TWMS J. App. and Eng. Math. V.13, N.4, 2023, pp. 1423-1433

DOMINATION CHANGING AND UNCHANGING SIGNED GRAPHS UPON THE VERTEX REMOVAL

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Abstract. A subset S of $V(\Sigma)$ is a dominating set of Σ if $|N^+(v) \cap S| > |N^-(v) \cap S|$ for all $v \in V - S$. This article is to start a study of those signed graphs that are stable and critical in the following way: If the removal of an arbitrary vertex does not change the domination number, the signed graph will be stable. The signed graph, on the other hand, is unstable if an arbitrary vertex is removed and the domination number changes. Specifically, we analyze the change in the domination of the vertex deletion and stable signed graphs.

Keywords: Signed graph, domination, domination stability, domination critical.

AMS Subject Classification: 05C22, 05C69.

1. Introduction

See [6, 10] for graph theory terminology and notation not covered here. We study the signed graph introduced by Harary [8]. A signed graph $\Sigma = (G, \sigma)$ is a graph $G = (V, E)$ together with a signing function $\sigma : E \to \{+, -\}$. For $S \subseteq V(\Sigma)$, $\Sigma[S]$ denotes the subgraph of Σ induced by S and let $v \in V(\Sigma)$. The *positive neighborhood* $N^+(v) = \{u \in V : uv \in E(G) \text{ and } \sigma(uv) = +\}$ and the *negative neighborhood* $N^-(v) = \{u \in V : uv \in E(G) \text{ and } \sigma(uv) = -\}$. Further, the *closed positive neighborhood* $N^+_\Sigma[v]$ of a vertex v in Σ is $N^+_\Sigma(v) \cup \{v\}$ and the *closed negative neighborhood* $N^-_\Sigma[v]$ of a vertex v in Σ is $N^-_\Sigma(v) \cup \{v\}$. The positive degree of v in Σ is $d^+(v) = |N^+(v)|$, the negative degree of v in Σ is $d^-(v) = |N^-(v)|$. The minimum positive (negative) and maximum positive (negative) degree among the vertices of Σ is denoted by δ^+ (δ^-) and Δ^+ $(\Delta^-,)$ respectively. A degree zero vertex of Σ is known as an *isolate* and a positive (negative) degree one vertex of Σ is known as a positive (negative) *leaf*, and its neighbor is called a positive (negative) *support vertex* and its incident edge is a positive (negative) *pendant edge*. If a vertex v

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§ Manuscript received: October 03, 2021; accepted: January 30, 2022.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.4
The first author is partially supported by Kalasalingam Academy of Research and Education.

is adjacent to two or more positive (negative) leaves, v is said to be a positive (negative) *strong support vertex*. The *distance d(x,y)* between vertices x and y of Σ is the length of a shortest $(x, y)-$ signed path in Σ. The maximum distance of any two vertices of Σ is the *diameter* of Σ, is referred as the *diam(Σ)*. We denote by P_n, C_n, K_n, $K_{1,n-1}$ and W_n, the signed *path*, the signed *cycle*, the signed *complete*, the signed *star* and the signed *wheel* on n vertices. A signed *tree* is an undirected signed graph in which any two vertices are connected by exactly one signed path.

We study domination in signed graphs introduced by Acharya [1]. In [11] we define $S \subseteq V(\Sigma)$ is known as a *dominating set* for Σ if $|N^+(v) \cap S| > |N^-(v) \cap S|$ for all $v \in V(\Sigma) - S$. The *domination number* $\gamma_s(\Sigma)$ is the minimum cardinality among all dominating sets of Σ. If S is a *dominating set* of Σ of size $\gamma_s(\Sigma)$, then we call S a $\gamma_s(\Sigma)$-set. Harary [9] was the first to propose the terms changing and unchanging. The critical vertex of domination in a graph G is the vertex with a reduction in the domination number, see [2,3,7,13–15,17–19] and graphs without a critical vertex, see [4,5,12,16].

In this paper, we study those signed graphs, in which when any vertex is removed, the dominion number increases. We study these signed graphs, in which when every vertex is removed the domination number remains unchanged.

2. Preliminary Results

Let $\Sigma - v$ represent the signed graph formed by removing the vertex v from Σ.
A signed graph is shown in the following classes.

$$\gamma_s(\Sigma - v) \neq \gamma_s(\Sigma) \text{ for all } v \in V(\Sigma)$$
$$\gamma_s(\Sigma - v) = \gamma_s(\Sigma) \text{ for all } v \in V(\Sigma)$$

Individual studies of these two classes have been conducted in the literature.
If $\gamma_s(\Sigma - v) \neq \gamma_s(\Sigma)$ for all $v \in V(\Sigma)$, then the signed graph is called a *vertex critical* signed graph.
If $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$ for all $v \in V(\Sigma)$, then the signed graph is called a *stable* signed graph.

Following that, we present some domination-related results that have already been demonstrated in [11].

Theorem 2.1. *[11] Consider a signed path $\Sigma = (v_1, v_2, ..., v_n)$ and let $n \geq 3$ by alternately assigning positive and negative signs to the edges. Then*

$$\gamma_s(\Sigma) = \begin{cases} \lceil \frac{n}{2} \rceil : \textit{if } n \textit{ is odd} \\ \frac{n}{2} : \textit{if } n \textit{ is even and both } v_1v_2 \textit{ and } v_{n-1}v_n \textit{ are in } E^+ \\ \frac{n}{2} + 1 : \textit{if } n \equiv 2(mod\,4) \textit{ and } v_1v_2 \textit{ and } v_{n-1}v_n \textit{ are in } E^- \\ \frac{n}{2} + 2 : \textit{if } n \equiv 0(mod\,4)\ v_1v_2 \textit{ and } v_{n-1}v_n \textit{ are in } E^- \end{cases}$$

Theorem 2.2. *[11] Consider a signed cycle $\Sigma = (v_1, v_2, ..., v_n, v_1)$ by alternately assigning positive and negative signs to the edges. Then*

$$\gamma_s(\Sigma) = \begin{cases} \frac{n}{2} : \textit{if } n \equiv 0(mod\,4) \textit{ and } n \equiv 3(mod\,4) \textit{ with } d^+(v_1) = 2, \\ \lfloor \frac{n}{2} \rfloor + 1 : \textit{if } n \equiv 2(mod\,4) \textit{ and } n \equiv 1(mod\,4), \\ \lceil \frac{n}{2} \rceil + 1 : \textit{if } n \equiv 3(mod\,4) \textit{ with } d^-(v_1) = 2. \end{cases}$$

