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NEW NOTIONS ON IDEAL CONVERGENCE OF TRIPLE SEQUENCES IN NEUTROSOPHIC NORMED SPACES

C. GRANADOS[1], C. CHOUDHURY[2*], §

Abstract. The usual convergence of sequences has many generalizations with the aim of providing deeper insights into the summability theory. In this paper, following a very recent and new approach, we introduce the notion of $\mathcal{I}_3$ and $\mathcal{I}_3^*-$convergence of triple sequences in neutrosophic normed spaces mainly as a generalization of statistical convergence of triple sequences. We investigate a few fundamental properties and study the relationship between the two notions. We also introduce and investigate the concept of $\mathcal{I}_3$ and $\mathcal{I}_3^*-$Cauchy sequence of triple sequences and show that the condition $(AP3)$ plays a crucial role to study the interrelationship between them.

Keywords: Statistical convergence, ideal, filter, $\mathcal{I}-$convergence, neutrosophic normed space, triple sequences.

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1. Introduction and Background

In 1951, the concept of statistical convergence was developed independently by Fast [4] and Steinhaus [32] to provide deeper insights into summability theory. Later on, it was further investigated from the sequence space point of view by Fridy [6, 7], Salat [29], and many researchers [30]. In 2003, Mursaleen and Edely [23] extended this concept over double sequences and mainly studied the relationship between statistical convergence and statistical Cauchy double sequences, statistical convergence, and strong Cesaro summable double sequences. Besides this, in [33], Tripathy studied various properties of the sequence spaces formed by statistical convergent double sequences and proved a decomposition theorem. Also, in 2007, Sahiner et. al. [27] investigated statistical convergence

[1] Estudiante de Doctorado en Matemáticas, Universidad de Antioquia, Medellín, Colombia.
e-mail: carlosgranadosortiz@outlook.es; ORCID: https://orcid.org/0000-0002-7754-1468.
[*] Corresponding author.
[2] Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India.
e-mail: chiranjibchoudhury123@gmail.com; ORCID: https://orcid.org/0000-0002-5607-9884.
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for triple sequences. Statistical convergence has many applications in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

The notion of $\mathcal{I}$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [19]. They used the notion of an ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ to define such a concept. For an extensive view of this article, we refer the reader to [3, 18, 35] where many more references can be found.

In 2005, Tripathy and Tripathy [34] introduced the concept of $\mathcal{I}-$convergence and $\mathcal{I}-$Cauchy sequence for double sequences and proved some properties related to solidity, symmetricity, completeness, and denseness. Furthermore, Das [2] and Kumar [20] also investigated the concept of $\mathcal{I}$ and $\mathcal{I}^*-$convergence of double sequences and proved some results more naturally. In 2008, Sahiner and Tripathy [28] studied ideal convergence in triple sequence settings. For more details on ideal convergence of triple sequences one may refer to [12] where many more references can be found.

The concept of fuzzy sets was first introduced by Zadeh [36] in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The theory of fuzzy sets cannot always cope with the lack of knowledge of membership degrees. To overcome the drawbacks, in 1986 Atanassov [1] introduced intuitionistic fuzzy sets as an extension of fuzzy sets. Intuitionistic fuzzy sets have been widely used to solve various decision-making problems.

Many times, decision-makers face some hesitations besides going to direct approaches (i.e., yes or no) in decision making. In addition, we can obtain a tricomponent outcome in some real events like sports, the procedure for voting, etc. Considering all in 2005, Smarandache [31] introduced the notion of Neutrosophic set as a generalization of both fuzzy set and intuitionistic fuzzy set. An element belonging to a neutrosophic set consists of a triplet namely truth-membership function (T), indeterminacy-membership function (F), and falsity-membership function (I). A neutrosophic set is determined as a set where every component of the universe has a degree of T, F, and I.

