0 S' + iA_1 S' + \cdots \right) e^{i\omega S(x)} \tag{3.15}$$

$$y'' = \left(A_0'' + \frac{1}{\omega} A_1'' + i\omega A_0' S' + i\omega A_0 S'' + iA_1' S' + iA_1 S'' \right.$$
$$\left. + i\omega A_0' S' + iA_1' S' - \omega^2 A_0 S'^2 - \omega A_1 S'^2 + \cdots \right) e^{i\omega S(x)}. \tag{3.16}$$

Substituting equations (3.16) and (3.14) into equation (3.13) and equating the coefficients of each power of ω to zero, we got the following infinite number of equations:

$$S'^2 A_0 - qA_0 = 0, \tag{3.17}$$

$$S'' A_0 + 2S' A_0' = 0, \tag{3.18}$$

$$A_0'' + iS'' A_1 + 2iS' A_1' = 0, \tag{3.19}$$

$$\cdots$$

Solutions of the equations (3.17) and (3.18) can be find easily:

$$S = \pm \int \sqrt{q(x)}\, dx, \tag{3.20}$$

$$A_0 = q(x)^{-1/4}. \tag{3.21}$$

We have therefore found first-term approximation of the general solution of equation (3.13) as follow:

$$y(x) \sim q(x)^{-1/4} \exp\left[\omega \left(\pm i \int \sqrt{q(x)} dx + O(\omega^{-2}) \right) \right], \tag{3.22}$$

thus, the general homogenious solution of the equation for oscillatory case is of the form:

$$y(x) \sim q(x)^{-1/4} \left\{ A \cos \omega \left[\int \sqrt{q(x)} dx + O(\omega^{-2}) \right] \right.$$
$$\left. + B \sin \omega \left[\int \sqrt{q(x)} dx + O(\omega^{-2}) \right] \right\}, \tag{3.23}$$

where A and B are arbitrary constants. When O-terms are omitted the solution (3.23) is called the WKB approximation.

Note that the non-oscillatory solution of the equation (3.13) can also be obtained simply by changing the sign of the function $q(x)$ in that equation. We have therefore found a first-term approximation of the general solution of (3.13) for non-oscillatory case as:

$$y(x) \sim q(x)^{-1/4}\left\{C\exp\left(-\omega\int\sqrt{q(x)}dx\right) + D\exp\left(\omega\int\sqrt{q(x)}dx\right)\right\}, \tag{3.24}$$

where C and D are arbitrary constants.

As stated above in deriving the WKB approximation, the points where $q(x)$ is zero, entirely change the behavior of the solution. So, the function $q(x)$ must be nonzero and the values of x where $q(x)$ become zero are called *turning points*. Suppose there exists a single turning point x_t where $q(x) > 0$ if $x > x_t$ and $q(x) < 0$ if $x < x_t$. This means that the solution of equation (3.13) will be oscillatory if $x < x_t$ and exponential if $x > x_t$ and we must use the WKB approximation on either side of the turning point separately to obtain the general solution of the problem.

3.4 Caustics

Classical asymptotic ray theory fails near singularities. The points of the ray, at which the ray Jacobian vanishes, are called *caustic* points. Theoretically, caustics correspond to points at which the amplitude of the signal tends to infinity. This can be viewed as limitations to the classical asymptotic ray theory. In most general form in three dimensional space caustics are the envelopes of the characteristics, however, as indicated above, the solution of the transport equation also might have a singularity, which is a caustic.

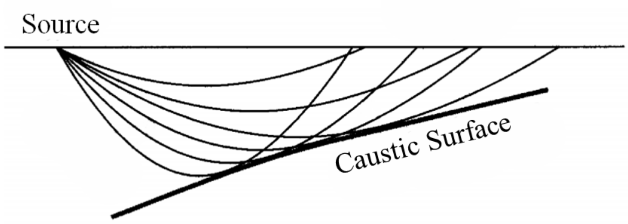


Figure 3.1 Formation of a caustic.

In optics the name caustic is given to the envelope of rays after their reflection or refraction, which is related to extraordinary brightness. The same idea applies to seismic rays associated with large amplitudes. Figure 3.1 shows a simple example of caustics. It shows the formation of a caustic as an envelope of *SH* rays reflected from the free surface of an elastic half-space in which the velocity is a linear function of depth (Ben-Menahem 1981).

It should be mentioned that the asymptotic solutions of wave equation close to caustics can also be derived, and correct amplitude of the ray can be computed, however, for these situations more detailed models such as uniform asymptotic expansions are needed (Borovikov and Kinber 1994).

Chapter 4

Numerical Results

4.1 Formulation of the Problem

Consider an elastic half-space covered by an elastic layer with thickness h as shown in Figure 4.1. Here we determine the positions of the points by the Cartesian system of coordinates $Oxyz$ with O being any point on the free surface. The layer and the half-space occupy the regions $\{-\infty < x < +\infty,\ -\infty < y < +\infty,\ 0 \le z \le h\}$ and $\{-\infty < x < +\infty,\ -\infty < y < +\infty,\ h \le z < +\infty\}$, respectively. We assume that Love wave propagate in the positive direction of x axis. Thus, the displacement component v along y direction is non-zero while the displacement components along x and z directions, u and w both are zero, i.e. $u = w = 0$, $v = v(x,z,t)$. Let the system be under initial normal compressive stress σ_z^0 along z direction and compressive or tensile initial stress σ_x^0 along x direction, respectively. The initial compressive stress σ_z^0 may be due to weight of the material of the layer and the half-space or some other external loads. However, the initial stress σ_x^0 might have been generated through other processes such as creep, temperature difference or some external forces. Note that the following notation will be used through the formulations: The values related with the covering layer and the half-space are denoted by upper indices 1 and 2, respectively. The values related to the initial stresses, though, are denoted by upper index 0.

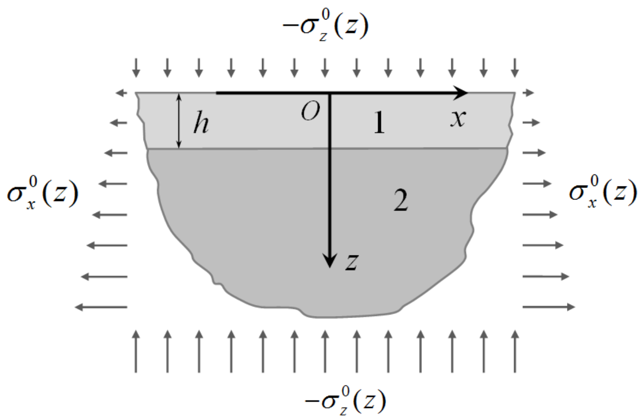


Figure 4.1 The geometry of the problem.

The dynamical equations of motion for initially stressed incompressible medium assuming small initial deformations are written as (Biot 1965),

$$\frac{\partial \sigma_{xy}^{(m)}}{\partial x}+\frac{\partial \sigma_{yz}^{(m)}}{\partial z}+\sigma_x^0\frac{\partial^2 v^{(m)}}{\partial x^2}+\sigma_z^0\frac{\partial^2 v^{(m)}}{\partial z^2}=\rho^{(m)}\frac{\partial^2 v^{(m)}}{\partial t^2}, \quad (m=1,2) \qquad (4.1)$$

where $\sigma_{xy}^{(m)}$ and $\sigma_{yz}^{(m)}$ are components of the Cauchy stress tensor, $v^{(m)}$ are the components of the displacement vector along y direction, $\rho^{(m)}$ are the mass density of the layer and the half-space, respectively. Note that, constitutive relations for a linear isotropic elastic solid are given by:

$$\sigma_{xy}^{(m)}=\mu^{(m)}\frac{\partial v^{(m)}}{\partial x}, \quad \sigma_{yz}^{(m)}=\mu^{(m)}\frac{\partial v^{(m)}}{\partial z}, \quad (m=1,2) \qquad (4.2)$$

where $\mu^{(m)}$ is the shear modulus or Lame's second parameter. The strain components in equation (4.2) can be calculated through the following formula:

$$\varepsilon_x^{(m)}=\frac{\partial u^{(m)}}{\partial x}, \quad \varepsilon_z^{(m)}=\frac{\partial w^{(m)}}{\partial z}, \quad \varepsilon_{zx}^{(m)}=\frac{1}{2}\left(\frac{\partial u^{(m)}}{\partial z}+\frac{\partial w^{(m)}}{\partial x}\right), \ (m=1,2). \qquad (4.3)$$

The displacements components of the considered system can be assumed to have the following form:

$$v^{(m)}(x,z,t) = V^{(m)}(z)\, e^{ik(x-ct)}, \quad (m = 1,2) \tag{4.4}$$

where k is the wavenumber, c the phase velocity of wave propagation, $i = \sqrt{-1}$, $V^{(1)}(z)$ and $V^{(2)}(z)$ are two undetermined functions with respect to z coordinate only. This way we obtain the following equation for $V^{(1)}(z)$ and $V^{(2)}(z)$ from the equation (4.1)–(4.4) as:

$$\frac{\mathrm{d}^2 V^{(1)}(z)}{\mathrm{d}z^2} + k^2 q^{(1)}(z)\, V^{(1)}(z) = 0, \tag{4.5}$$

$$\frac{\mathrm{d}^2 V^{(2)}(z)}{\mathrm{d}z^2} - k^2 q^{(2)}(z)\, V^{(2)}(z) = 0, \tag{4.6}$$

where

$$q^{(1)}(z) = \frac{-\mu^{(1)} - \sigma_x^0(z) + \rho^{(1)} c^2}{\mu^{(1)} + \sigma_z^0(z)}, \tag{4.7}$$

$$q^{(2)}(z) = \frac{\mu^{(2)} + \sigma_x^0(z) - \rho^{(2)} c^2}{\mu^{(2)} + \sigma_z^0(z)}. \tag{4.8}$$