3. Effects of vertex removal

Removing the vertex from the signed graph may increase, decrease or remain the signed graph's domination number. For example, consider a signed *star graph* Σ of order $n \geq 2$

with $m \geq \lceil \frac{n}{2} \rceil$ negative edges and Σ' is a signed graph obtained when a vertex from Σ has been removed. Then $\gamma_s(\Sigma) = m + 1$ and

$$\gamma_s(\Sigma') = \begin{cases} \gamma_s(\Sigma) : \text{if the vertex is a positive pendant vertex} \\ \gamma_s(\Sigma) - 1 : \text{if the vertex is a negative pendant vertex} \\ n : \text{if the vertex is a center vertex} \end{cases}$$

Definition 3.1. *Let Σ be a signed graph, we define a partition of $V(\Sigma) = V^0(\Sigma) \cup V^+(\Sigma) \cup V^-(\Sigma)$ of its vertex set, where*

$$V^0(\Sigma) = \{v \in V(\Sigma) : \gamma_s(\Sigma - v) = \gamma_s(\Sigma)\}$$
$$V^+(\Sigma) = \{v \in V(\Sigma) : \gamma_s(\Sigma - v) > \gamma_s(\Sigma)\}$$
$$V^-(\Sigma) = \{v \in V(\Sigma) : \gamma_s(\Sigma - v) < \gamma_s(\Sigma)\}$$

We simply write V^0, V^+, and V^-.

We observe that some of these sets may be empty. For instance, consider any signed graph with $V(\Sigma) = V^-$ so that $V^+ = V^0 = \emptyset$.

4. Domination changing signed graphs

This section examines the increasing and decreasing domination numbers when a vertex is removed from the signed graph.
We begin by giving some useful properties of V^+ and V^-.

Proposition 4.1. *(a) If I denotes the collection of all isolated vertices of Σ, then $I \subseteq V^-$.*
(b) $|V^+| \leq \gamma_s(\Sigma)$.

Proof. (a) Let S be a γ_s-set of Σ. If v is an isolated vertex, then v should be in S. Otherwise, $|N^-(v) \cap S| = 0 = |N^+(v) \cap S|$ and $\gamma_s(\Sigma - \{v\}) = \gamma_s(\Sigma) - 1 < \gamma_s(\Sigma)$. Hence $v \in V^-$ and hence the result follows.
(b) Let $v \in V - S$. For any vertex $w \in \Sigma' = \Sigma - \{v\}$, $|N^+(w) \cap S|$ in Σ is same as $|N^+(w) \cap S|$ in Σ' and $|N^-(w) \cap S|$ in Σ is same as $|N^-(w) \cap S|$ in Σ' so the removal of any vertex v in $V - S$ cannot increase the domination number. Hence $|V^+| \leq \gamma_s(\Sigma)$. $\square$

Proposition 4.2. *Let $v \in V(\Sigma)$ be a pendant vertex with $deg^-(v) = 1$. Then $v \in V^-$.*

Proof. We know that every negative pendant vertex is in every γ_s-set of Σ. Hence v is in a every γ_s-set of Σ. Removal of v, decrease domination number exactly one. Thus $\gamma_s(\Sigma - v) = \gamma_s(\Sigma) - 1 < \gamma_s(\Sigma)$. Hence $v \in V^-$. $\square$

Theorem 4.1. *A vertex $v \in V^-(\Sigma)$ if and only if $N^-(v) \cap S \neq \emptyset$ for some γ_s-set S containing v.*

Proof. Let $v \in V^-$. Let D be a γ_s-set of $\Sigma - \{v\}$. Then there is a γ_s-set S of Σ which properly contains D. Let $u \in D$. If $N^+(u) \cap D \supseteq \{v\}$, then D is a γ_s-set of Σ, contradiction. Hence $u \in N^-(v)$, $N^-(v) \cap S \neq \emptyset$.
Conversely, suppose v belongs to some γ_s-set S of Σ and $N^-(v) \cap S \neq \emptyset$. Then $S - \{v\}$ is a dominating set of $\Sigma - \{v\}$. Hence $v \in V^-$. $\square$

Theorem 4.2. *A vertex $v \in V^-(\Sigma)$ if and only if there exists some $\gamma_s(\Sigma)$-set S and a vertex $u \in S$ such that $v \notin S$ and $S \cap N^+(v) = \{u\}$.*

Proof. Let $v \in V^-(\Sigma)$. Let D be a $\gamma_s(\Sigma - v)$-set. For each vertex $u \in (\Sigma - \{v\}) - D$, satisfies the domination condition and D does not contain the neighbor of v. Then $D \cap N(v) = \emptyset$. Let $u \in N^+(v)$. Take $S = D \cup \{u\}$. Since D is a $\gamma_s(\Sigma - v)$-set, $S \cap N^+(v) = \{u\}$ and $S \cap N^-(v) = \emptyset$ and so S dominates Σ and $v \notin S$.

Conversely, suppose that there is a γ_s-set S in Σ such that $v \notin S$ with $S \cap N^+(v) = \{u\}$ where $u \in S$. $D = S - \{u\}$ is a dominating set in $\Sigma - \{v\}$ because u is a positive neighbor of v. Then $|D| = \gamma_s(\Sigma) - 1$ and so $v \in V^-(\Sigma)$. □

Proposition 4.3. *If $\gamma_s(\Sigma) = 1$, then $V^- = \emptyset$.*

Proof. If $w \in V^-$, then $\gamma_s(\Sigma - w) < \gamma_s(\Sigma)$. But $\gamma_s(\Sigma) = 1$, so $\gamma_s(\Sigma - w) = 0$, a contradiction. $V^- = \emptyset$ and the result follows. □

Proposition 4.4. *Every vertex $v \in V^+$ belong to each γ_s-set of Σ.*

Proof. Let $v \in V^+$ and let S be a γ_s-set of Σ. If $v \notin S$, then S is a γ_s-set of $\Sigma - v$. As a result, $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$, contradiction. □

Theorem 4.3. *A vertex v belongs $V^+(\Sigma)$ if and only if*
(a) v is in every γ_s-set of Σ.
(b) There is no subset $S \subseteq V(\Sigma) - N[v]$ with cardinality $\gamma_s(\Sigma)$ that dominates $\Sigma - v$.

