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ENERGY PRESERVING INTEGRATION OF KDV-KDV SYSTEMS

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ABSTRACT. Coupled Korteweg de Vries (KdV) equations in Hamiltonian form are integrated by the energy preserving average vector field (AVF) method. Numerical results confirm long term preservation of the energy and the quadratic invariants. Produced generalized solitary waves are similar to those in the literature for larger mesh sizes and time steps. Numerical and continuous dispersion relations of the linearized equations are compared to analyze the behavior of the traveling waves and the interaction of the solitons.

Keywords: Energy preservation, bi-Hamiltonian systems, Poisson structure, KdV equation, dispersion.

AMS Subject Classification: 65P10, 65L06, 37K10.

1. INTRODUCTION

The coupled equations of Boussinesq type, KdV-KdV system [4]

$$\begin{aligned} u_t + uu_x + v_x + \tfrac{1}{6}v_{xxx} &= 0, \\ v_t + (uv)_x + u_x + \tfrac{1}{6}u_{xxx} &= 0 \end{aligned} \tag{1}$$

and the symmetric KdV-KdV system [5]

$$\begin{aligned} u_t + \tfrac{3}{2}uu_x + \tfrac{1}{2}vv_x + v_x + \tfrac{1}{6}v_{xxx} &= 0, \\ v_t + \tfrac{1}{2}(uv)_x + u_x + \tfrac{1}{6}u_{xxx} &= 0. \end{aligned} \tag{2}$$

model surface water waves. They represent approximation to two dimensional Euler equations for surface wave propagation along a horizontal channel. The space and time variables x and t represent the position and the elapsed time, respectively along the channel, where u is the horizontal velocity and v is the deviation of the free surface from its rest position. Both systems are Hamiltonian partial differential equations (PDEs) of the form

$$\frac{\partial u}{\partial t} = \mathcal{J}\frac{\delta \mathcal{H}}{\delta u} \tag{3}$$

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in the domain $\Omega = (x,t) \in \mathbb{R} \times \mathbb{R}$. $\mathcal{H}$ and $\mathcal{J}$ denote the Hamiltonian functional and the skew-adjoint Hamiltonian operator, respectively [11]. The variational derivative is given by

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{\partial \mathcal{H}}{\partial u} - \partial_x \left(\frac{\partial \mathcal{H}}{\partial u_x} \right) + \partial_x^2 \left(\frac{\partial \mathcal{H}}{\partial u_{xx}} \right) - \cdots .$$

The corresponding skew-adjoint operators and Hamiltonians for the KdV-KdV system (1) are [4]

$$\mathcal{J} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2} \int \left(-v^2 - u^2 - u^2 v + \frac{1}{6}(u_x^2 + v_x^2) \right) dx. \tag{4}$$

The symmetric KdV-KdV system (2) can also be written in Hamiltonian form as [8]

$$\mathcal{J} = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2} \int \left(-uv - \frac{1}{4} uv^2 - \frac{u^3}{4} - \frac{1}{6} uv_{xx} \right) dx, \tag{5}$$

where D_x denotes the partial derivative with respect to the space variable x.

Both KdV-KdV systems (1, 2) under periodic boundary conditions possess generalized solitary waves", consisting of a solitary pulse decaying symmetrically to oscillations of small, constant amplitude. In other words, for both systems the solutions are in form of traveling waves with main pulses like the classical solitary waves and dispersive oscillations following the main pulses. The small amplitude oscillations appear immediately and propagate in the system with a higher speed than the main pulses. As a result of this propagation, the main pulses start to decay and the energy is transferred to the small oscillations, the ripples. These waves are named as "radiating solitary waves". The formation of the ripples, stability and the interaction and long time behavior of the wave forms are studied in [4, 5] by discretizing (1 and 2) in space with the Galerkin finite element method by using smooth splines. The resulting stiff system of ordinary differential equations (ODEs) are discretized with the fourth order accurate two-stage implicit Gauss-Legendre Runge-Kutta method.