The notion of fuzzy normed space was introduced by Felbin [5] in the year 1992. Later on, in 2006 the concept of intuitionistic fuzzy normed spaces was introduced by Saadati and Park [26]. In 2008, statistical convergence in intuitionistic fuzzy normed spaces was introduced and investigated by Karakus et. al. [9]. For more details on statistical convergence and its related generalizations in intuitionistic fuzzy normed spaces, one may refer to [10, 15, 22, 24]. Recently, Kirisci and Simsek [16] introduced neutrosophic normed linear space and investigated the notion of statistical convergence. Following their work, several researchers investigated various notions of convergence of sequences in the neutrosophic normed space. For more details, one may refer to [11, 13, 14].

Very recently, Kisi [17] extended the notion of statistical convergence to $\mathcal{I}$ and $\mathcal{I}^*-$convergence of sequences in neutrosophic normed spaces. In this paper, our main aim is to generalize it to $\mathcal{I}_3$ and $\mathcal{I}_3^*-$convergence of triple sequences which is also a generalization of statistical convergence of triple sequences in the neutrosophic normed spaces developed by Granados and Das [8].

2. Definitions and Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively and by the convergence of a triple sequence we mean the convergence in Pringsheim's [25] sense.

Definition 2.1. [25] *A triple sequence (x_{ijg}) is said to be convergent to l if, for any $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $i, j, g \geq k_0$, $|x_{ijg} - l| < \varepsilon$.*

Definition 2.2. [27] *Let $K \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $K_{m,n,p}$ denotes the set*

$$\{(i,j,g) \in K : i \leq m, j \leq n, g \leq p\}.$$

The triple natural density of K is denoted and defined by $\delta^3(K) = \lim_{m,n,p\to\infty} \frac{|K_{m,n,p}|}{mnp}$. Here, $|K_{m,n,p}|$ denotes the cardinality of the set $K_{m,n,p}$.

Definition 2.3. [27] *A triple sequence (x_{ijg}) is said to be statistical convergent to l if for each $\varepsilon > 0$, $\delta^3(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : | x_{ijg} - l | \geq \varepsilon\}$.*
In this case, l is called the statistical limit of the triple sequence (x_{ijg}) and symbolically it is expressed as $x_{ijg} \xrightarrow{st} l$.

Definition 2.4. [27] *A triple sequence (x_{ijg}) is said to be statistical Cauchy if for each $\varepsilon > 0$, there exists three positive integers $M = M(\varepsilon)$, $N = N(\varepsilon)$ and $P = P(\varepsilon)$ such that*

$$\delta^3(A(\varepsilon)) = 0, \text{ where } A(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : | x_{ijg} - x_{MNP} | \geq \varepsilon\}.$$

Definition 2.5. [19] *Let X is a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if*
(i) $\emptyset \in \mathcal{I}$;
(ii) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(iii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called non-trivial if $X \notin \mathcal{I}$.

Definition 2.6. [19] *Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if*
(i) $\emptyset \notin \mathcal{F}$;
(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a non-trivial ideal in X with $X \neq \emptyset$, then the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ forms a filter on X, known as the filter associated with $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

A non-trivial ideal $\mathcal{I}$ in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is called strongly admissible if for any $k \in \mathbb{N}$, $\mathbb{N}\times\{k\}\times\mathbb{N}$, $\{k\}\times\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}\times\{k\}$ belong to $\mathcal{I}$.

Throughout the paper, we consider $\mathcal{I}_3$ as a non-trivial admissible ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Definition 2.7. [28] *A triple sequence (x_{ijg}) is said to be $\mathcal{I}_3-$convergent to l if for each $\varepsilon > 0$,*

$$A(\varepsilon) \in \mathcal{I}_3, \text{ where } A(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : | x_{ijg} - l | \geq \varepsilon\}.$$

In this case, l is called the $\mathcal{I}_3-$limit of the triple sequence (x_{ijg}) and symbolically it is expressed as $x_{ijg} \to l(\mathcal{I}_3)$.