Proof. Let $v \in V^+$. By Theorem 4.2, (a) holds. Since $v \in S$, removal of v from Σ increase the domination number, (b) holds. Let $v \notin S$. Suppose there exists a γ_s-set S subset of $V(\Sigma) - N[v]$ with cardinality $\gamma_s(\Sigma) = |S|$ dominates $\Sigma - v$. Then $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$, a contradiction. Hence (b) holds.
Conversely, suppose (a) and (b) hold. Let $v \in S$. Then $\gamma_s(\Sigma - v) > \gamma_s(\Sigma)$, and so $v \in V^+$. Let $v \notin S$. Let D be γ_s-set in $\Sigma - \{v\}$. Suppose D contains the neighbor of v. For $v \in V - D$, $|N^+(v) \cap D| > |N^-(v) \cap D|$, D is a dominating set in Σ with cardinality $|D| > \gamma_s(\Sigma)$, a contradiction. Hence $v \in V^+$. □

For any γ_s-set S and $v \in S$, define $A_s(v) = \{u : u \notin S \, and \, N(u) \cap S = \{v\}\}$.

Proposition 4.5. *If $A_s(v)$ has at least two vertices, then $\gamma_s(\Sigma - v) > \gamma_s(\Sigma)$.*

Proof. Let S denote a γ_s-set of Σ and $v \in S$. Suppose there are vertices $a, b \in A_s(v)$, such that $N(a) \cap S = \{v\}$ and $N(b) \cap S = \{v\}$. Suppose a and b are non-adjacent in Σ. Since a, $b \in A_s(v)$, there is no vertex in $S - \{v\}$ such that $N(a) \cap S - \{v\} = \emptyset = N(b) \cap S - \{v\}$. Then $\gamma_s(\Sigma - v) > \gamma_s(\Sigma)$. Suppose a and b are adjacent in Σ. As above the argument, we obtain the results as follows, $\gamma_s(\Sigma - v) > \gamma_s(\Sigma)$. □

Observation 4.1. *If $v \in V^+$, then v has at least one negative neighbor in S.*

As a consequence of the Observation 4.1, we have

Proposition 4.6. *If $v \in V^+$ and $u \in V^-$, then uv is an edge.*

Proof. Let $v \in V^+$ and $u \in V^-$. Suppose uv is not an edge of Σ. Let S be a $\gamma_s(\Sigma - \{v\})$-set. Then by Theorem 4.2, $S \cap N^+(v) \neq \{u\}$. Hence $u \notin S$. Let $w \in N^+(v)$ and $w \neq u$. Then $D = S \cup \{w\}$ is a γ_s-set of Σ and so u is not belong to every γ_s-set of Σ, a contradiction to $v \in V^+$. Hence uv is an edge of Σ. □

The impact of deleting any vertex in Σ on the domination number of a signed graph is our next result.

Theorem 4.4. *Let Σ be a signed graph of order n. Then for every vertex $v \in V(\Sigma)$,*
$\gamma_s(\Sigma) - 2 \leq \gamma_s(\Sigma - v) \leq \gamma_s(\Sigma) - 1 + deg(v) + k, \; k < n.$

Proof. First, we set the lower bound. Let S be any $\gamma_s(\Sigma - v)$-set. If $N^-(v) \cap S \neq \emptyset$, then $S \cup \{v\}$ is a γ_s-set. If $N^+(u) \cap S \neq \emptyset$, then $S \cup \{u\}$ is a γ_s-set where $u \in N^+(v)$. If $N(v) \cap S = \emptyset$, then $S \cup \{u, v\}$ is a γ_s-set where $u \in N^-(v)$. From the above three cases, $\gamma_s(\Sigma) \leq |S| + 2$ and so $\gamma_s(\Sigma) - 2 \leq \gamma_s(\Sigma - v)$. Let S represent any $\gamma_s(\Sigma)$-set of Σ.

If $v \notin S$, then S is a γ_s-set of $\Sigma - v$ and so $\gamma_s(\Sigma - v) \leq \gamma_s(\Sigma)$. Suppose $v \in S$. Let B be the set of vertices in $V - S$ that are dominated by v. Then $|B| \leq deg(v)$. Let A be the set of vertices in $V - S$ such that each vertex $v \in A$ is the negative neighbor of $w \in B$. Then A is dominated by $S - \{v\}$. It follows that $S \cup A \cup B - \{v\}$ is a γ_s-set of $\Sigma - v$ and hence $\gamma_s(\Sigma - v) \leq |S| - 1 + |A| + |B| \leq \gamma_s(\Sigma) - 1 + deg(v) + k$. □

We begin by defining the $i-$ stability and $d-$ stability of a signed graph.

Definition 4.1. *The $i-$ stability, denoted as $\gamma_s^+(\Sigma)$, is the minimum number of vertices that must be removed to increase the domination number in Σ.*
The $d-$ stability, denoted as $\gamma_s^-(\Sigma)$, is the minimum number of vertices that must be removed to reduce the domination number in Σ.

Observation 4.2. *For a signed path Σ on n vertices where $n \equiv 0(mod\,4)$ whose edges are arranged in a positive and negative sign alternatively, $i-$ stability does not exist.*

We prove that for sufficiently large n, $\gamma_s^+(\Sigma)$ and $\gamma_s^-(\Sigma)$ are constant for signed paths and signed cycles.