Many partial differential equations like the Korteweg de Vries equation, non-linear Schrödinger equation, sine-Gordon equation are Hamiltonian PDEs of the form (3). In the last two decades various geometric integrators were developed for solving Hamiltonian PDEs by preserving the integrals and symplectic/multisymplectic structure very accurately in long time integration [10]. In this paper we consider energy preserving methods for integrable evolutionary equations in Hamiltonian form like the coupled KdV-KdV systems (1) and (2). The energy preserving discrete gradient methods rely on appropriate approximation of the Hamiltonian $\mathcal{H}$ and the skew-adjoint operator $\mathcal{J}$. After suitable spatial discretization of $\mathcal{J}$ and $\mathcal{H}$ in (3), the finite dimensional Hamiltonian system

$$\dot{\mathbf{u}} = J \nabla H(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^N \tag{6}$$

is obtained. J stands for the $N \times N$ skew-symmetric structure matrix, representing the discrete approximation of the skew-adjoint operator $\mathcal{J}$ and $H(\mathbf{u})$ for the discretized Hamiltonian. Recently linear energy preserving collocation methods were developed for Hamiltonian ODEs (6) known as average vector field methods (AVF's) [9, 6] as an extension to the mid-point rule. The finite-difference semi-discretized KdV-KdV systems (1) and (2) have skew-symmetric constant structure matrices J. The energy preserving AVF integrator [9] for these semi-discretized Hamiltonian PDEs is given by

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} = J\int_0^1 \nabla H(\mathbf{u}^n + \tau(\mathbf{u}^{n+1}-\mathbf{u}^n))d\tau. \tag{7}$$

Higher order AVF methods are constructed by using the Gaussian quadrature and they are interpreted as Runge-Kutta method with continuous stages for canonical and non-canonical Hamiltonian systems. The computation of the integrals occurring in the AVF method may be time consuming compared to the symplectic integrators like the mid-pint rule and implicit Runge-Kutta methods of Gauss-Legendre type. When the Hamiltonian is polynomial as the coupled KdV equations (1) and (2), the integrals can be computed exactly and the computational cost is comparable with the symplectic Gauss-Legendre Runge-Kutta methods.

The dispersion and group velocity analysis of the discretized linear wave equations are essential tools to understand the behavior of the numerical solutions and to determine how accurately the nonlinear dynamics is resolved by the discretization. Numerical errors in the dispersion can destroy the qualitative features of the solutions. In order to analyze the wave forms of the solitary traveling waves we have performed a dispersion analysis. The dispersive properties of symplectic and multisymplectic methods were investigated for the KdV equation [1, 2]. We investigate here the energy preserving AVF integrator (7) under the aspect of preservation of the dispersion properties for the linearized KdV-KdV systems (1) and (2).

The structure of the paper is as follows. In the next section, the formulation of the AVF method for the KdV-KdV systems is presented with some numerical experiments, illustrating the energy preservation in long term integration. In Section 3, the numerical dispersion relations of the AVF method are investigated for linearized KdV-KdV systems. The paper ends with some conclusions.

2. Average vector field integration of coupled KdV equations

We consider periodic boundary conditions such that no additional boundary terms will appear after semi-discretization. For the discretization of (3) in space, it is crucial to preserve the skew-adjoint structure of $\mathcal{J}$ to obtain the Hamiltonian system ODEs. The integrals in the Hamiltonians can be approximated either by the rectangle or trapezoidal rule. We use here the forward finite difference approximation for the first order derivatives in the Hamiltonians and the rectangle rule for approximation of the integrals.

For semi-discrete systems we use the notation u_j^n where the index j corresponds to increments in space and n to increments in time. The discrete approximation of $u(j\Delta x, n\Delta t)$ is denoted by $\mathbf{u}^n = (u_1^n, \dots, u_j^n, \dots, u_N^n)^T$.