Equations (4.5) and (4.6) are second order differential equation with variable coefficients and in general obtaining the exact solution of the problem is very difficult. However, for high-frequency waves whose wavenumber is very large, i.e. $k >> 1$, the WKB asymptotic approximation method can be applied to obtain approximate solution of the problem. Thus by assuming that k is a large number then $\varepsilon = 1/k << 1$ will be a very small number and the equations (4.5) and (4.6) in general form can be recast as:

$$\varepsilon^2 \frac{\mathrm{d}^2 V(z)}{\mathrm{d}z^2} + q(z)\, V(z) = 0. \tag{4.9}$$

We are looking for the solution of equation (4.9) in the following form:

$$V(z) \sim e^{i\phi(z)/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n(z), \tag{4.10}$$

where ϕ and $S_i, i = 0,1,2,3,...$ are undetermined functions of *z*. Substituting equation (4.10) into (4.9) and equating the coefficients of each power of ε to zero, we get an infinite number of equations:

$$\phi'^2 S_0 - q S_0 = 0, \tag{4.11}$$

$$\phi'' S_0 + 2\phi' S_0' = 0, \tag{4.12}$$

$$S_0'' + i\phi'' S_1 + 2i\phi' S_1' = 0, \tag{4.13}$$

$$\dots$$

where superscript $'$ denotes differentiation with respect to the coordinate *z*. Equation (4.11) is a first order nonlinear differential equation and is called the *eikonal* equation. Its solutions can be find easily:

$$\phi = \pm \int \sqrt{q(z)}\, dz. \tag{4.14}$$

The other equations are linear and determine the higher order terms in the expansion. The second equation is called the *transport* equation, then we got the following expression for S_0 from equation (4.12),

$$S_0 = q^{-1/4}. \tag{4.15}$$

We have therefore found that a first-term approximation of the general solution of equations (4.5) and (4.6) are:

$$V^{(1)}(z) \sim q^{(1)}(z)^{-1/4} \left(A_1 e^{-\frac{i}{\varepsilon}\int \sqrt{q^{(1)}(z)}dz} + B_1 e^{\frac{i}{\varepsilon}\int \sqrt{q^{(1)}(z)}dz} \right), \tag{4.16}$$

$$V^{(2)}(z) \sim q^{(2)}(z)^{-1/4} \left(A_2 e^{-\frac{1}{\varepsilon}\int \sqrt{q^{(2)}(z)}dz} \right), \tag{4.17}$$

where A_1, B_1 and A_2 are arbitrary constants. Note that, the solution of the second equation in (4.17) satisfies the decay condition i.e. $V^{(2)}(z) \to 0$ as $z \to \infty$. Hence, the displacement components in the covering layer and half-space are given by:

$$v^{(1)}(x,z,t) \sim q^{(1)}(z)^{-1/4}\left(A_1 e^{-ik\int\sqrt{q^{(1)}(z)}dz} + B_1 e^{ik\int\sqrt{q^{(1)}(z)}dz} \right) e^{ik(x-ct)}, \tag{4.18}$$

$$v^{(2)}(x,z,t) \sim q^{(2)}(z)^{-1/4}\left(A_2 e^{-k\int\sqrt{q^{(2)}(z)}dz} \right) e^{ik(x-ct)} \ . \tag{4.19}$$

4.2 Boundary Conditions and Dispersion Equation

We assume that the following boundary conditions on the free face plane of the covering layer satisfy:

$$\sigma_{yz}^{(1)}\Big|_{z=0} = 0 \, . \tag{4.20}$$

Now we consider the formulation of the imperfect contact conditions on the interface plane between the covering layer and the half-space. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities of the displacements and forces across the mentioned interface. According to the discussion made in the previous subsection and according to Martin (1992) the mathematical formulation of the imperfectness of the contact conditions and these conditions are written as follows:

$$\sigma_{yz}^{(1)}\Big|_{z=h} = \sigma_{yz}^{(2)}\Big|_{z=h} \, , \tag{4.21}$$

$$v^{(2)}\Big|_{z=h} - v^{(1)}\Big|_{z=h} = F\,\frac{h}{\mu^{(2)}}\,\sigma_{yz}^{(2)}\Big|_{z=h}, \quad F > 0 \tag{4.22}$$

where F is a non-dimensional shear-spring parameter and $0 \le F \le \infty$. Note that the case where $F \to 0$ means that the displacement component across the interface is continuous and therefore the half-space and the covering layer are perfectly bonded together or to say that they are in welded contact condition. At the other extreme, $F \to \infty$ implies that the half-space and the covering layer are completely unbounded together and the full slipping condition is satisfied. Thus, any other finite positive values of F expresses different imperfect interface conditions in the problem.

Substituting of the equations (4.18) and (4.19) and their corresponding stress displacement components into the equations of motion (4.1) and considering the boundary conditions (4.20)-(4.22) yields the system of three homogenous algebraic equations for A_1, B_1 and A_2. For a nontrivial solution the determinant of the coefficients must vanish giving the dispersion equation of Love wave propagation,

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i, j = 1, 2, 3. \tag{4.23}$$

Since the explicit expressions of the α_{ij} in the dispersion equation (4.23) are cumbersome we are omitting these details. This completes the formulation of the problem and in the case where $\sigma_x^0 = \sigma_z^0 = 0$ this formulation transforms to the corresponding one made within the scope of the classical linear theory of elastodynamics.

4.3 Numerical Results

Now we perform numerical calculations here to study the quantitative and qualitative influence of initial stresses and imperfect bonding conditions on dispersion of Love wave propagation. In the following numerical example, we assume that $\rho^{(1)} = 2800\,\text{kg/m}^3$, $\beta^{(1)} = 3000\,\text{m/s}$ and $\rho^{(2)} = 3200\,\text{kg/m}^3$, $\beta^{(2)} = 5000\,\text{m/s}$ and $h = 10\,\text{km}$ (Aki and Richards 2002); where $\rho^{(1)}$, $\rho^{(2)}$ are the mass density and $\beta^{(1)}$, $\beta^{(2)}$ are the shear wave velocities in the layer and the half-space, respectively, and h is the thickness of the crustal layer. We assume that all the initial stresses in the medium are zero (i.e. $\sigma_x^0 = \sigma_z^0 = 0$), and study the effect of imperfectness of the contact conditions first. After programming of the dispersion equation (4.23), we obtain Love wave dispersion curves as Figure 4.2, which shows the dependencies between non-dimensional phase velocity $c/\beta^{(1)}$ and non-dimensional wavenumber kh for different imperfect contact conditions for the considered example. Figure 4.2 gives dispersion curves for the first four modes of Love wave propagation in the

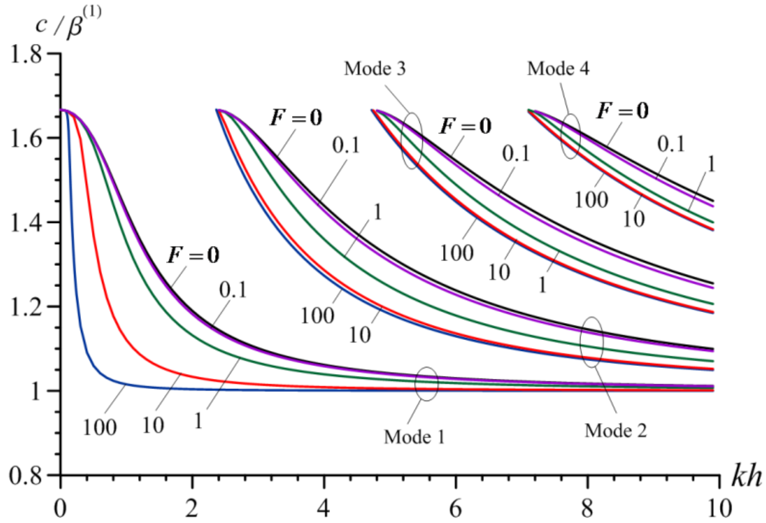


Figure 4.2 Dispersion curves for the first four mode of Love wave propagation for the considered example. The numbers in the figure fields show the values of the parameter F .

system consisting a covering layer and a half-space as described in above example. Note that, the numbers labeled on the curves correspond to the dimensionless imperfect parameter F of the related curves. Also, note that the curves given in the figure for welded contact condition, i.e. where $F=0$, coincide with corresponding ones in classical theory for perfect contact condition. It follows from Figure 4.2 that the imperfectness between the constituents causes to decrease of the wave propagation velocity for all modes of propagation of Love waves. In this cases values of the velocity decrease monotonically with the shear-spring parameter F . It is seen from the figure that the dimensionless wavenumber kh has cut-off values for the higher modes of wave propagation, however, the imperfectness of the interface conditions do not effect or change these cut-off values for the higher mode of propagation. In addition, as the figure illustrates the low and the high wavenumber limit values of the wave propagation velocities as $kh \rightarrow 0$ and $kh \rightarrow \infty$, do not depend on the imperfectness of the interface, i.e. on the parameter F . It is also seen that, in general, in lower wavenumbers the higher modes of Love wave propagation are less sensitive to interface conditions than the first mode of propagation. In other words, higher the mode number, less sensitivity to imperfectness of the interface conditions.