Theorem 4.5. *Consider a signed path $\Sigma = (v_1, v_2, ..., v_n)$ on n vertices, and the edges of Σ are assigned as $+$ or $-$, respectively. Then*

$$V^+(\Sigma) + V^-(\Sigma) = \begin{cases} \emptyset : \textit{if } n \equiv 0(mod\,4), n > 8 \\ \emptyset : \textit{if } n \equiv 2(mod\,4), \textit{ with } \delta^+(\Sigma) = 0 \\ \lfloor \frac{n}{4} \rfloor + 1 : \textit{if } n \textit{ is odd} \\ \lceil \frac{n}{3} \rceil + 2 : \textit{if } n \equiv 2(mod\,4) \textit{ with } \delta^+(\Sigma) = 0 \\ \frac{n}{2} + 2 : \textit{if } n \equiv 0(mod\,4) \textit{ with } \delta^+(\Sigma) = 0 \end{cases}$$

Proof. Let Σ be a signed path.
Case 1: n is even.
By Theorem 2.1, $\gamma_s(\Sigma) = \frac{n}{2}$. If i is odd, $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an even signed path on $i-1$ vertices and edges of Σ_{i-1} are alternatively arranged in $+$ and $-$ sign and Σ_{n-i} is an odd signed path on $n-i$ vertices and edges of Σ_{n-i} are arranged in alternatively $-$ and $+$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2}$ and hence $\gamma_s(\Sigma - \{v_i\}) = \gamma_s(\Sigma)$. If i is even, $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an odd signed path on $i-1$ vertices and edges of Σ_{i-1} are alternatively arranged in $+$ and $-$ sign and Σ_{n-i} is an even signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $+$ and $-$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2}$ and hence $\gamma_s(\Sigma - \{v_i\}) = \gamma_s(\Sigma)$.
Hence for any vertex $v \in V(\Sigma)$, $\gamma_s(\Sigma - \{v\}) = \gamma_s(\Sigma)$ and hence $V(\Sigma) = V^0$.
Case 2: n is odd.
Subcase 1: $n \equiv 1(mod\,4)$.
By Theorem 2.1, $\gamma_s(\Sigma) = \lceil \frac{n}{2} \rceil$. If $i \equiv 1(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an even signed path on $i-1$ vertex and edges of Σ_{i-1} are alternatively arranged in $+$ and $-$ signs and Σ_{n-i} is an even signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $-$ and $+$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2} + 2$ and hence $\gamma_s(\Sigma - \{v_i\}) > \gamma_s(\Sigma)$.
Subcase 2: $n \equiv 3(mod\,4)$.
If $i \equiv 3(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an even signed path on $i-1$ vertex and edges of Σ_{i-1} are alternatively arranged in $+$ and $-$ signs and Σ_{n-i} is an even signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $-$ and $+$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2} + 2$ and

hence $\gamma_s(\Sigma - \{v_i\}) > \gamma_s(\Sigma)$. From the above two cases, for any vertex $v_i \in V(\Sigma)$, $i \equiv 1(mod\,4)$ and $i \equiv 3(mod\,4)$ is removed from Σ, increase the domination number and so $|V^+| = \lfloor \frac{n}{4} \rfloor$. By the assumption, Σ contains exactly one negative pendant vertex say $\{v_n\}$. By Proposition 4.2, $v_n \in V^-$ and hence $|V^-| = 1$. Hence the result follows.

Case 3: $n \equiv 2(mod\,4)$ with $\delta^+(\Sigma) = 0$.

By Theorem 2.1, $\gamma_s(\Sigma) = \frac{n}{2} + 1$. Let $v_i \in V(\Sigma)$. Since the two pendant vertices of Σ are negative, $\{v_1\}$ and $\{v_2\}$ are in V^- and so $|V^-| = 2$. If $i \equiv 2(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an odd signed path on $i-1$ vertices and edges of Σ_{i-1} are alternatively arranged in $-$ and $+$ sign and Σ_{n-i} is an even signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $-$ and $+$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2} + 2$ and hence $\gamma_s(\Sigma - \{v_i\}) > \gamma_s(\Sigma)$. If $i \equiv 1(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an even signed path on $i-1$ vertices and edges of Σ_{i-1} are alternatively arranged in $-$ and $+$ sign and Σ_{n-i} is an odd signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $+$ and $-$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2} + 2$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2}$ and hence $\gamma_s(\Sigma - \{v_i\}) > \gamma_s(\Sigma)$. Hence for any vertex $v_i \in V(\Sigma)$ is removed from Σ, increase the domination number and so $V^+ = \{v_i : i \equiv 1\, and\, (mod\,4)\}$ and $|V^+| = \lceil \frac{n}{4} \rceil$. Hence the result follows.

Case 4: $n \equiv 0(mod\,4)$ with $\delta^+(\Sigma) = 0$.

By Theorem 2.1, $\gamma_s(\Sigma) = \frac{n}{2} + 2$. Since Σ contains two negative pendant vertices, $\{v_1\}$ and $\{v_2\}$ are in V^-. Let $v_i \in V(\Sigma)$. If $i \equiv 2(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an odd signed path on $i-1$ vertex and edges of Σ_{i-1} are arranged in $-$ and $+$ signs alternatively and Σ_{n-i} is an even signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $-$ and $+$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2}$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2} + 2$ and hence $\gamma_s(\Sigma - \{v_i\}) < \gamma_s(\Sigma)$. If $i \equiv 3(mod\,4)$, then $\Sigma - \{v_i\} = \Sigma_{i-1} \cup \Sigma_{n-i}$ where Σ_{i-1} is an even signed path on $i-1$ vertices and edges of Σ_{i-1} are alternatively arranged in $-$ and $+$ sign and Σ_{n-i} is an odd signed path on $n-i$ vertices and edges of Σ_{n-i} are alternatively arranged in $+$ and $-$ sign. Then by Theorem 2.1, $\gamma_s(\Sigma_{i-1}) = \frac{i-1}{2} + 2$ and $\gamma_s(\Sigma_{n-i}) = \frac{n-i}{2}$ and hence $\gamma_s(\Sigma - \{v_i\}) < \gamma_s(\Sigma)$. Hence $V^- = \{v_i : i \equiv 2\; and\, 3(mod\,4)\} \cup \{v_1, v_n\}$ and $|V^-| = \frac{n}{2} + 2$ and $|V^+| = \emptyset$. As a consequence, the outcome is as follows. □

Theorem 4.6. *Consider a signed path $\Sigma = (v_1, v_2, ..., v_n)$ on n vertices, and the edges of Σ are assigned as $+$ or $-$, respectively. Then*

$$\gamma_s^+(\Sigma) = \begin{cases} 1 : \text{if } n \text{ is odd} \\ 2 : \text{if } n \text{ is even } n \geq 6 \text{ with } \delta^-(\Sigma) = 0 \\ 3 : \text{if } n \equiv 0(mod\,4), n > 8 \text{ with } \delta^+(\Sigma) = 0 \\ 1 : \text{if } n \equiv 2(mod\,4), \text{ with } \delta^+(\Sigma) = 0 \end{cases}$$

Proof. Case 1: $n \equiv 1(mod\,4)$.