Discrete forms of the skew-adjoint Hamiltonian operator $\mathcal{J}$ and the Hamiltonian (4) are

$$J = \frac{1}{2\Delta x}\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, \quad H = \frac{1}{2}\sum_{j=1}^{N}\left(-v_j^2 - u_j^2 - u_j^2 v_j + \frac{1}{6\Delta x^2}((u_{j+1}-u_j)^2 + (v_{j+1}-v_j)^2)\right)\Delta x,$$

$$A=\begin{pmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{pmatrix},$$

where A is an $N\times N$ tridiagonal circulant matrix due to the periodic boundary conditions. Applying the AVF method to (1) gives

$$\begin{aligned} \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} &= -\frac{1}{4\Delta x}A\left(\mathbf{v}^n+\frac{1}{3}\hat{\mathbf{u}}^n\right)-\frac{1}{24\Delta x^3}A\left(\tilde{\mathbf{v}}^n+\tilde{\mathbf{v}}^{n+1}\right), \\ \frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t} &= -\frac{1}{4\Delta x}A\left(\mathbf{u}^n+\widetilde{\mathbf{uv}}\right)-\frac{1}{24\Delta x^3}A\left(\tilde{\mathbf{u}}^n+\tilde{\mathbf{u}}^{n+1}\right), \end{aligned}$$

where the vectors $\hat{u}^n, \tilde{u}^n$, and $\widetilde{uv}$ are defined as

$$\begin{aligned} \hat{\mathbf{u}}^n &= (\ldots,(u_j^n)^2+u_j^nu_j^{n+1}+(u_j^{n+1})^2,\ldots)^T, \\ \tilde{\mathbf{u}}^n &= (\ldots,u_{j+1}^n-2u_j^n+u_{j-1}^n,\ldots)^T, \\ \widetilde{\mathbf{uv}} &= (\ldots,\frac{2}{3}u_j^nv_j^n+\frac{1}{3}\left(u_j^nv_j^{n+1}+u_j^{n+1}v_j^n\right)+\frac{2}{3}u_j^{n+1}v_j^{n+1},\ldots)^T, \end{aligned}$$

with the corresponding vectors $\tilde{u}^{n+1}, \tilde{v}^n, \tilde{v}^{n+1}$. The corresponding semi-discrete skew-symmetric J matrix and the disretized Hamiltonian H for the symmetric KdV-KdV system (2) are

$$J=\frac{1}{2\Delta x}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad H=\frac{1}{2}\sum_{j=1}^{N}\left(-u_jv_j-\frac{1}{4}u_jv_j^2-\frac{u_j^3}{4}-\frac{u_j(v_{j+1}-2v_j+v_{j-1})}{6\Delta x^2}\right)\Delta x.$$

Application of the AVF method leads to

$$\begin{aligned} \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} &= -\frac{1}{4\Delta x}A\left(\mathbf{v}^n+\mathbf{v}^{n+1}\right)-\frac{1}{2\Delta x}A\left(\frac{1}{4}\hat{\mathbf{u}}+\frac{1}{12}\hat{\mathbf{v}}\right)-\frac{1}{12\Delta x^3}A\left(\tilde{\mathbf{v}}^n+\tilde{\mathbf{v}}^{n+1}\right), \\ \frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\Delta t} &= -\frac{1}{4\Delta x}A\left(\mathbf{u}^n+\mathbf{u}^{n+1}\right)-\frac{1}{8\Delta x}A\widetilde{\mathbf{uv}}-\frac{1}{12\Delta x^3}A\left(\tilde{\mathbf{u}}^n+\tilde{\mathbf{u}}^{n+1}\right). \end{aligned}$$

The resulting nonlinear system of equations are solved by Newton's method.

Both KdV-KdV systems (1) and (2) are solved with the AVF integrator with the initial condition in [4]

$$u(x,0)=0, \quad v(x,0)=0.3e^{-(x+100)^2/25} \tag{8}$$

in the interval $x\in[-150,150]$ for coarser mesh sizes $N=500$ and larger time steps $\Delta t=0.04$ than in [4] up to $t=100$. The errors in the Hamiltonian functional in Figure 1 show that energies in both systems are conserved up to the machine accuracy.

Other invariants are $\mathcal{I}_1=\int uv\,dx$ for the KdV-KdV system (1) and $\mathcal{I}_1=\int\left(u^2+v^2\right)dx$ for the symmetric KdV-KdV system (2). It is well known that the quadratic invariants are preserved by symplectic integrators with a small error in long time integration. The numerical errors for these quadratic invariants in Figure 2 indicate that this is the case for the energy preserving AVF integrator, too. There are also trivial linear invariants, which are $\mathcal{I}_2=\int u\,dx$ and $\mathcal{I}_3=\int v\,dx$. Since the preservation of the linear invariants is a strict condition for the energy and the quadratic invariants preservation, they are not shown particularly.