Definition 2.8. [28] *A triple sequence (x_{ijg}) is said to be $\mathcal{I}_3-$Cauchy if for each $\varepsilon > 0$, there exists three positive integers $M = M(\varepsilon)$, $N = N(\varepsilon)$ and $P = P(\varepsilon)$ such that*

$$A(\varepsilon) \in \mathcal{I}_3, \text{ where } A(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : | x_{ijg} - x_{MNP} | \geq \varepsilon\}.$$

Definition 2.9. [2] *An admissible ideal $\mathcal{I}_3 \subset P(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ is said to satisfy the condition $(AP3)$ if for every countable family of mutually disjoint sets $\{D_1, D_2, ...\}$ belonging to $\mathcal{I}_3$, there exists a countable family of sets $\{E_1, E_2, ...\}$ such that for all $m \in \mathbb{N}$, the symmetric differences $D_m \triangle E_m$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ (i.e., $D_m \triangle E_m \subseteq \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (\exists k_0 \in \mathbb{N})(i,j,g \geq k_0 \Rightarrow (i,j,g) \notin A)\}$) and $E = \bigcup_{m=1}^{\infty} E_m \in \mathcal{I}_3$.*

Definition 2.10. [21] *A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-norm if the following conditions are satisfied:*

i. *$\odot$ is associative and commutative,*
ii. *$\odot$ is continuous,*
iii. *$s \odot 1 = s$, for all $s \in [0,1]$,*
iv. *$s \odot t \le u \odot v$ whenever $s \le u$ and $t \le v$, for all $s, t, u, v \in [0,1]$.*

Definition 2.11. [21] *A binary operation $\circledast : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-conorm if the following conditions are satisfied:*

i. *$\circledast$ is associative and commutative,*
ii. *$\circledast$ is continuous,*
iii. *$s \circledast 0 = s$, for all $s \in [0,1]$,*
iv. *$s \circledast t \le u \circledast v$ whenever $s \le u$ and $t \le v$, for all $s, t, u, v \in [0,1]$.*

From the above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 1$ such that $\varepsilon_1 \odot \varepsilon_3 \ge \varepsilon_2, \varepsilon_1 \ge \varepsilon_4 \circledast \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0,1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \odot \varepsilon_6 \ge \varepsilon_5$ and $\varepsilon_7 \circledast \varepsilon_7 \le \varepsilon_5$.

Definition 2.12. [31] *Let X be the universe of discourse. Then, the set $A_{NS} \subseteq X$ defined by*

$$A_{NS} = \{< u, \mathcal{S}_A(u), \mathcal{T}_A(u), \mathcal{W}_A(u) >: u \in X\}$$

is called a neutrosophic set, where $\mathcal{S}_A(u), \mathcal{T}_A(u), \mathcal{W}_A(u) : X \to [0,1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-membership respectively, with $0 \le \mathcal{S}_A(u) + \mathcal{T}_A(u) + \mathcal{W}_A(u) \le 3$.

Definition 2.13. [16] *Let F be a vector space and $\mathcal{N} = \{< u, \mathcal{S}(u), \mathcal{T}(u), \mathcal{W}(u) >: u \in F\}$ be a normed space (NS) such that $\mathcal{S}, \mathcal{T}, \mathcal{W} : F \times \mathbb{R}^+ \to [0,1]$. Let $\odot$ and $\circledast$ be the continuous t-norm and continuous t-conorm, respectively. Then the four-tuple $V = (F, \mathcal{N}, \odot, \circledast)$ is called neutrosophic normed space (NNS) if the following conditions hold, for all $u, v \in F$ and $\eta, \nu > 0$ and for each $\sigma \neq 0$:*