In this case, $\{v_1\}$ is a positive pendant vertex and $\{v_n\}$ is negative pendant vertex of Σ. When we take $\{v_1\}$ out of Σ, we get $\Sigma - \{v_1\} = \Sigma^*$, which is a signed path with $n-1$ vertices and edges that alternately have $-$ and $+$ signs. By Theorem 2.1, $\gamma_s(\Sigma^*) = \frac{n-1}{2} + 2$ and $\gamma_s(\Sigma^*) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) = 1$.

Case 2: $n \equiv 3(mod\,4)$.

If we remove $\{v_3\}$ from Σ, we obtain two components Σ_1 and Σ_2 signed paths with $|\Sigma_1| = 2$ and $|\Sigma_2| = n-3$. Since Σ_1 has only positive edge, $\gamma_s(\Sigma) = 1$. Since Σ_2 is an even signed path and edges that alternatively in $-$ and $+$ sign and by Theorem 2.1, $\gamma_s(\Sigma_2) = \frac{n-3}{2} + 2$. Then $\gamma_s(\Sigma_1) + \gamma_s(\Sigma_2) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) = 1$.

Case 3: n is even and $n \geq 6$ with $\delta^-(\Sigma) = 0$.

If we take away $\{v_1\}$ and $\{v_n\}$ from Σ, we get the signed path Σ^* with $n-2$ vertices and edges that alternate between $-$ and $+$ sign. By Theorem 2.1, $\gamma_s(\Sigma^*) = \frac{n-2}{2}+2$ and $\gamma_s(\Sigma^*) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) \leq 2$. By Theorem 4.5, $\gamma_s^+(\Sigma) \geq 2$. Hence $\gamma_s^+(\Sigma) = 2$.
Case 4: $n \equiv 0(mod\,4)$, $n > 8$ with $\delta^+(\Sigma) = 0$.
If we remove $\{v_2\}$, $\{v_7\}$ and $\{v_8\}$ from Σ, we obtain three components of Σ say Σ_1, Σ_2 and Σ_3 signed paths with $|\Sigma_1| = 2$, $|\Sigma_2| = 4$ and $|\Sigma_3| = n-8$. Since Σ_1 has only positive edge, $\gamma_s(\Sigma) = 1$. Since the edges of Σ_2 and Σ_3 alternate between $-$ and $+$ signs, by Theorem 2.1, $\gamma_s(\Sigma_2) = 4$ $\gamma_s(\Sigma_3) = \frac{n-8}{2}+2$. Then $\gamma_s(\Sigma_1)+\gamma_s(\Sigma_2) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) \leq 3$. By theorem 4.5, $\gamma_s^+(\Sigma) \geq 2$. Since $\gamma_s(\Sigma_{n-1}) = \gamma_s(\Sigma_{n-2}) = \lceil \frac{n}{2} \rceil < \gamma_s(\Sigma_n) = \lceil \frac{n}{2} \rceil + 2$ and so any pairs of vertices are removed from Σ decrease the domination number, $\gamma_s^+(\Sigma) = 3$.
Case 5: $n \equiv 2(mod\,4)$ with $\delta^+(\Sigma) = 0$.
If we remove $\{v_2\}$ from Σ, we obtain two signed path graphs say Σ_1 and Σ_2 with $|\Sigma_1| = 1$ and $|\Sigma_2| = n-2$ respectively, and the edges of Σ_2 alternately arranged in $-$ and $+$ sign. By Theorem 2.1, $\gamma_s(\Sigma - \{v_2\}) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) = 1$. □

Theorem 4.7. *Consider a signed path $\Sigma = (v_1, v_2, ..., v_n)$ on n vertices, and the edges of Σ are assigned as $+$ or $-$, respectively. Then*

$$\gamma_s^-(\Sigma) = \begin{cases} 1: \text{if } n \text{ is odd} \quad \text{or } n \equiv 0(mod\,4), n > 8 \\ 1: \text{if } n \equiv 2(mod\,4), \text{ with } \delta^+(\Sigma) = 0 \\ 2: \text{if } n \text{ is even } n \geq 6 \text{ with } \delta^-(\Sigma) = 0 \end{cases}$$

Proof. Case 1: n is odd.
In this case, $\{v_1\}$ is a positive pendant vertex and $\{v_n\}$ is a negative pendant vertex of Σ. Since every negative pendant vertex belongs to V^-, $\gamma_s(\Sigma - \{v_n\}) < \gamma_s(\Sigma)$ and hence $\gamma_s^-(\Sigma) = 1$.
Case 2: n is even, $n \geq 6$ with $\delta^-(\Sigma) = 0$.
If we remove $\{v_3\}$ and $\{v_4\}$ from Σ, we obtain the two components of signed paths Σ' and Σ'' with $|\Sigma'| = 2$ and $|\Sigma''| = n-4$. Since Σ' has only positive edge, $\gamma_s(\Sigma') = 1$. Now the signed path Σ'' is an odd signed path on $n-4$ vertices whose edges are alternatively in $-$ and $+$ sign. By Theorem 2.1, $\gamma_s(\Sigma'') = \frac{n-4}{2}+2$. Now $\gamma_s(\Sigma')+\gamma_s(\Sigma'') < \gamma_s(\Sigma)$. Hence $\gamma_s(\Sigma) \leq 2$. By Theorem 4.6, $\gamma_s^-(\Sigma) \neq 1$ and hence $\gamma_s^-(\Sigma) > 1$.
Case 3: $n \equiv 0$ *or* $2(mod\,4)$, $n > 8$ with $\delta^+(\Sigma) = 0$.
If we remove either $\{v_1\}$ or $\{v_n\}$ from Σ, we obtain the signed path Σ_1 with $|\Sigma_1| = n-1$. By Theorem 2.1, $\gamma_s(\Sigma_1) < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) = 1$. □

Observation 4.3. *Consider a signed cycle $\Sigma = \{v_1, v_2, ..., v_n\}$ on n vertices and the edges of Σ are alternately assigned to $+$ and $-$.*