Now we analyze the numerical results related to the influence of the initial stresses in the constituents of the medium on the wave propagation velocity. However, before to obtain explicit expressions for dispersion equation (4.23) and consequently to obtain related dispersion curves we need to determine the variation pattern of the initial stresses in the system. It is known that approximately initial stress magnitudes in the Earth's crust show a linear increase with depth (Zang and Stephansson 2010). Accordingly, here we also assume that the initial stresses vary linearly with depth and the variation pattern of the initial stresses in both normal and transverse directions are taken as the following relations:

$$\sigma_x^0(z) = \sigma_x^0 \cdot (1 + mz) \tag{4.24}$$

$$\sigma_z^0(z) = \sigma_z^0 \cdot (1 + nz) \tag{4.25}$$

where σ_x^0, σ_z^0 denote the magnitudes and m, n denote the gradient coefficients of the inhomogeneous initial stresses in the transverse *Ox* and in the normal *Oz* directions, respectively.

For now, we assume that the magnitudes of the both initial stresses in the normal and the transverse directions do not change in depth, i.e. σ_x^0 and σ_z^0 both are constants or saying that, $m = n = 0$. We will consider different possible combinations of the initial stresses in the system. So in the following we will study the results for the five initial stress cases as:

$$\begin{aligned}
&\text{Case 1.} \quad \sigma_x^0 > 0, \quad \sigma_z^0 = 0; \\
&\text{Case 2.} \quad \sigma_x^0 < 0, \quad \sigma_z^0 = 0; \\
&\text{Case 3.} \quad \sigma_x^0 = 0, \quad \sigma_z^0 < 0; \\
&\text{Case 4.} \quad \sigma_x^0 > 0, \quad \sigma_z^0 < 0; \\
&\text{Case 5.} \quad \sigma_x^0 < 0, \quad \sigma_z^0 < 0.
\end{aligned} \tag{4.26}$$

For estimation of the magnitude of the initial stresses we also introduce a new dimensionless parameters as:

$$\psi_1 = \frac{\sigma_x^0}{\mu^{(1)}}, \quad \psi_2 = \frac{\sigma_z^0}{\mu^{(1)}}. \tag{4.27}$$

Moreover, we introduce the notation:

$$\eta = \frac{c\big|_{\psi_1 \neq 0;\ \psi_2 \neq 0} - c\big|_{\psi_1 = 0;\ \psi_2 = 0}}{c\big|_{\psi_1 = 0;\ \psi_2 = 0}}, \tag{4.28}$$

for estimation the influence of the initial stresses in the constituents, i.e. the influence of the parameters ψ_1 and ψ_2 on the wave propagation velocity. Thus, through the graphs of the dependencies between η (4.28) and kh constructed for various values of the parameters ψ_1, ψ_2 and F we analyze the effect of the imperfectness of the contact conditions between the covering layer and the half-space on the influence of the initial stresses in the constituents on the wave propagation velocity in the cases noted in (4.26).

Numerical results obtained for the above mentioned five initial stress cases are given in Figures 4.3 through 4.7, respectively. Note that in these figures the graphs indicated by letter *a* correspond to the first and the graphs indicated by letters *b* correspond to the second mode of the wave propagation. Also note that the numbers in the figure fields show the values of the imperfectness parameter F for the related curves.

Figure 4.3 shows the influence of the imperfect bonding conditions and initial stresses on the dispersion of Love wave for the first and second modes of the wave propagation in Case 1 ($\sigma_x^0 > 0, \sigma_z^0 = 0$). The graphs indicate that the initial stretching stresses in the covering layer and in the half-space causes to increase the wave propagation velocity and the velocity increase monotonically with ψ_1. Also, the graphs show that the wave propagation velocity increase monotonically with the parameter F. Consequently, the imperfectness of the contact conditions causes to increase the influence of the initial stretching in the covering layer and in the half-space on the wave propagation velocity related to the first and second modes of the propagation.

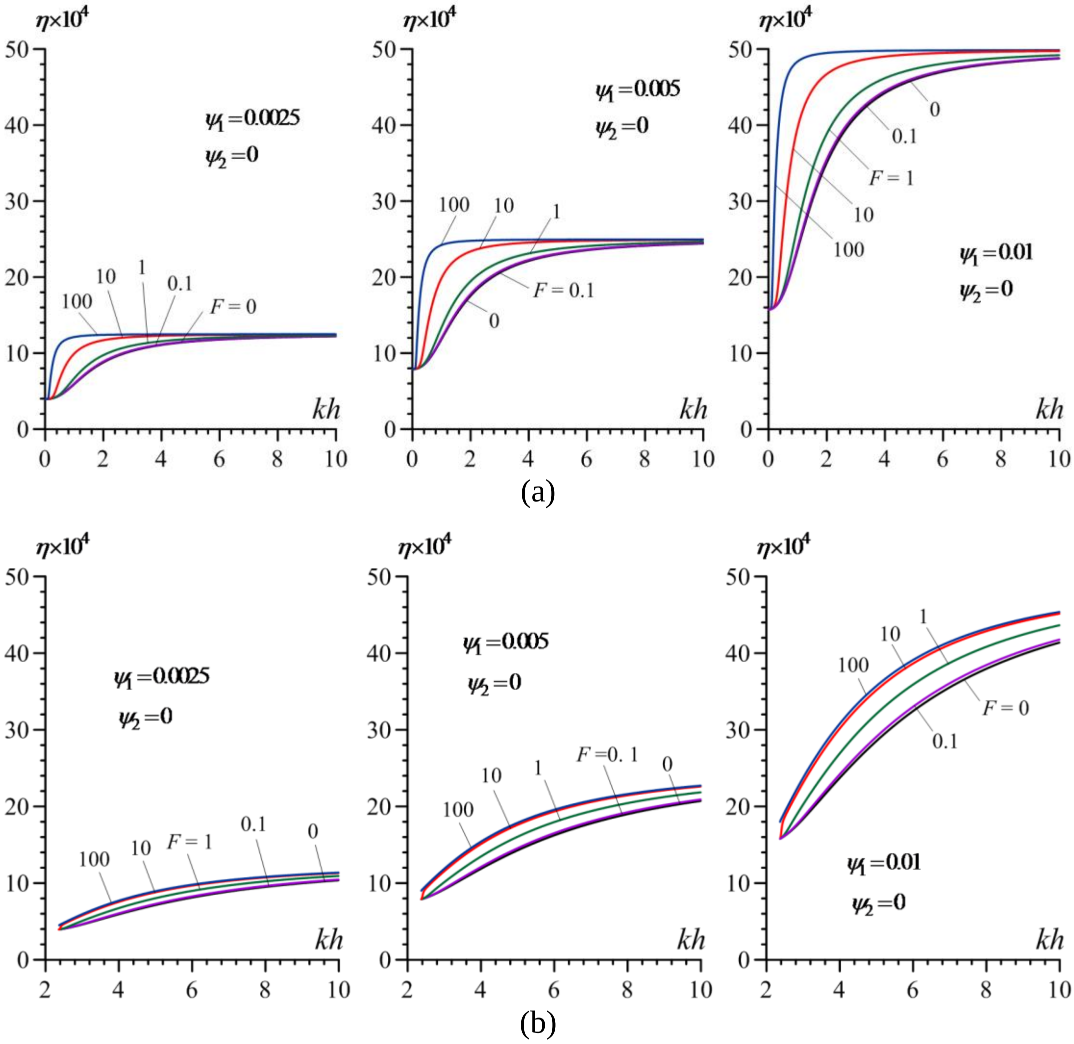


Figure 4.3 The influence of the imperfect bonding conditions and the initial stresses on the dispersion of Love wave for (a) the first and (b) the second mode of the propagation for Case 1. The numbers in the figure fields show the values of the parameter F .

Figure 4.4 shows the influence of the imperfect contact conditions and initial stresses on the dispersion of Love wave for the first and second modes of the wave propagation in Case 2 ($\sigma_x^0 < 0$, $\sigma_z^0 = 0$). These graphs show that as a result of the initial compressing stresses in the covering layer and in the half-space, unlike the initial stretching stresses as considered in previous case (Figure 4.3), the wave propagation velocity decreases. In this case the imperfectness of the contact conditions also decrease the wave propagation velocity related to the first and the second modes of propagation and the velocity decrease monotonically with ψ_1.