i. $0 \le \mathcal{S}(u,\eta) \le 1$, $0 \le \mathcal{T}(u,\eta) \le 1$, $0 \le \mathcal{W}(u,\eta) \le 1$,
ii. $\mathcal{S}(u,\eta) + \mathcal{T}(u,\eta) + \mathcal{W}(u,\eta) \le 3$,
iii. $\mathcal{S}(u,\eta) = 1$ *(for $\eta > 0$) if and only if* $u = 0$,
iv. $\mathcal{S}(\sigma u,\eta) = \mathcal{S}(u, \frac{\eta}{|\sigma|})$,
v. $\mathcal{S}(u,\eta) \odot \mathcal{S}(v,\nu) \le \mathcal{S}(u+v, \eta+\nu)$,
vi. $\mathcal{S}(u,.)$ *is a continuous non-decreasing function,*
vii. $\lim_{\eta\to\infty} \mathcal{S}(u,\eta) = 1$,
viii. $\mathcal{T}(u,\eta) = 0$ *(for $\eta > 0$) iff* $u = 0$,
ix. $\mathcal{T}(\sigma u,\eta) = \mathcal{T}(u, \frac{\eta}{|\sigma|})$,
x. $\mathcal{T}(u,\eta) \circledast \mathcal{T}(v,\nu) \ge \mathcal{T}(u+v, \eta+\nu)$,
xi. $\mathcal{T}(u,.)$ *is a continuous and non-increasing function,*
xii. $\lim_{\eta\to\infty} \mathcal{T}(u,\eta) = 0$,
xiii. $\mathcal{W}(u,\eta) = 0$ *(for $\eta > 0$) if and only if* $u = 0$,
xiv. $\mathcal{W}(\sigma u,\eta) = \mathcal{W}(u, \frac{\eta}{|\sigma|})$,
xv. $\mathcal{W}(u,\eta) \circledast \mathcal{W}(v,\nu) \ge \gamma(u+v, \eta+\nu)$,
xvi. $\mathcal{W}(u,.)$ *is a continuous non-increasing function,*
xvii. $\lim_{\eta\to\infty} \mathcal{W}(u,\eta) = 0$,
xviii. *If* $\eta \le 0$, *then* $\mathcal{S}(u,\eta) = 0, \mathcal{T}(u,\eta) = 1$ *and* $\mathcal{W}(u,\eta) = 1$.

Then, $\mathcal{N} = (\mathcal{S}, \mathcal{T}, \mathcal{W})$ is called Neutrosophic norm (NN).

Definition 2.14. [17] *Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}-$convergent to l with respect to the neutrosophic norm (NN), if for every $0 < \varepsilon < 1$ and $\eta > 0$,*

$$K(\varepsilon) \in \mathcal{I}, \text{ where}$$
$$K(\varepsilon) = \{k \in \mathbb{N} : \mathcal{S}(x_k - l, \eta) \leq 1 - \varepsilon, \mathcal{T}(x_k - l, \eta) \geq \varepsilon \text{ and } \mathcal{W}(x_k - l, \eta) \geq \varepsilon\}.$$

Symbolically it is denoted as $\mathcal{I} - \mathcal{N} - \lim x_k = l$ or $x_k \to l(\mathcal{I} - \mathcal{N})$.

Definition 2.15. [17] *Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}^*-$convergent to l with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that $x_{k,k\in M} \to l(\mathcal{N})$.*

In this case, we write $\mathcal{I}^* - \mathcal{N} - \lim x_k = l$ or $x_k \to l(\mathcal{I}^* - \mathcal{N})$.

Definition 2.16. [17] *Let (x_k) be a sequence in a NNS V and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then, (x_k) is said to be $\mathcal{I}-$Cauchy with respect to neutrosophic norm (NN), if for any $0 < \varepsilon < 1$ and $\eta > 0$, there exists $M = M(\varepsilon)$ such that $KC(\varepsilon) \in \mathcal{I}$, where $KC(\varepsilon) = \{k \in \mathbb{N} : \mathcal{S}(x_k - x_M, \eta) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x_M, \eta) \geq \varepsilon, \mathcal{W}(x_k - x_M, \eta) \geq \varepsilon\}$.*

Definition 2.17. [17] *Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}^*-$Cauchy with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that $(x_{k,k\in M})$ is an ordinary Cauchy sequence with respect to neutrosophic norm (NN).*