- *If $n \equiv 0(mod\,4)$, then $V(\Sigma) = V^0$.*
- *If $n \equiv 2(mod\,4)$, then $V(\Sigma) = V^-$.*
- *If $n \equiv 1(mod\,4)$ with $d^+(v_1) = 2$, then $|V^+| + |V^-| = 1 + \frac{n}{2}$.*
- *If $n \equiv 3(mod\,4)$ with $d^+(v_1) = 2$, then $|V^+| = \frac{n}{2}$.*
- *If $n \equiv 1(mod\,4)$ with $d^-(v_1) = 2$, then $|V^+| + |V^-| = \frac{n}{2} + 1$.*
- *If $n \equiv 3(mod\,4)$ with $d^-(v_1) = 2$, then $|V^-| = \frac{n}{2} + 2$.*

Theorem 4.8. *Let Σ be a signed cycle on n vertices and the edges of Σ be arranged in alternatively $+$ and $-$ signs and vice verse. Then*

$$\gamma_s^+(\Sigma) = \begin{cases} 1 : \textit{if } n \textit{ is odd with } \delta^-(\Sigma) = 0 \\ 1 : \textit{if } n \equiv 1(mod\, 4) \textit{ with } \delta^+(\Sigma) = 0 \\ 3 : \textit{if } n \equiv 3(mod\, 4),\ n \geq 11 \textit{ with } \delta^+(\Sigma) = 0 \\ 2 : \textit{if } n \equiv 0(mod\, 4),\ n \geq 12 \\ 4 : \textit{if } n \equiv 2(mod\, 4),\ n \geq 14 \end{cases}$$

Proof. Case 1: n is odd with $\delta^-(\Sigma) = 0$.
By Theorem 2.2, for $n \equiv 3(mod\, 4)$ with $d^+(v_1) = 2$, $\gamma_s(\Sigma) = \frac{n}{2}$ and for $n \equiv 1(mod\, 4)$ with $d^+(v_1) = 2$, $\gamma_s(\Sigma) = \lfloor \frac{n}{2} \rfloor + 1$. Let $\Sigma - \{v_1\} = \Sigma^*$ be a signed path on $n-1$ vertices with the edges of Σ^* alternately arranged in $-$ and $+$ sign when $\{v_1\}$ is removed from Σ. By Theorem 2.1, $\gamma_s(\Sigma^*) = \frac{n-1}{2} + 2$ and $\gamma_s(\Sigma^*) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) = 1$.
Case 2: $n \equiv 1(mod\, 4)$ with $\delta(\Sigma)^+ = 0$.
By Theorem 2.2, $\gamma_s(\Sigma) = \lfloor \frac{n}{2} \rfloor + 1$. Let $\Sigma - \{v_2\} = \Sigma'$ be a signed path on $n-1$ vertices with the edges of Σ' alternately arranged in $-$ and $+$ sign when $\{v_2\}$ is removed from Σ. By Theorem 2.1, $\gamma_s(\Sigma') = \frac{n-1}{2} + 2$ and $\gamma_s(\Sigma') > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) = 1$.
Case 3: $n \equiv 3(mod\, 4)$, $n \geq 11$ with $\delta(\Sigma)^+ = 0$.
By Theorem 2.2, $\gamma_s(\Sigma) = \lceil \frac{n}{2} \rceil + 1$. If we remove $\{v_5\}$, $\{v_6\}$ and $\{v_n\}$ from Σ, we obtain two components of even signed paths Σ' and Σ'' with $|\Sigma'| = 4$, $|\Sigma''| = n-7$ and Σ' and Σ" have their edges alternately arranged in $-$ and $+$ sign. By Theorem 2.1, $\gamma_s(\Sigma') = \gamma_s(\Sigma'')$ and $\gamma_s(\Sigma') + \gamma_s(\Sigma'') > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) \leq 3$. By the Observation 4.3, $V^+(\Sigma) = \emptyset$ and $\gamma_s^+(\Sigma) \geq 2$. Since any pairs of vertices are removed from Σ decrease the domination number, $\gamma_s^+(\Sigma) \geq 3$.
Case 4: $n \equiv 0(mod\, 4)$, $n \geq 12$.
By Theorem 2.2, $\gamma_s(\Sigma) = \frac{n}{2}$. If we remove $\{v_4\}$ and $\{v_{11}\}$ from Σ, we obtain two components of even signed paths Σ' and Σ'' with $|\Sigma'| = 4$ and $|\Sigma''| = n-6$. By Theorem 2.1, $\gamma_s(\Sigma') = \frac{4}{2} + 2$ and $\gamma_s(\Sigma'') = \frac{n-6}{2} + 1$. Then $\gamma_s(\Sigma') + \gamma_s(\Sigma'') > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) \leq 2$. By the Observation 4.3, $V^+(\Sigma) = \emptyset$ and $\gamma_s^+(\Sigma) \geq 2$.
Case 5: $n \equiv 2(mod\, 4)$, $n \geq 14$.
By Theorem 2.2, $\gamma_s(\Sigma) = \lceil \frac{n}{2} \rfloor + 1$. If we remove $\{v_1, v_6, v_7, v_{12}\}$ from Σ, we obtain three components of even signed paths Σ_1, Σ_2 and Σ_3, with $|\Sigma_1| = 4$, $|\Sigma_2| = 4$, and $|\Sigma_3| = n - 12$. By Theorem 2.1, $\gamma_s(\Sigma_1) = \frac{4}{2} + 2 = \gamma_s(\Sigma_2)$ and $\gamma_s(\Sigma_3) = \frac{n-12}{2}$. Then $\gamma_s(\Sigma_1) + \gamma_s(\Sigma_2) + \gamma_s(\Sigma_3) > \gamma_s(\Sigma)$ and so $\gamma_s^+(\Sigma) \leq 4$. By the Observation 4.3, $V^+(\Sigma) = \emptyset$ and $\gamma_s^+(\Sigma) \geq 2$. Since any three vertices of Σ are removed from Σ decrease the domination number, $\gamma_s^+(\Sigma) \geq 4$. $\square$