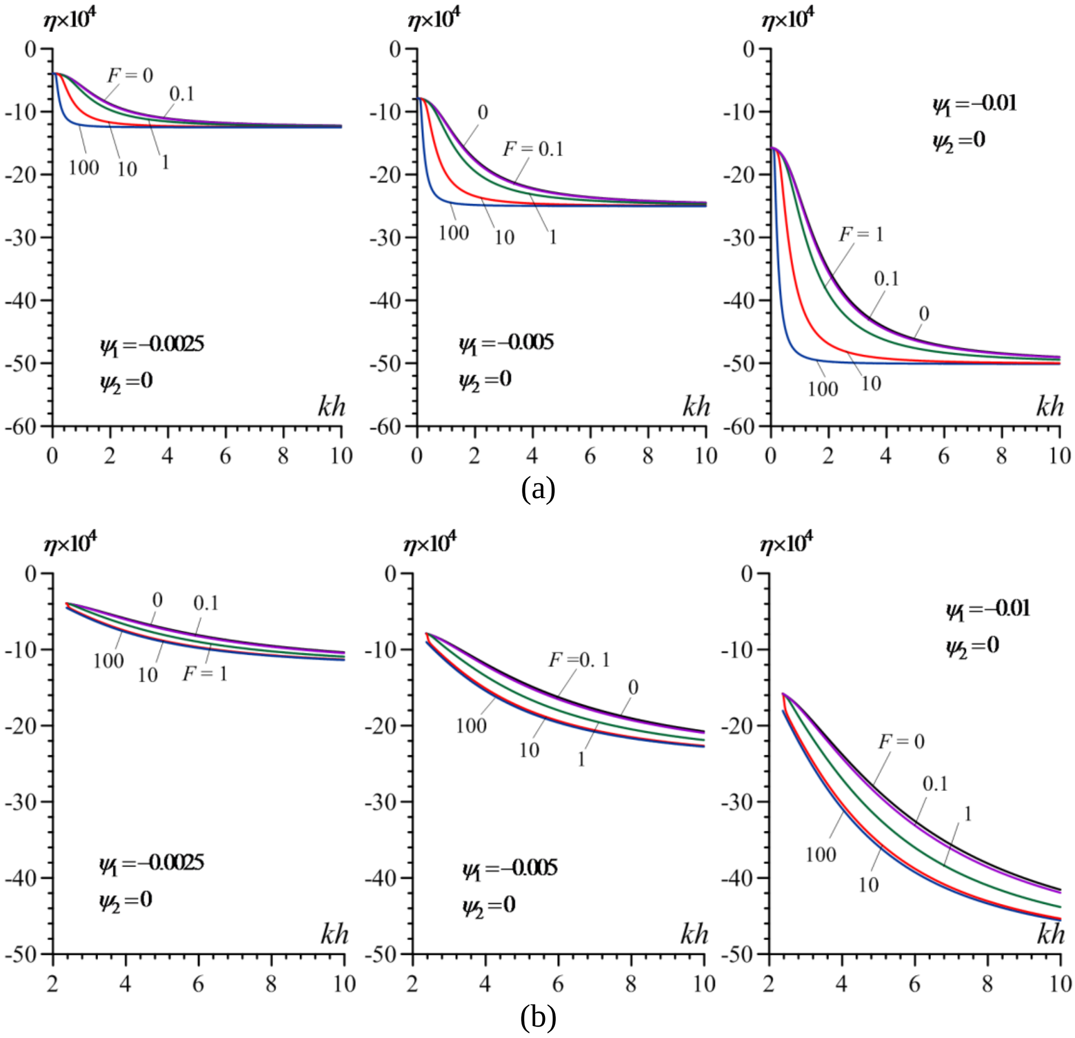


Figure 4.4 The influence of the imperfect bonding conditions and the initial stresses on the dispersion of Love wave for (a) the first and (b) the second mode of the propagation for Case 2. The numbers in the figure fields show the values of the parameter F .

Consequently, the imperfectness of the contact conditions, as in the previous case, also increase the influence of the initial compressing in the system. Figure 4.5 illustrates the influence of the imperfect interface conditions when there exists only initial compressive stresses in normal direction, Case 3 ($\sigma_x^0 = 0$, $\sigma_z^0 < 0$), on the dispersion of Love wave velocity for the first and second modes of the wave propagation. The figure shows that as a result of the initial compression in normal direction the wave propagation velocity increases. In this case the imperfectness of the contact conditions, in general, causes to increase the wave propagation velocity related to the first and the second modes. However, the imperfectness of the contact

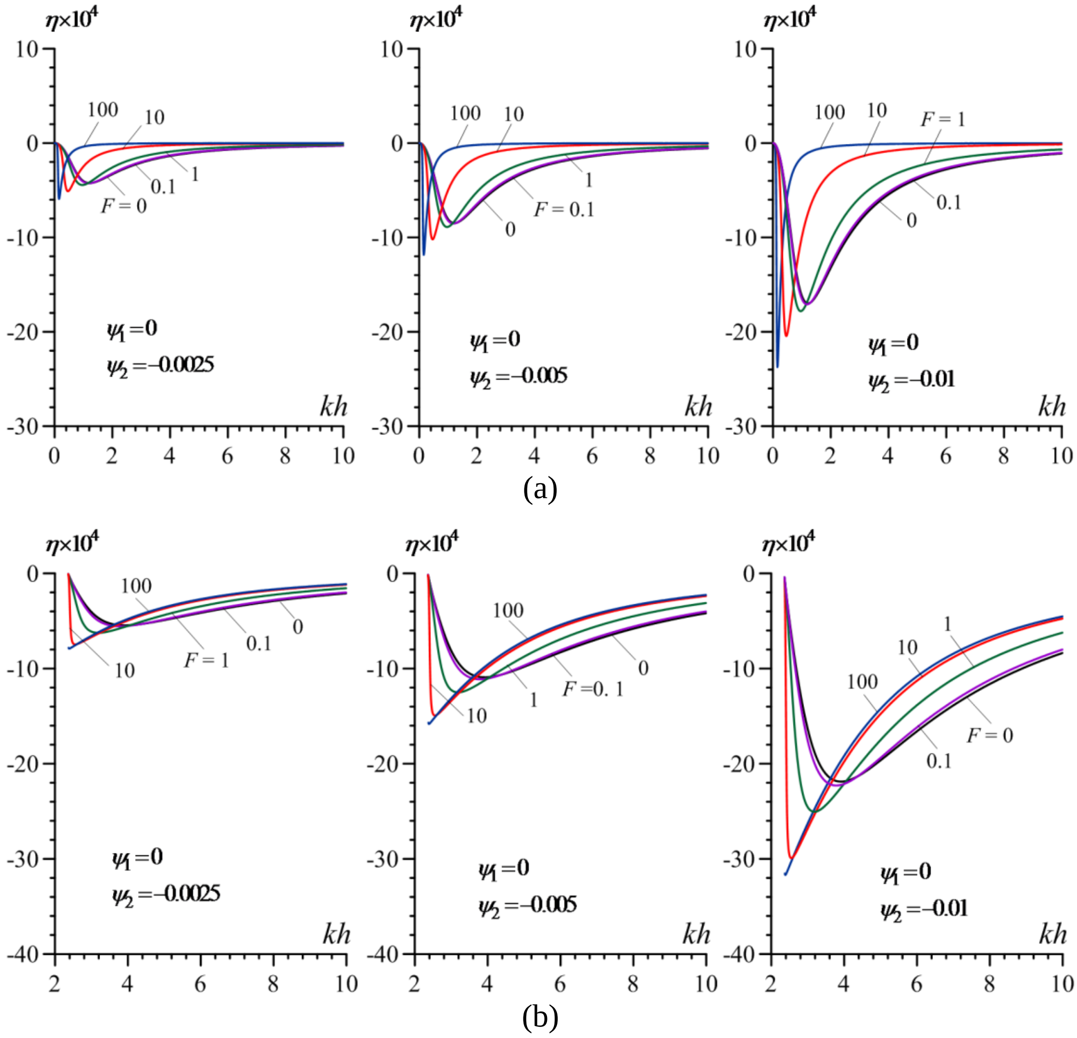


Figure 4.5 The influence of the imperfect bonding conditions and the initial stresses on the dispersion of Love wave for (a) the first and (b) the second mode of the propagation for Case 3. The numbers in the figure fields show the values of the parameter F .

conditions before a certain value of the kh causes to decrease the wave propagation velocity. In other words, the influence of the parameter F on the graphs between η and kh which are shown in Figure 4.5, has a complicate character. For instance, in the first mode of propagation at interval $0 < kh < 0.5$; and in the second mode at interval $0 < kh < 4$, the imperfectness of the contact conditions can cause to change the character of the dispersion curve. Consequently, the shear-spring type imperfectness between the constituents can acts on the dispersion curves not only quantitatively, but also qualitatively. Moreover, the influence of the parameter F on