Example 2.1. [16] *Suppose $(F, ||\cdot||)$ be a NS. For $s, t \in [0,1]$, define the t-norm $\odot$ and the t-conorm $\circledast$ as $s \odot t = st$ and $s \circledast t = s + t - st$, respectively. For $\eta > ||u||$, let*

$$\mathcal{S}(u,\eta) = \frac{\eta}{\eta + ||u||},\ \mathcal{T}(u,\eta) = \frac{||u||}{\eta + ||u||},\ \mathcal{W}(u,\eta) = \frac{||u||}{\eta}\ \forall u \in F \text{ and } \eta > 0$$

and for $\eta \leq ||u||$, let $\mathcal{S}(u,\eta) = 0, \mathcal{T}(u,\eta) = 1$ and $\mathcal{W}(u,\eta) = 1$. Then, $(F, \mathcal{N}, \odot, \circledast)$ is a NNS.

Definition 2.18. [8] *Let V be a NNS. A triple sequence (x_{ijg}) is said to be statistical convergent to l with respect to the neutrosophic norm (NN), if for every $0 < \varepsilon < 1$ and $\eta > 0$, $\delta^3(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon, \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon \text{ and } \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\}$. Symbolically it is denoted as $st^3 - \mathcal{N} - \lim x_{ijg} = l$ or $x_{ijg} \to l(st^3 - \mathcal{N})$.*

Definition 2.19. [8] *Let (x_{ijg}) be a triple sequence in a NNS V. Then, (x_{ijg}) is said to be statistical Cauchy if for any $0 < \varepsilon < 1$ and $\eta > 0$, there exists $M = M(\varepsilon), N = N(\varepsilon)$ and $P = P(\varepsilon)$ such that $\delta^3(KC(\varepsilon)) = 0$, where*

$$KC(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - x_{MNP}, \eta) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_{ijg} - x_{MNP}, \eta) \geq \varepsilon, \mathcal{W}(x_{ijg} - x_{MNP}, \eta) \geq \varepsilon\}.$$

3. Main Results

Definition 3.1. *Let V be a NNS. A triple sequence (x_{ijg}) is said to be $\mathcal{I}_3-$convergent to l with respect to neutrosophic norm (NN), if for every $0 < \varepsilon < 1$ and $\eta > 0$, $K(\varepsilon) \in \mathcal{I}_3$, where*

$$K(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon \text{ and } \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\}.$$

In this case, we write, $\mathcal{I}_3 - \mathcal{N} - \lim x_{ijg} = l$ or $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$.

In particular, if we take $\mathcal{I}_3 = \mathcal{I}_{\delta^3} = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \delta^3(A) = 0\}$, then the above definition coincides with the definition of statistical convergence of triple sequences in NNS, which was recently investigated by Granados and Das [8].

Example 3.1. *Let $(F, ||\cdot||)$ be a NS. For all $s,t \in [0,1]$, define the continuous t-norm $s \odot t = st$ and the continuous t-conorm $s \circledast t = \min\{s+t, 1\}$. We take $\mathcal{S}, \mathcal{T}, \mathcal{W}$ from Example 2.1, for all $\eta > 0$. Then V is a NNS. Define the triple sequence (x_{ijg}) as*

$$x_{ijg} = \begin{cases} 1, & i\, is\, a\, perfect\, cube,\; for\, all\, (j,g) \in \mathbb{N}\times\mathbb{N} \\ 0, & otherwise \end{cases}.$$

Then, $x_{ijg} \to 0(\mathcal{I}_{\delta^3} - \mathcal{N})$.
***Justification**: For every $0 < \varepsilon < 1$, we have*

$$K_{m,n,p}(\varepsilon) = \{i \leq m, j \leq n, g \leq p : \mathcal{S}(x_{ijg} - 0, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - 0, \eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ijg} - 0, \eta) \geq \varepsilon\}.$$