Since the numerical results for the initial condition (8) for both KdV-KdV systems are similar, the solitary wave solutions of the KdV-KdV system (1) are shown in Figure 3 at $t=100$. Two wave trains are moving in opposite directions and producing solitary wave

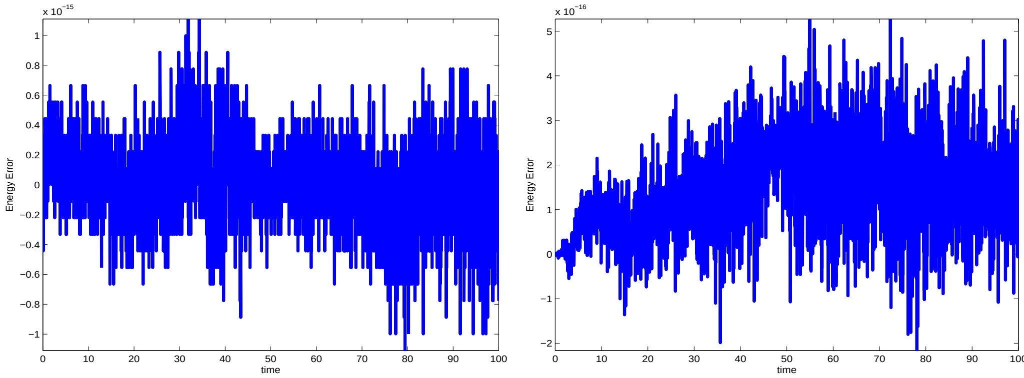


FIGURE 1. Error for the energy of the KdV-KdV (left) and the symmetric KdV-KdV (right)

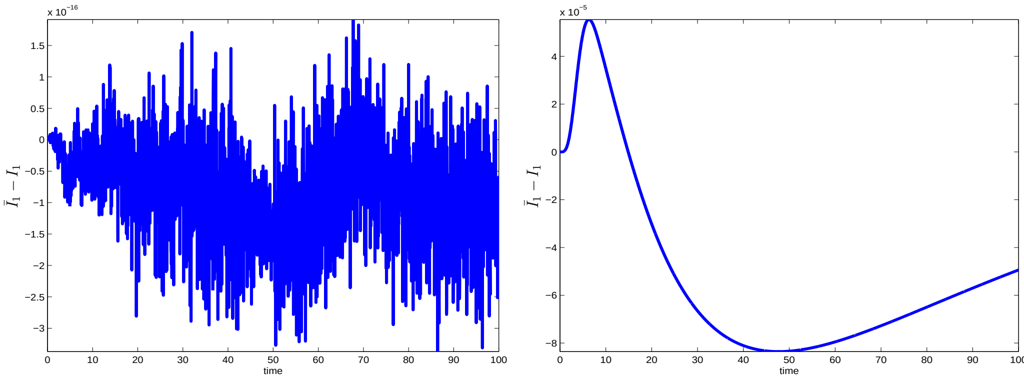


FIGURE 2. Conserved quantities $\mathcal{I}_1$ of the KdV-KdV (left) and the symmetric KdV-KdV (right)

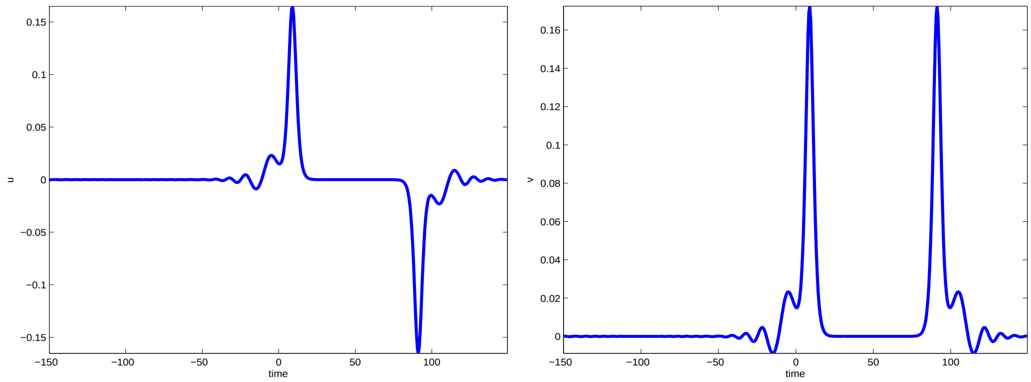


FIGURE 3. Solutions of the KdV-KdV systems for u (left) and v (right)

solutions with ripples decaying symmetrically around the solitons as in [4].