Theorem 4.9. *Consider a signed cycle $\Sigma = \{v_1, v_2, ..., v_n\}$ on n vertices, and the edges of Σ are assigned as $+$ or $-$, respectively. Then*

$$\gamma_s^-(\Sigma) = \begin{cases} 1 : \textit{if } n \textit{ is odd with } \delta^+(\Sigma) = 0 \\ 1 : \textit{if } n \equiv 1(mod\, 4) \textit{ with } \delta^-(\Sigma) = 0 \\ 3 : \textit{if } n \equiv 3(mod\, 4), \textit{ with } \delta^-(\Sigma) = 0 \\ 2 : \textit{if } n \equiv 0(mod\, 4),\ n \geq 12 \\ 1 : \textit{if } n \equiv 2(mod\, 4),\ n \geq 14 \end{cases}$$

Proof. Case 1: $n \equiv 1(mod\, 4)$ with $\delta^+(\Sigma) = 0$.
If we remove $\{v_1\}$ from Σ, we obtain $\Sigma - \{v_1\} = \Sigma^*$ is a signed path on $n-1$ vertices with alternately arranged in the $+$ and $-$ sign. By Theorem 2.1, $\gamma_s(\Sigma^*) = \frac{n-1}{2}$ and by Theorem 2.2, $\gamma_s(\Sigma^*) < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) = 1$.
Case 2: $n \equiv 3(mod\, 4)$ with $\delta^+(\Sigma) = 0$.

When we subtract $\{v_2\}$ from Σ, we get $\Sigma - \{v_2\} = \Sigma^*$, which is a signed path with $n-1$ vertices. By Theorem 2.1, $\gamma_s(\Sigma^*) = \frac{n-1}{2}$ and by Theorem 2.2, $\gamma_s(\Sigma^*) < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) = 1$.
Case 3: $n \equiv 1(mod\,4)$ and $\delta^-(\Sigma) = 0$.
By Theorem 2.2, $\gamma_s(\Sigma) = \lfloor \frac{n}{2} \rfloor + 1$. If we remove $\{v_2\}$ from Σ, we obtain $\Sigma - \{v_2\} = \Sigma'$ is a signed path on $n-1$ vertices with edges are alternately arranged in the $+$ and $-$ sign. By Theorem 2.1, $\gamma_s(\Sigma') = \frac{n-1}{2}$ and $\gamma_s(\Sigma') < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) = 1$.
Case 4: $n \equiv 3(mod\,4)$, and $\delta^-(\Sigma) = 0$.
By Theorem 2.2, $\gamma_s(\Sigma) = \frac{n}{2}$. If we remove $\{v_3\}$, $\{v_{n-2}\}$ and $\{v_{n-1}\}$ from Σ, we obtain two components of signed paths Σ' and Σ'' with $|\Sigma'| = 3$ and $|\Sigma''| = n-6$ and the edges of Σ'' are alternately arranged in a $-$ and $+$ sign. Since Σ' is isomorphic to unsigned path on 3 vertices, $\gamma_s(\Sigma') = 1$. By Theorem 2.1, $\gamma_s(\Sigma'') = \lceil \frac{n-6}{2} \rceil$ Then $\gamma_s(\Sigma') + \gamma_s(\Sigma'') < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) \leq 3$. By the Observation 4.3, $V^-(\Sigma) = \emptyset$ and $\gamma_s^-(\Sigma) \geq 2$. Since any pairs of vertices are removed from Σ increase the domination number, $\gamma_s^-(\Sigma) \geq 3$.
Case 5: $n \equiv 0(mod\,4)$, $n \geq 12$.
By Theorem 2.2, $\gamma_s(\Sigma) = \frac{n}{2}$. If we remove $\{v_1, v_2\}$ from Σ, we obtain an even signed path Σ' with $|\Sigma'| = n-2$. By Theorem 2.1, $\gamma_s(\Sigma') = \frac{n-2}{2}$. Then $\gamma_s(\Sigma') < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) \leq 2$. By the Observation 4.3, $V^-(\Sigma) = \emptyset$ and $\gamma_s^-(\Sigma) \geq 2$.
Case 6: $n \equiv 2(mod\,4)$, $n \geq 14$.
By Theorem 2.2, $\gamma_s(\Sigma) = \lfloor \frac{n}{2} \rfloor + 1$. If we remove $\{v_2\}$ from Σ, we obtain an isolated vertex v_1 and an odd signed path Σ' with $|\Sigma'| = n-2$. By Theorem 2.1, $\gamma_s(\Sigma') = \lceil \frac{n-2}{2} \rceil$. Then $1 + \gamma_s(\Sigma') < \gamma_s(\Sigma)$ and so $\gamma_s^-(\Sigma) \leq 1$. By the Observation 4.3, $V^-(\Sigma) = V(\Sigma)$ and $\gamma_s^-(\Sigma) \geq 1$. $\square$

5. Domination stable signed graphs

This section examines how the domination number remains stable when a vertex is removed from a signed graph and we characterize the k-stable signed graph.
We make a useful definitions and we will use the following observations.

Definition 5.1. *A vertex $v \in V(\Sigma)$ is stable if $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$.*

Definition 5.2. *A signed graph Σ is γ_s-stable if $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$ for any vertex $v \in V(\Sigma)$. If so, then $V = V^0$.*

Definition 5.3. *A signed graph Σ is k-stable if it is stable and $\gamma_s(\Sigma) = k$.*

We begin with the following observations.

Observation 5.1. *If Σ has either an isolated vertex or negative pendant vertex, then Σ is not γ_s-stable signed graph. So we consider the signed graphs without isolated vertex and negative pendant vertex.*

As a direct result of the Observation 5.1, we have,

Corollary 5.1. *A signed graph Σ is γ_s-stable graph if and only if for each vertex v either (a) v is in every γ_s-set and there exist a dominating set S in $\Sigma - v$ such that $|S| = \gamma_s(\Sigma)$ and $S \subseteq V(\Sigma) - N[v]$ or*
(b) There is no $\gamma_s(\Sigma)$-set S such that $v \notin S$ and $S \cap N^+(v) = \{u\}$ for some vertex $u \in S$.

We are now in a position to characterize the connected γ_s-stable signed graph Σ.