This implies that,

$$\begin{aligned} K_{m,n,p}(\varepsilon) &= \{i \leq m, j \leq n, g \leq p : \frac{\eta}{\eta + ||x_{ijg}||} \leq 1 - \varepsilon,\ \frac{||x_{ijg}||}{\eta + ||x_{ijg}||} \geq \varepsilon\, and\, \frac{||x_{ijg}||}{\eta} \geq \varepsilon\} \\ &= \{i \leq m, j \leq n, g \leq p : ||x_{ijg}|| \geq \frac{\eta\varepsilon}{1-\varepsilon}\, and\, ||x_{ijg}|| \geq \eta\varepsilon\} \\ &= \{i \leq m, j \leq n, g \leq p : x_{ijg} = 1\}. \end{aligned}$$

Then we have, $\delta^3(K(\varepsilon)) = \lim_{n,m,p\to\infty} \frac{|K_{m,n,p}(\varepsilon)|}{mnp} \leq \lim_{n,m,p\to\infty} \frac{\sqrt[3]{mnp}}{mnp} = 0.$
Hence, $x_{ijg} \to 0(\mathcal{I}_{\delta^3} - \mathcal{N})$.

Lemma 3.1. *Let V be a NNS. Then, for any $0 < \varepsilon < 1$, the following statements are equivalent:*

i. $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$;
ii. $\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon\} \in \mathcal{I}_3, \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon\} \in \mathcal{I}_3$, *and* $\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\} \in \mathcal{I}_3$;
iii. $\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) > 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) < \varepsilon\, and\, \mathcal{W}(x_{ijg} - l, \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I}_3)$;
iv. $\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) > 1 - \varepsilon\} \in \mathcal{F}(\mathcal{I}_3), \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ijg} - l, \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I}_3)$, *and* $\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ijg} - l, \eta) < \varepsilon\} \in \mathcal{F}(\mathcal{I}_3)$;
v. $\mathcal{S}(x_{ijg} - l, \eta) \to 1(\mathcal{I}_3 - \mathcal{N})$, $\mathcal{T}(x_{ijg} - l, \eta) \to 0(\mathcal{I}_3 - \mathcal{N})$ *and* $\mathcal{W}(x_{ijg} - l, \eta) \to 0(\mathcal{I}_3 - \mathcal{N})$.

Theorem 3.1. *Let V be a NNS and let (x_{ijg}) be a triple sequence such that $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$. Then l is unique.*

Proof. If possible let $x_{ijg} \to l_1(\mathcal{I}_3 - \mathcal{N})$ and $x_{ijg} \to l_2(\mathcal{I}_3 - \mathcal{N})$ for $l_1 \neq l_2$. Then, for a given $0 < \varepsilon < 1$, we can choose $\nu > 0$ satisfying $(1-\varepsilon) \odot (1-\varepsilon) > 1 - \nu$ and $\varepsilon \circledast \varepsilon < \nu$.

Now, for any $\eta > 0$ we define the following sets:

$$K_{\mathcal{S}_1}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg}-l_1,\frac{\eta}{2}) \leq 1-\varepsilon\},$$
$$K_{\mathcal{S}_2}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg}-l_2,\frac{\eta}{2}) \leq 1-\varepsilon\},$$
$$K_{\mathcal{T}_1}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ijg}-l_1,\frac{\eta}{2}) \geq \varepsilon\},$$
$$K_{\mathcal{T}_2}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ijg}-l_2,\frac{\eta}{2}) \geq \varepsilon\},$$
$$K_{\mathcal{W}_1}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ijg}-l_1,\frac{\eta}{2}) \geq \varepsilon\},$$
$$K_{\mathcal{W}_2}(\varepsilon,\eta) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ijg}-l_2,\frac{\eta}{2}) \geq \varepsilon\}.$$

Since $x_{ijg} \to l_1(\mathcal{I}_3-\mathcal{N})$, so by Lemma 3.1, for any $\eta > 0$ we have,

$$K_{\mathcal{S}_1}(\varepsilon,\eta), K_{\mathcal{T}_1}(\varepsilon,\eta),\ and\ K_{\mathcal{W}_1}(\varepsilon,\eta) \in \mathcal{I}_3.$$