We have also solved the symmetric KdV-KdV system (2) with the initial conditions Eq. 4.1 in [5]

$$u_0 = \phi(x), \qquad v_0 = \phi(x) + \frac{1}{4}\phi^2(x),$$

where $\phi(x) = Asech^2\left(\frac{\sqrt{3A}}{2}x\right)$, with $A = 0.6$ in order to show the emergence of so called radiating waves.

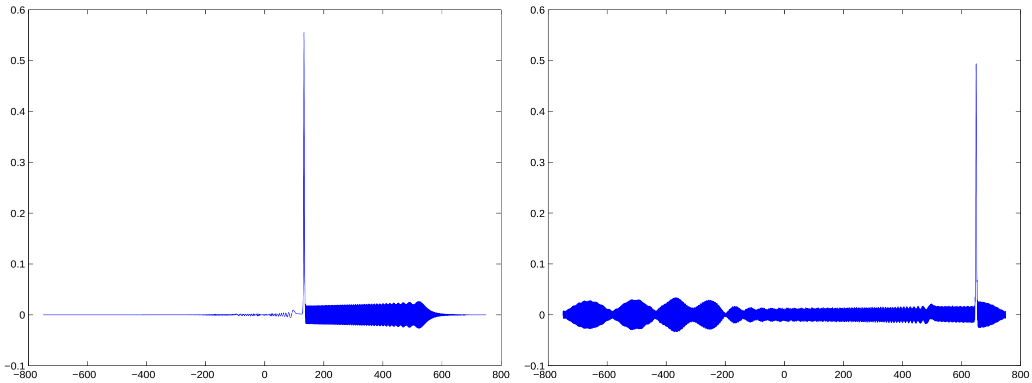


FIGURE 4. Solutions of the symmetric KdV-KdV system for u at $t = 100$ and at $t = 500$ (left)

The numerical results in Figure 4 are very similar for long term integration to those in [5].

As a summary, for both KdV-KdV system the AVF method preserves the Hamiltonians and the quadratic invariants in long term integration well. The numerical results indicate that using second order accurate finite difference discretization and second order in time AVF integrator numerically similar results are obtained as in [4, 5]. We would like to remark that in both papers a fourth order space and time discretization is used which increases the computational cost for solving the implicit equations by Newton's method.

3. Dispersion analysis

In this section, the dispersion relations and the group velocities of the linearized KdV-KdV (1) and the symmetric KdV-KdV (2) systems are examined. Nonlinear PDEs such as KdV-KdV systems are dispersive, i.e. the wave packets with different wave numbers travel with different velocities. The preservation of the energy in long time alone can not explain the accuracy of the solutions. The behavior of a nonlinear PDE can be determined by the dispersion relation in regions where linearized PDE is a valid approximation to the nonlinear PDE. A dispersion relation $\omega = \omega(k)$ of a constant coefficient linear evolution equation determines how time oscillations $e^{i\omega t}$ of the frequencies are linked to spatial oscillations e^{ikx} of a wave number k. Any linear constant coefficients PDE has a solution of the form

$$u(x,t) = \int_{\infty}^{\infty} A(k)e^{i(kx+\omega(k)t)}dk, \qquad i = \sqrt{-1}. \tag{9}$$

The dispersion relation $D(\omega, k) = 0$ is obtained by assuming that each wave $Ae^{i(kx+\omega t)}$ itself is a solution of the linear PDE. Each wave travels with the phase velocity $\omega_p(k) = \omega/k$, characterizing the speed of the wave front. The speed of the energy transport of the composite wave packet is characterized by the the group velocity $\omega'(k)$. The non-vanishing group velocity dispersion causes spatial spreading of the wave packet. Numerical errors in the dispersion relation and the group velocities can lead to the propagation of the numerical wave with different velocity and can destroy the qualitative feature of the solutions. In numerical simulations, it is important to preserve the sign of the group velocity in order to avoid spurious solutions. Recently, dispersive properties of symplectic and multisymplectic integrators for the KdV equation are examined in [1, 2]. It was shown that the multisymplectic Preissman box scheme qualitatively preserves the dispersion relation of the KdV equation [1, 2].