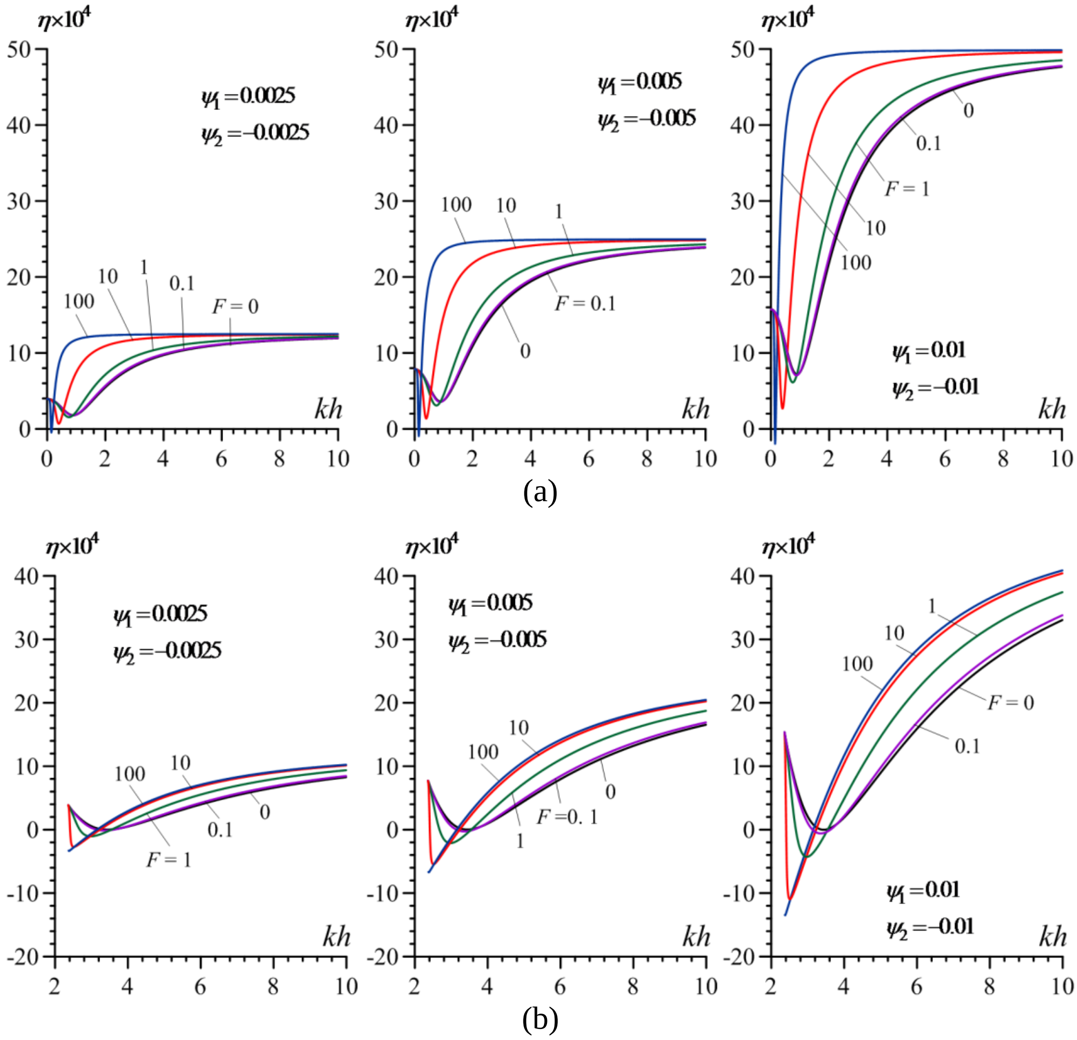


Figure 4.6 The influence of the imperfect bonding conditions and the initial stresses on the dispersion of Love wave for (a) the first and (b) the second mode of the propagation for Case 4. The numbers in the figure fields show the values of the parameter F .

wave propagation velocity is non-monotonic. In is also worth to notice that in this case, i.e. when only initial compressive stress in normal direction acts on a system the low and the high wavenumber limit values of the wave propagation velocities as $kh \to 0$ and $kh \to \infty$, unlike other four considered cases, do not depend on the imperfectness of the interface, i.e. on the parameter F .

Figure 4.6 shows the influence of the imperfect contact conditions and initial stresses on the dispersion of Love wave for the first and second modes of the wave

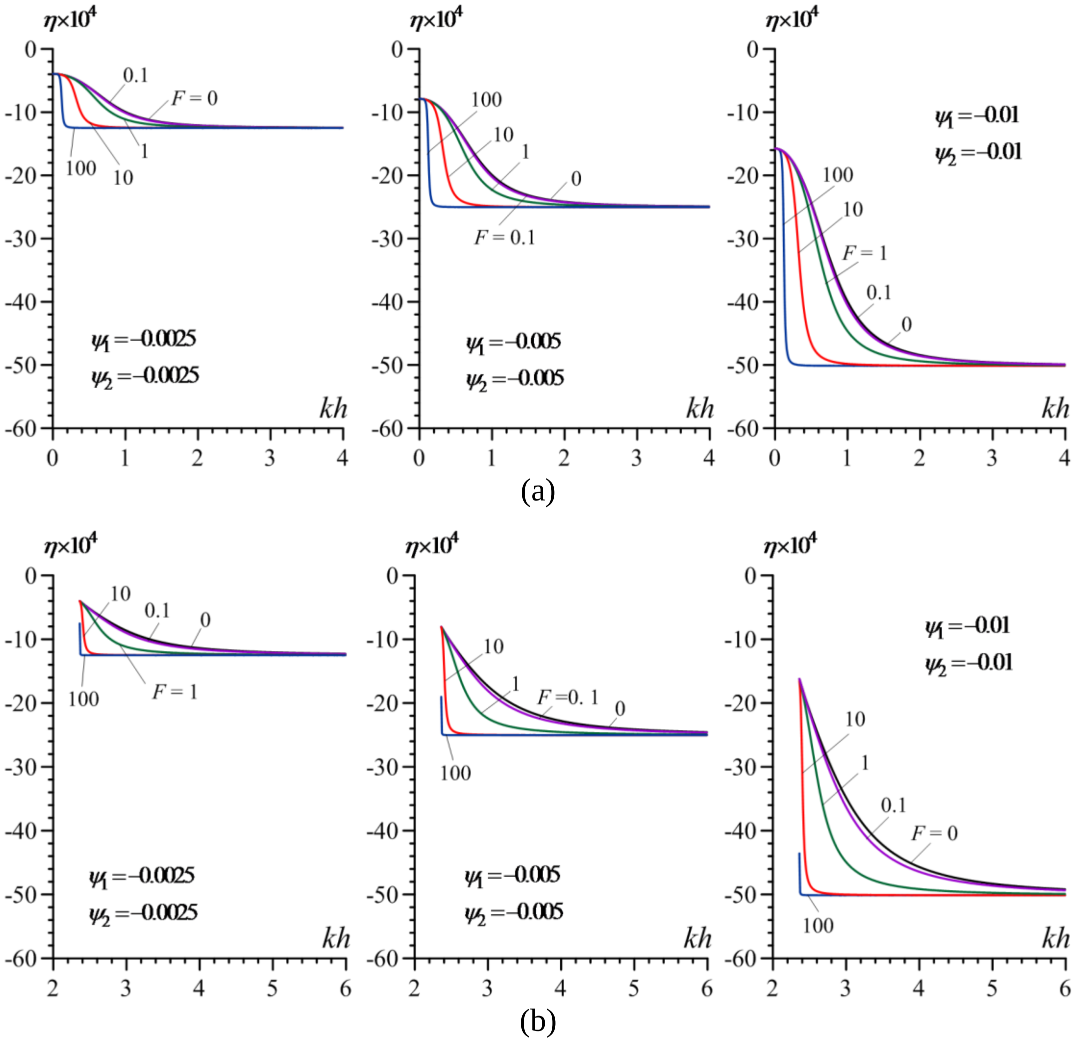


Figure 4.7 The influence of the imperfect bonding conditions and the initial stresses on the dispersion of Love wave for (a) the first and (b) the second mode of the propagation for Case 5. The numbers in the figure fields show the values of the parameter F .

propagation in Case 4 ($\sigma_x^0 > 0$, $\sigma_z^0 < 0$). The figure shows that as a result of this initial stress pattern the wave propagation velocity also increases. However, just as the previous case, the imperfectness of the contact conditions has a complicate character and before a certain value of the kh causes to decrease the wave propagation velocity. For instance, in the second mode at interval $0 < kh < 4$, the imperfectness of the contact conditions change the character of the dispersion curve and decreases the velocity of wave propagation. Similarly, the influence of the parameter F on wave propagation velocity also is non-monotonic.

Finally, Figure 4.7 shows the influence of the imperfect contact conditions and initial stresses on the dispersion of Love wave for the first and second modes of the wave propagation in Case 5 ($\sigma_x^0 < 0$, $\sigma_z^0 < 0$). The results of this initial stress pattern are similar to the case 2 (Figure 4.4). Figure 4.7 show that as a result of this initial stress pattern the wave propagation velocity decreases. In this case also the imperfectness of the contact conditions decrease the wave propagation velocity related to the first and the second modes of propagation and the velocity decrease monotonically.

This completes our analyses of the numerical results related to constant initial stresses in the constituents of the system and now we are going to study the effect of inhomogeneous initial stress on dispersion of Love wave propagation on a system consisting of an elastic layer over an elastic half-space.