Theorem 5.1. *Let Σ be a connected signed graph of order n having at least one positive pendant vertex and no negative pendant vertex. Then Σ is k-stable signed graph if and only if $\Sigma \cong cor(\Sigma^*)$ for some connected signed graph Σ^* of order k, $\delta^-(\Sigma^*) \geq 1$.*

Proof. Suppose $\Sigma \cong cor(\Sigma^*)$ for some connected signed graph Σ^* of order k, $\delta^-(\Sigma^*) \geq 1$. Let S be a γ_s-set of Σ with $|S| = \gamma_s(\Sigma) = |V(\Sigma^*)| = k$. Let $v \in V(\Sigma) - S$ be a positive pendant vertex of Σ. Then $V(\Sigma^*)$ dominates $\Sigma - v$ and so $\gamma_s(\Sigma - v) = |V(\Sigma^*)| = \gamma_s(\Sigma) = k$. Hence Σ is a k-stable signed graph. Now, suppose that Σ is a k-stable signed graph with positive pendant vertex. Let v_1 be a positive pendant vertex and let v be a neighbor vertex of v_1. Suppose there exists a negative pendant vertex $v_2 \in N(v) - \{v_1\}$ with $v_2 \notin V(\Sigma^*)$. Then by Theorem 4.2, $v \in V^-$, a contradiction to Σ is a stable signed graph. Hence each vertex $N(v) - \{v_1\}$ is a support vertex of Σ and $\Sigma \cong cor(\Sigma^*)$ for for some connected signed graph Σ^* of order k, also $\gamma_s(\Sigma) = |V(\Sigma^*)| = k$. Now suppose induced Σ^* is a positive signed graph. Let $v \in V(\Sigma) - V(\Sigma^*)$. Then v is adjacent to a vertex u of Σ^* by a positive edge. Since $\sigma(uv) = +$, $V(\Sigma^*) - \{u\}$ dominates $\Sigma - \{v\}$ and so $\gamma_s(\Sigma - v) < V(\Sigma^*) = \gamma_s(\Sigma) - 1$, a contradiction. Hence induced Σ^* is not a positive signed graph and each vertex $v \in V(\Sigma^*)$ has at least one negative neighbor in Σ^*. □

We begin by showing that the removal of any stable vertex from a signed tree Σ leaves the domination number unchanged.

Lemma 5.1. *If v is a stable vertex of a signed tree Σ, $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$.*

Proof. Let $v \in V(\Sigma)$. If v is a positive pendant vertex, then $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$. Suppose v is not a positive pendant vertex. Then v is either a support vertex or a non-support vertex. Since v is stable, every $\gamma_s(\Sigma)$-set is a $\gamma_s(\Sigma)$-set for $\Sigma - v$ and $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$. Suppose v is adjacent to a positive end vertex u and at least one negative end vertex w. Let $\Sigma' = \Sigma - v - w$. Then $\gamma_s(\Sigma') \leq \gamma_s(\Sigma - u) \leq \gamma_s(\Sigma)$. If $\gamma_s(\Sigma') = \gamma_s(\Sigma) - 1$, then $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$. Suppose not, $\gamma_s(\Sigma') = \gamma_s(\Sigma) = \gamma_s(\Sigma - u)$. If v is not in any γ_s-set of Σ. Suppose every γ_s-set of Σ is a γ_s-set for $\Sigma - v$. Then $\gamma_s(\Sigma - v) \leq \gamma_s(\Sigma)$. Assume that $\gamma_s(\Sigma - v) = \gamma_s(\Sigma) - 1$. Let S' be a $\gamma_s(\Sigma - v)$-set. Then S' contains no neighbor of v. Let $w \in N^+(v)$. Then $S' \cup w$ would be $\gamma_s(\Sigma)$-set, contradiction. Suppose $\gamma_s(\Sigma - v) = \gamma_s(\Sigma) - 2$. Let D be a $\gamma_s(\Sigma - v)$-set. If either $N^+(v) \cap S \neq \emptyset$, or $N^-(v) \cap S \neq \emptyset$, then either $S \cup \{v\}$ or $S \cup \{u\}$, $u \in N^+(v)$ is a $\gamma_s(\Sigma)$-set of Σ, contradiction. Hence $N(v) \cap S = \emptyset$ and also $S \cup \{u, v\}$, $u \in N^-(v)$ is a $\gamma_s(\Sigma)$-set of Σ, contradiction to v is not in any $\gamma_s(\Sigma)$-set of Σ. Suppose $\gamma_s(\Sigma - v) = \gamma_s(\Sigma) - 1$. Let S_1 be a γ_s-set of $\Sigma - v$. If $N^-(v) \cap S_1 = \{v\}$, then $S_1 \cup \{v\}$ is a γ_s-set of Σ, contradiction to v is not in any γ_s-set. Suppose $N^+(v) \cap S_1 = \{u\}$. Then S_1 contains the neighbor of v and so $S_1 \cup \{u\}$ is a γ_s-set of Σ, and by Theorem 4.2, $v \in V^-$, contradiction. Hence $\gamma_s(\Sigma - v) = \gamma_s(\Sigma)$. □

Now we are ready to characterize γ_s-critical.

Theorem 5.2. *A signed tree Σ of order $n \geq 3$ is γ_s-critical if and only if each vertex v of Σ is either a negative leaf or a support vertex or $d^-(v) \geq d^+(v)$.*

Proof. Suppose Σ is a critical signed tree. Suppose that v is neither a support vertex nor a negative leaf such that $d^+(v) > d^-(v)$. Let $N(v) = \{u_1, u_2, .., u_k\}$. Let $\Sigma_1, .., \Sigma_k$ be the component of $\Sigma - v$. Since v is not a support vertex, each $u_i \in V(\Sigma_i)$ is a support vertex with at least one negative leaf. Also, every vertex Σ_i is either support or a negative leaf. Let S_i be a γ_s-set of Σ_i, $i = 1, .., k$. Let S be γ_s-set of Σ. Let $\Sigma' =_{i=1}^{k} \cup \Sigma_i = \Sigma - v$. Then $S' =_{i=1}^{k} \cup S_i$ is a γ_s-set of Σ and hence $\gamma_s(\Sigma) \leq |S'| = \gamma_s(\Sigma - v)$, contradiction. Thus Σ is a γ_s-critical signed graph. □

6. Conclusions

We discovered in this paper that when every vertex is removed, the domination number rises. We also looked at signed graphs with no difference in the domination number when a vertex is removed.

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