In this section, the dispersion relations and the group velocities of the linearized KdV-KdV (1) and the symmetric KdV-KdV (2) systems are examined. The linearized KdV-KDV system (1) around the constant solutions $\bar{u}, \bar{v}$ becomes

$$\begin{array}{rcl} u_t + \frac{1}{6} v_{xxx} + \bar{u} u_x + v_x & = & 0, \\ v_t + \frac{1}{6} u_{xxx} + (1+\bar{v}) u_x + \bar{u} v_x & = & 0. \end{array} \tag{10}$$

The continuous dispersion relations and group velocities are

$$\omega_1(k) = \frac{c}{6}k^3 - (a+c)\,k, \qquad \omega_2(k) = \frac{1}{6c}k^3 - \left(\frac{1+b}{c} + a\right)k, \tag{11}$$

$$\frac{d\omega_1(k)}{dk} = \frac{c}{2}k^2 - (a+c)\,, \qquad \frac{d\omega_2(k)}{dk} = \frac{1}{2c}k^2 - \left(\frac{1+b}{c} + a\right), \tag{12}$$

with $a = \bar{u}$, $b = \bar{v}$ and $c = \bar{u}/\bar{v}$.

Numerical dispersion relations are for the discrete version of the Fourier mode (9) given as:

$$\tilde{u}_j^n = \hat{u} e^{i(jk\Delta x + n\omega\Delta t)} = \hat{u} e^{i(j\bar{k} + n\bar{\omega})}, \tag{13}$$

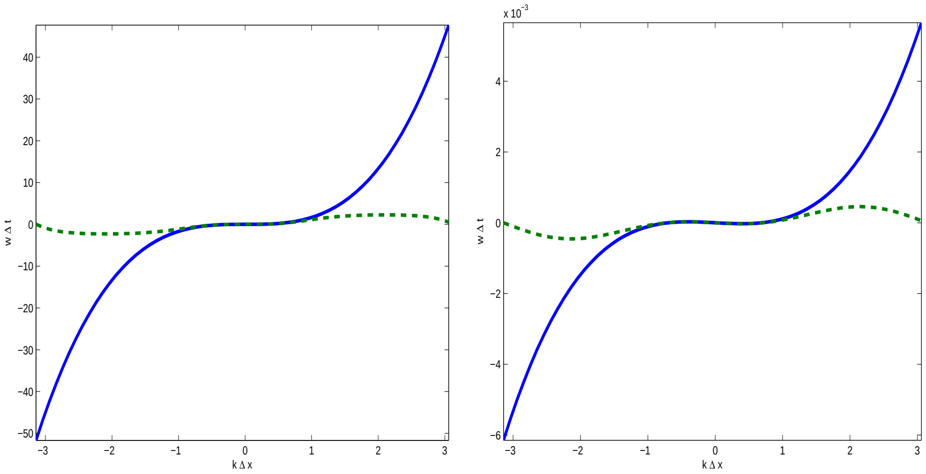
where $\bar{k} = k\Delta x$ and $\bar{\omega} = \omega\Delta t$, denote the numerical wavenumber and the numerical velocity, respectively in the range $-\pi \le \bar{k}, \bar{\omega} \le \pi$.

Numerical dispersion relations are obtained by solving the linearized KdV-KdV equation (10) with the AVF method are

$$\begin{array}{rcl} \bar{\omega}_1(\bar{k}) & = & 2\arctan\left(\frac{-\lambda c}{6\Delta x^2}(\sin\bar{k}(\cos\bar{k} - 1)) - (a+c)\,\frac{\lambda}{2}\sin\bar{k}\right), \\ \bar{\omega}_2(\bar{k}) & = & 2\arctan\left(\frac{-\lambda}{6\Delta x^2 c}(\sin\bar{k}(\cos\bar{k} - 1)) - \left(\frac{1+b}{c} + a\right)\frac{\lambda}{2}\sin\bar{k}\right). \end{array} \tag{14}$$