As discussed in the foregoing sections we assume that the initial stresses vary linearly with depth and the variation pattern of the initial stresses in both normal and transverse directions are taken as equations (4.24) and (4.25). Substituting these equations into the equations (4.7) and (4.8), respectively, we got:

$$q^{(1)}(z) = \frac{-\mu^{(1)} - \sigma_x^0 \cdot (1+mz) + \rho^{(1)} c^2}{\mu^{(1)} + \sigma_z^0 \cdot (1+nz)}, \tag{4.29}$$

$$q^{(2)}(z) = \frac{\mu^{(2)} + \sigma_x^0 \cdot (1+mz) - \rho^{(2)} c^2}{\mu^{(2)} + \sigma_z^0 \cdot (1+nz)}. \tag{4.30}$$

To obtain the displacement components in the covering layer and in the half-space as given by equations (4.18) and (4.19) we have to integrate $\sqrt{q^{(1)}(z)}$ and $\sqrt{q^{(2)}(z)}$. However, without loss of generality, we assume that, $q(z) = \dfrac{a\,z+b}{c\,z+d}$, to simplfy the calculations and integarte it in the following general form:

$$\int \sqrt{\frac{a\,z+b}{c\,z+d}} = \frac{\sqrt{a^2d^2 - 2abcd + b^2c^2}}{\sqrt{-ac^3}} \cdot \arctan\left(\frac{c^2 \sqrt{q(z)} \sqrt{a^2d^2 - 2abcd + b^2c^2}}{\sqrt{-ac^3}\,(ad-bc)} \right) + \sqrt{\frac{c}{a}}\,(ad-bc) \left(\sqrt{a}\,c^{3/2} - \frac{c^{5/2}(az+b)}{\sqrt{a}\,(cz+d)} \right)^{-1} \sqrt{q(z)},$$

where in the case of $q^{(1)}(z)$ and $q^{(2)}(z)$, the expressions for the parameters *a*, *b*, *c* and *d* are given by:

$$q^{(1)}(z): \begin{cases} a = -\sigma_x^0\, m \\ b = -\mu^{(1)} - \sigma_x^0 + \rho^{(1)} c^2 \\ c = \sigma_z^0\, n \\ d = \mu^{(1)} + \sigma_z^0 \end{cases}, \quad q^{(2)}(z): \begin{cases} a = \sigma_x^0\, m \\ b = \mu^{(2)} + \sigma_x^0 - \rho^{(2)} c^2 \\ c = \sigma_z^0\, n \\ d = \mu^{(2)} + \sigma_z^0 \end{cases}.$$

Inserting these results to the displacement components in the covering layer and the half-space, equations (4.18) and (4.19), and considering the boundary conditions, equations (4.20)-(4.22), yields the system of three homogenous algebraic equations as discussed in the earlier sections. For a nontrivial solution the determinant of the coefficients must vanish giving the dispersion equation of Love wave propagation. Since the expressions for the components of the matrix of the corresponding dispersion determinant are cumbersome we are omitting here these details. The explicit expressions of the α_{ij} in the dispersion equation (4.23) for two typical examples when $\sigma_x^0(z) = m\,z$, $\sigma_z^0 = 0$ and $\sigma_x^0 = 0$, $\sigma_z^0 = n\,z$ as special cases are presented in the end of this chapter.

Now, again we consider the numerical results in our pervious geophysical example; this time we assume that the initial stresses are not constant and vary linearly with depth, equations (4.24)-(4.25). We assume that $\rho^{(1)} = 2800\,\text{kg/m}^3$, $\beta^{(1)} = 3000\,\text{m/s}$ and $\rho^{(2)} = 3200\,\text{kg/m}^3$, $\beta^{(2)} = 5000\,\text{m/s}$ and thickness of the Earth's crustal layer is $h = 10\,\text{km}$. As discussed in chapter 2, in the Earth's crust the lithostatic stress magnitudes increases linearly with depth, therefore we assume that the gradient coefficients of the inhomogeneous initial stresses in the transverse and in the normal directions are $m = 9000\,\text{MPa/km}$ and $n = 27000\,\text{MPa/km}$, respectively. Figure 4.8 through Figure 4.11 show the dispersion curves for the first four modes of Love wave propagation in this example for the initial stress cases as described before by equation (4.26). Note that each curve in the graphs is obtained for different value of the initial stress values σ_x^0 and σ_z^0 as indicated in the figures.

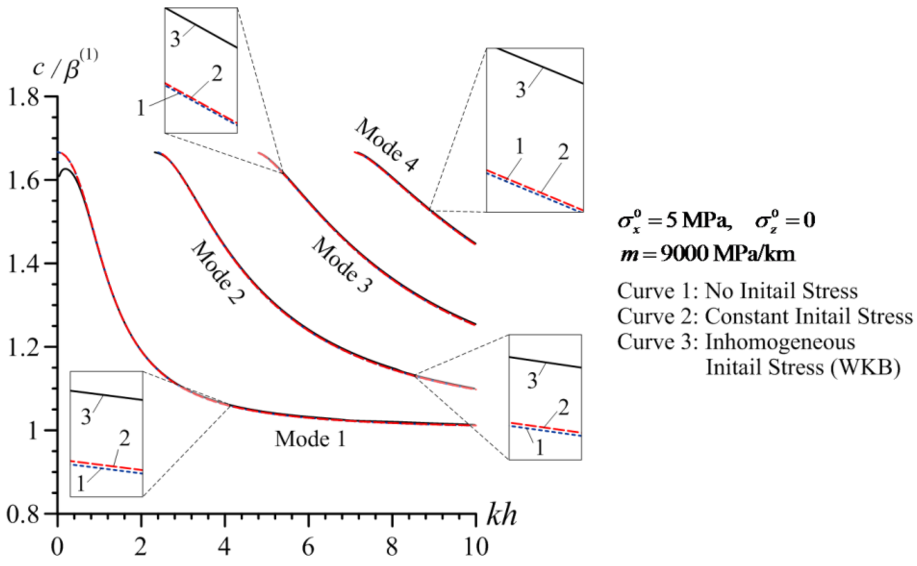


Figure 4.8 The influence of the inhomogeneous initial stress on the dispersion curves for the first four mode of Love wave propagation for Case 1.

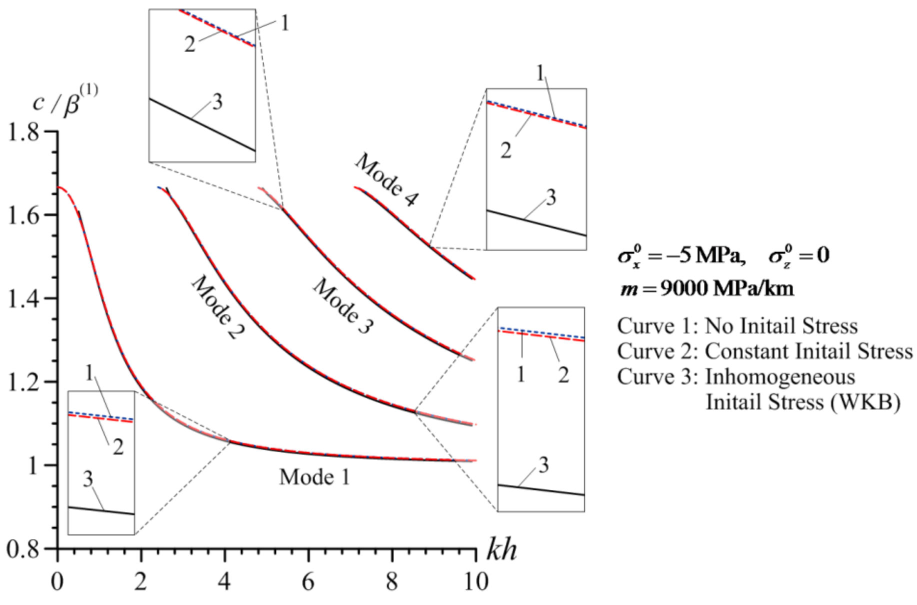


Figure 4.9 The influence of the inhomogeneous initial stress on the dispersion curves for the first four mode of Love wave propagation for Case 2.

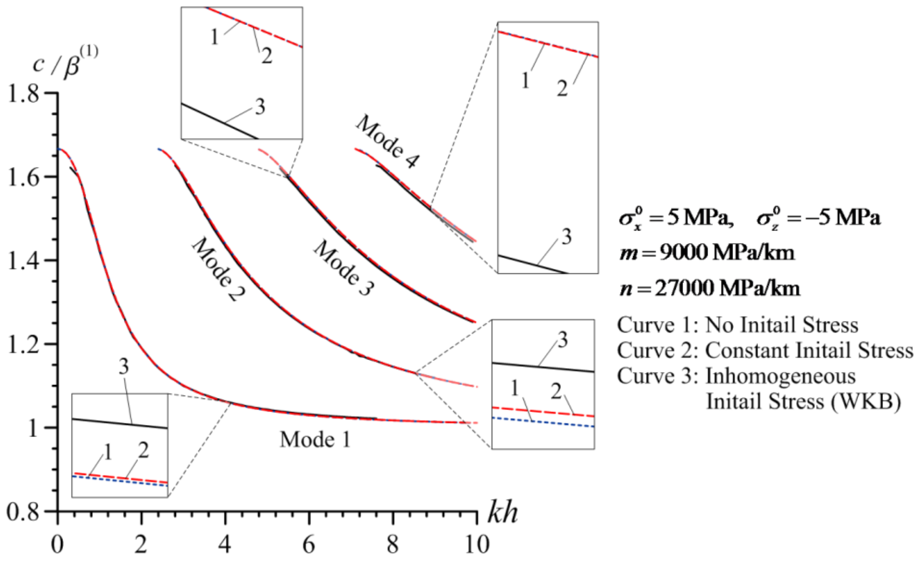


Figure 4.10 The influence of the inhomogeneous initial stress on the dispersion curves for the first four mode of Love wave propagation for Case 4.