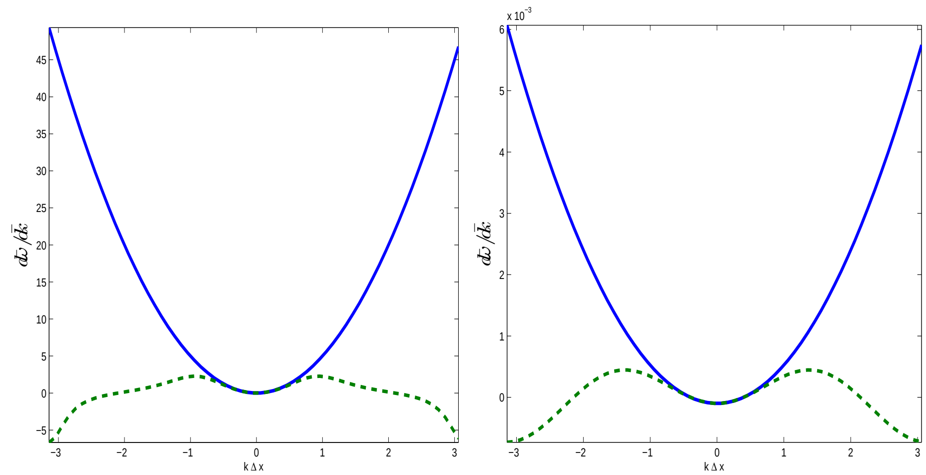
The continuous (11) (solid curves) and the discrete (14) (dotted curves) dispersion relations are compared in Figure 5 by choosing $a = b = c = 0.1$, $\Delta t = 0.1$ for $\lambda = 0.1$ (left plot), and for $\lambda = 0.005$ (right plot). For small wavenumbers $\bar{k}$, both the analytical and the numerical dispersion behave similarly. For small wavenumbers each frequency $\bar{\omega}$ corresponds to a particular value of $\bar{k}$, but for large values of wavenumber multiple discrete wavenumbers correspond to a single frequency. This is due the fact that for the AVF integrator there exists no diffeomorphism between the continuous and discrete dispersion relations as opposed to the symplectic and multisymplectic integrators for the KdV equation [1, 2, 3]. Artificial modes may also exist for large wave numbers.

The group velocity of the linearized KdV-KdV systems can be obtained by taking the first derivative of the dispersion relations (14) with respect to $\bar{k}$. The difference between the continuous and the discrete group velocities can be observed in Figure 6. Similar to the dispersion relations, for different values of λ, the numerical group velocities are close to the analytical ones for small wavenumbers, corresponding to long waves. That is, the sign of the group velocity, i.e. the direction of the flow for the waves, is well preserved for long waves. For large $\bar{k}$ values AVF integrator produces traveling waves where the numerical results are slower than the continuous ones. Again, because for the AVF integrator there exists no diffeomorphism between the continuous and discrete dispersion relations unlike for symplectic and multisymplectic integrators for the KdV equation, the sign of the group velocity is not preserved for all wave numbers.



(a) (b)

Figure 5. Dispersion curves for u of the linearized KdV-KdV system (1)



(a) (b)

Figure 6. Group velocities for u of the linearized KdV-KdV system (1)

The continuous and discrete dispersion relations and group velocities for the symmetric KdV-KdV equations (2) are similar to those for the KdV-KdV equation (1), and they are not given here.

4. Conclusions

The numerical results confirm the long-term preservation of the energy (Hamiltonian) and the integrals of the underlying equations. The numerical results compare well with those in the literature obtained by other methods. Dispersion analysis reveals that there does not exist such a diffemorphism between the continuous and discrete dispersion relations for the AVF method. Therefore for some wavenumber parasitic waves may exist.

Because the energy preserving methods are implicit as symplectic and multisymplectic integrators, the resulting nonlinear equations must be solved within the round-off error, to preserve symplecticity or the energy. This limits the applicability of these methods to large-scale systems. In these situations, either splitting can be used which are based on the splitting of the vector field in linear and nonlinear parts or linearly implicit methods which require the accurate solution of a linear system of equations in each time step [7].

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