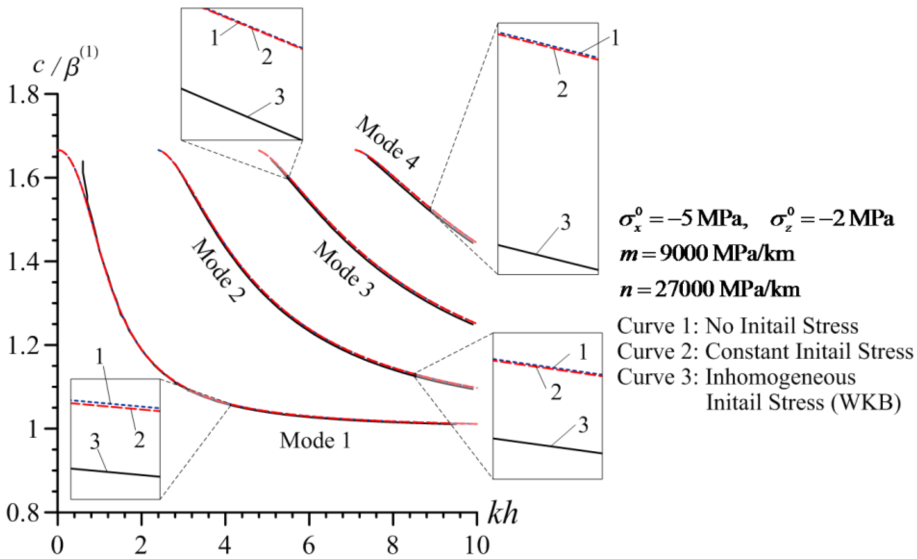


Figure 4.11 The influence of the inhomogeneous initial stress on the dispersion curves for the first four mode of Love wave propagation for Case 5.

As Figure 4.8 through Figure 4.11 show, the effect of inhomogeneous initial stresses on the dispersion of Love wave are almost similar to the constant initial stresses qualitatively, as discussed in forgoing sections for first two modes of the propagation, but with more intensive magnitudes, as might expected. For example, both Figure 4.7 and Figure 4.11, which show the effect of initial stresses on the dispersion curves of Love wave propagation for Case 5, indicate that as a result of this initial stress pattern the wave propagation velocity decreases. However, as mentioned above the effect of inhomogeneous initial stress are more significant in comparison to constant one.

Now we consider the same example, but here we assume that the initial stresses have the following form: $\sigma_x^0(z) = m\,z$ and $\sigma_z^0(z) = n\,z$, to exactly overlap with Earth's crustal stress model as discussed in the previous sections. Figure 4.12 illustrate the geometry, mechanical constants and initial stress patterns related to that problem. Love wave dispersion curves for the first four modes of propagation for this example is also shown in Figure 4.13. To study an example of Love wave propagation at high frequency, we concentrate on a wave with wavelength equal to, say 1/4 of the layer thickness for instance, i.e. $kh = 8\pi$. We are focusing on the first fundamental mode of propagation (Figure 2.13). We find numerically that it propagates with speed $c \approx 1.002\beta^{(1)}$, then $c \approx 3006\,\text{m/s}$. Roots of $q^{(1)}(z) = 0$, have vital importance because at these points differential equation of the motion changes its characteristic

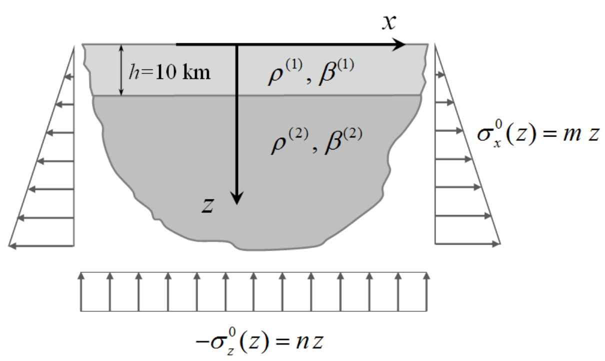


Figure 4.12 The geometry of the geophysical example.

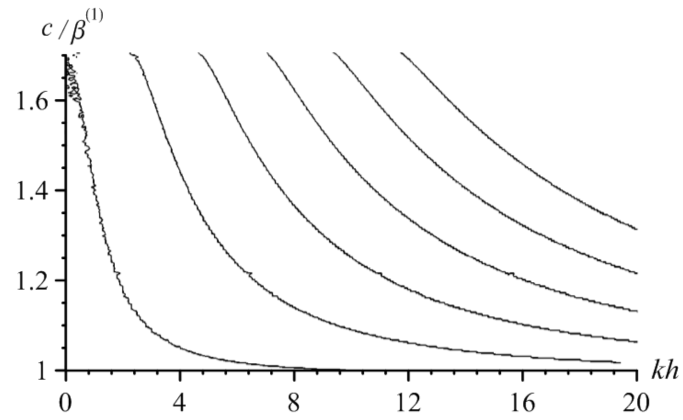


Figure 4.13 Dispersion curves for the first four mode of Love wave propagation for the considered example in Figure 4.12 (WKB approximation).

and as equation (4.15) shows mathematically amplitude of the wave at these points goes to infinity and as we discussed in chapter 3, these are called caustic points. So, we are interested on the roots of $q^{(1)}(z)$, which we repeated here for convenience, equation (4.7):

$$q^{(1)}(z) = \frac{-\mu^{(1)} - \sigma_x^0(z) + \rho^{(1)}c^2}{\mu^{(1)} + \sigma_z^0(z)}.$$

Here, $q^{(1)}(z) = 0$, requires that $-\mu^{(1)} - \sigma_x^0(z) + \rho^{(1)}c^2 = 0$ or $\sigma_x^0(z) = \rho^{(1)}c^2 - \mu^{(1)}$. Now if we substitute the values of $\rho^{(1)}$, $\mu^{(1)}$ and c in this equation, we find that $\sigma_x^0(z) = 100.9\,\text{MPa}$. Thus substituting this again into equation $\sigma_x^0(z) = \rho^{(1)}c^2 - \mu^{(1)}$, and solving for z we obtain $z = 11.21\,\text{km}$. This value is outside the acceptable region for *z*, which for covering layer was, $0 < z < 10\,\text{km}$. Therefore this point cannot be a caustic point for this example, however, if the gradient coefficient of the inhomogeneous initial stress *m* has bigger values, then this may create such caustic points in the region. For instance, if the gradient coefficient of the inhomogeneous initial stress for this example be $m = 12\,\text{MPa/km}$, then at $z = 8.41\,\text{km}$, $q^{(1)}(z) = 0$ and this point will be caustic point at the region. The character of differential equation at that point will change completely and the amplitude of the wave as can be seen from equation (4.15) tends to infinity. Physically this is not possible and though the intensity of the wave at such caustic points increases dramatically the amplitude of the wave cannot be infinity and this is the most important limitation and shortcoming of the classical WKB asymptotic approximation method. As discussed in the chapter 3 it should be noted that the asymptotic solutions of wave equation near the caustics points can also be obtained, and correct amplitude of the ray can be computed, however, for these situations more detailed methods such as uniform asymptotic expansions will be needed.

The expressions of the components $\alpha_{ij},\ i, j=1,2,3.$

For the case when $\sigma_x^0(z)=m\,z$ and $\sigma_z^0=0$:

$$\alpha_{11}=-k\exp\left(\frac{k\mu_1\,\xi_1^{\frac{3}{2}}\,2i}{3m\sigma_x^0}\right)\xi_1^{\frac{1}{4}}i+\frac{m\exp\left(\frac{k\mu_1\,\xi_1^{\frac{3}{2}}\,2i}{3m\sigma_x^0}\right)\sigma_x^0}{4\mu_1\,\xi_1^{\frac{5}{4}}},$$

$$\alpha_{12}=\frac{k\,\xi_1^{\frac{1}{4}}\,i}{\exp\left(\frac{k\mu_1\,\xi_1^{\frac{3}{2}}\,2i}{3m\sigma_x^0}\right)}+\frac{m\,\sigma_x^0}{4\mu_1\,\xi_1^{\frac{5}{4}}\exp\left(\frac{k\mu_1\xi_1^{\frac{3}{2}}\,2i}{3m\sigma_x^0}\right)},\qquad \alpha_{13}=0,$$

$$\alpha_{21}=-\mu_1 k\exp\left(\frac{k\,\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)\xi_2^{\frac{1}{4}}\,i+\frac{m\exp\left(\frac{k\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)\sigma_x^0}{4\xi_2^{\frac{5}{4}}},$$

$$\alpha_{22}=\frac{k\mu_1\,\xi_2^{\frac{1}{4}}\,i}{\exp\left(\frac{k\,\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)}+\frac{m\sigma_x^0}{4\xi_2^{\frac{5}{4}}\exp\left(\frac{k\,\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)},$$

$$\alpha_{23}=\frac{k\mu_2\,\xi_3^{\frac{1}{4}}}{\exp\left(\frac{2k\,\xi_3^{\frac{3}{2}}\,\mu_2}{3m\sigma_x^0}\right)}+\frac{m\sigma_x^0\,\xi_3^{-\frac{5}{4}}}{4\exp\left(\frac{2k\,\xi_3^{\frac{3}{2}}\,\mu_2}{3m\sigma_x^0}\right)},$$

$$\alpha_{31}=\frac{\exp\left(\frac{k\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)}{\xi_2^{\frac{1}{4}}},$$

$$\alpha_{32}=\frac{\left(\exp\left(\frac{k\xi_2^{\frac{3}{2}}\,\mu_1 2i}{3m\sigma_x^0}\right)\right)^{-1}}{\xi_2^{\frac{1}{4}}},$$

$$\alpha_{33}=-\frac{\left(\exp\left(\frac{2k\xi_3^{\frac{3}{2}}\mu_2}{3m\sigma_x^0}\right)\right)^{-1}}{\xi_3^{\frac{1}{4}}},$$

where

$$\xi_1=\frac{\rho^{(1)}c^2-\mu^{(1)}}{\mu^{(1)}},\qquad \xi_2=\frac{\rho^{(1)}c^2-\mu^{(1)}-hm\sigma_x^0}{\mu^{(1)}},\qquad \xi_3=\frac{\mu^{(2)}-\rho^{(2)}c^2+hm\sigma_x^0}{\mu^{(2)}}.$$

And for the case when $\sigma_x^0=0$ and $\sigma_z^0=nz$:

$$\alpha_{11}=-\frac{\left(\exp\left(\frac{k\mu_1\sqrt{\zeta_1}2i}{n\sigma_z^0}\right)\right)^{-1}\left(k\sqrt{\zeta_1}2i-\frac{k\zeta_1 i}{\sqrt{\zeta_1}}\right)}{\zeta_1^{\frac{1}{4}}}+\frac{n\left(\exp\left(\frac{k\mu_1\sqrt{\zeta_1}2i}{n\sigma_z^0}\right)\right)^{-1}\zeta_1\sigma_z^0}{4\mu_1\zeta_1^{\frac{5}{4}}},$$

$$\alpha_{12}=\frac{\exp\left(\frac{k\mu_1\sqrt{\zeta_1}2i}{n\sigma_z^0}\right)\left(k\sqrt{\zeta_1}2i-\frac{k\zeta_1 i}{\sqrt{\zeta_1}}\right)}{\zeta_1^{\frac{1}{4}}}+\frac{n\exp\left(\frac{k\mu_1\sqrt{\zeta_1}2i}{n\sigma_z^0}\right)\zeta_1\sigma_z^0}{4\mu_1\zeta_1^{\frac{5}{4}}},\qquad \alpha_{13}=0,$$

$$\alpha_{21}=-\frac{\left(k\sqrt{\zeta_2}2i-\frac{k\zeta_2 i}{\sqrt{\zeta_2}}\right)\mu_1}{\exp\left(\frac{k\left(\mu^{(1)}+hn\sigma_z^0\right)\sqrt{\zeta_2}2i}{n\sigma_z^0}\right)\zeta_2^{\frac{1}{4}}}+\frac{n\sigma_z^0\zeta_2\mu_1\left(\mu^{(1)}+hn\sigma_z^0\right)^{-1}}{4\exp\left(\frac{k\left(\mu^{(1)}+hn\sigma_z^0\right)\sqrt{\zeta_2}2i}{n\sigma_z^0}\right)\zeta_2^{\frac{5}{4}}},$$

$$\alpha_{22}=\left(\frac{\left(k\sqrt{\zeta_2}2i-\frac{k\zeta_2 i}{\sqrt{\zeta_2}}\right)\mu_1}{\zeta_2^{\frac{1}{4}}}-\frac{n\sigma_z^0\zeta_2\mu_1}{4\left(\mu^{(1)}+hn\sigma_z^0\right)\zeta_2^{\frac{5}{4}}}\right)\exp\left(\frac{k\left(\mu^{(1)}+hn\sigma_z^0\right)\sqrt{\zeta_2}2i}{n\sigma_z^0}\right),$$

$$\alpha_{23}=\frac{\left(2k\sqrt{\zeta_3}-\frac{k\zeta_3}{\sqrt{\zeta_3}}\right)\mu^{(2)}}{\exp\left(\frac{2k\left(\mu^{(2)}+hn\sigma_z^0\right)\sqrt{\zeta_3}}{n\sigma_z^0}\right)\zeta_3^{\frac{1}{4}}}-\frac{n\mu^{(2)}\zeta_3\sigma_z^0}{4\exp\left(\frac{2k\left(\mu^{(2)}+hn\sigma_z^0\right)\sqrt{\zeta_3}}{n\sigma_z^0}\right)\left(\mu^{(2)}+hn\sigma_z^0\right)\zeta_3^{\frac{5}{4}}},$$

$$\alpha_{31}=\left(\exp\left(\frac{k\left(\mu^{(1)}+hn\sigma_z^0\right)\sqrt{\zeta_2}2i}{n\sigma_z^0}\right)\zeta_2^{\frac{1}{4}}\right)^{-1},$$

$$\alpha_{32} = \frac{\exp\left(\frac{k\left(\mu^{(1)} + hn\sigma_z^0\right)\sqrt{\zeta_2}\,2i}{n\sigma_z^0}\right)}{\zeta_2^{\frac{1}{4}}},$$

$$\alpha_{33} = -\left(\exp\left(\frac{2k\left(\mu^{(2)} + hn\sigma_z^0\right)\sqrt{\zeta_3}}{n\sigma_z^0}\right)\zeta_3^{\frac{1}{4}}\right)^{-1},$$

where

$$\zeta_1 = \frac{\rho^{(1)}c^2 - \mu^{(1)}}{\mu^{(1)}}, \qquad \zeta_2 = \frac{\rho^{(1)}c^2 - \mu^{(1)}}{\mu^{(1)} + hn\sigma_z^0}, \qquad \zeta_3 = \frac{\mu^{(2)} - \rho^{(2)}c^2}{\mu^{(2)} + hn\sigma_z^0}.$$

Chapter 5

Conclusions and Recommendations

Classical closed form solution of many interesting physical and practical problems such as elastodynamics problems do not exist and most of the time accuracy of numerical methods are in doubt. Asymptotic analysis, then, is an analytical approximate method which provides a new and modern alternative tools to solve such complex problems. In this thesis we exploited one of the most powerful and famous asymptotic methods so called WKB high-frequency analysis to solve seismic Love wave dispersion problem under the effect of some special inhomogeneous initial stress pattern. Theoretical derivations are obtained in the framework of classical linearized theory of elastic waves in initially stressed bodies for small deformations and numerical examples are given and discussed. The influence of the imperfectness of the contact conditions on Love wave propagation velocity has also been studied through the influence of the interface imperfectness parameter F, equation (4.22), on the wave propagation velocity. In the vicinity of caustic points also some possible asymptotic solutions of the wave problem is discussed. From these discussions the following main conclusions are derived:

- The imperfectness of the contact conditions cause to decrease of the wave propagation velocity of Love waves.
- The low wavenumber and high wavenumber limit values of the wave propagation velocity do not depend on the imperfectness of the contact conditions.
- The imperfectness of the contact conditions causes to increase the influence of the initial stresses in the wave propagation velocity.

- Influence of the imperfectness of contact conditions of Love wave propagation velocity in general is complex and frequency dependent. In this case before (after) a certain value of the wavenumber, the influence of the parameter F causes to increase (decrease) the wave propagation velocity.
- In general, the graphs of the dependence between the imperfectness parameter and the influence of the initial stresses on the wave propagation velocity cannot be limited with the corresponding ones obtained at complete contact or full slipping ones.
- For linearly varying initial stress pattern, as we considered in our example, for some big values of gradient coefficients of the linearly varying initial stresses a caustic points can be created which then it completely will change the behavior of wave propagation in the medium.

Rapid development of electronic devices, specially, has revolutionized elastic surface waves applications in many engineering fields through the last decade. Indeed, variety of mechanical, material and structural properties, presence of damages or cracks, etc. make the study of these wave processes an active field of research. In our study, we consider some of those parameters, namely the effect of initial stresses and contact conditions. For the future works it is desirable to consider the effect of different patterns of initial stresses rather than only the linear one. It is also possible to consider the effects of lateral inhomogeneity in material properties of the medium. And finally, we consider a one dimensional aspect of the problem in our study, however in general the related wave propagation problem is in three dimensional space and it create caustics which are some two dimensional surfaces in the space. This study can also be extended to include such interesting type of problems.

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