Again since $x_{ijg} \to l_2(\mathcal{I}_3-\mathcal{N})$, so by Lemma 3.1, for any $\eta > 0$ we have,

$$K_{\mathcal{S}_2}(\varepsilon,\eta), K_{\mathcal{T}_2}(\varepsilon,\eta),\ and\ K_{\mathcal{W}_2}(\varepsilon,\eta) \in \mathcal{I}_3.$$

Suppose $K(\varepsilon,\eta) = (K_{\mathcal{S}_1}(\varepsilon,\eta)\cup K_{\mathcal{S}_2}(\varepsilon,\eta))\cap(K_{\mathcal{T}_1}(\varepsilon,\eta)\cup K_{\mathcal{T}_2}(\varepsilon,\eta))\cap(K_{\mathcal{W}_1}(\varepsilon,\eta)\cup K_{\mathcal{W}_2}(\varepsilon,\eta))$. Then, we have $K(\varepsilon,\eta) \in \mathcal{I}_3$ and consequently $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon,\eta) \in \mathcal{F}(\mathcal{I}_3)$. Thus, the set $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon,\eta)$ is non-empty. Choose $(s,t,f) \in \mathbb{N} \setminus K(\varepsilon,\eta)$. Then, there are three possibilities:

i. $(s,t,f) \in ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{S}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{S}_2}(\varepsilon,\eta))$;
ii. $(s,t,f) \in ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{T}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{T}_2}(\varepsilon,\eta))$;
iii. $(s,t,f) \in ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{W}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{W}_2}(\varepsilon,\eta))$.

If we consider (i), then we have the following

$$\mathcal{S}(l_1-l_2,\eta) \geq \mathcal{S}(x_{stf}-l_1,\frac{\eta}{2}) \odot \mathcal{S}(x_{stf}-l_2,\frac{\eta}{2}) > (1-\varepsilon)\odot(1-\varepsilon) > 1-\nu. \quad (1)$$

Now since ν is arbitrary so from Equation (1), for any $\eta > 0$, we obtain $\mathcal{S}(l_1-l_2,\eta) = 1$ i.e., $l_1 = l_2$.

Again, if we consider (ii), then we have the following

$$\mathcal{T}(l_1-l_2,\eta) \leq \mathcal{T}(x_{stf}-l_1,\frac{\eta}{2}) \circledast \mathcal{T}(x_{stf}-l_2,\frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu. \quad (2)$$

Now, since ν is arbitrary so from Equation (2), for any $\eta > 0$, we obtain $\mathcal{T}(l_1-l_2,\eta) = 0$ i.e., $l_1 = l_2$.

Finally, if we consider (iii), then we have the following

$$\mathcal{W}(l_1-l_2,\eta) \leq \mathcal{W}(x_{stf}-l_1,\frac{\eta}{2}) \circledast \mathcal{W}(x_{stf}-l_2,\frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu. \quad (3)$$

Now, since ν is arbitrary so from Equation (3), for any $\eta > 0$, we obtain $\mathcal{W}(l_1-l_2,\eta) = 0$ i.e., $l_1 = l_2$. Thus in all cases we obtain $l_1 = l_2$ and this completes the proof. □

Theorem 3.2. *Let (x_{ijg}) and (y_{ijg}) be two triple sequences in the NNS V such that $x_{ijg} \to l_1(\mathcal{I}_3-\mathcal{N})$ and $y_{ijg} \to l_2(\mathcal{I}_3-\mathcal{N})$. Then,*
(i) $x_{ijg}+y_{ijg} \to l_1+l_2(\mathcal{I}_3-\mathcal{N})$ and (ii) $\alpha x_{ijg} \to \alpha l_1(\mathcal{I}_3-\mathcal{N})$ where $\alpha \in \mathbb{R}$.

Proof. (i) Suppose $x_{ijg} \to l_1(\mathcal{I}_3 - \mathcal{N})$ and $y_{ijg} \to l_2(\mathcal{I}_3 - \mathcal{N})$. Then, by definition for any $0 < \varepsilon < 1$, $K(\varepsilon), K'(\varepsilon) \in \mathcal{I}_3$, where

$$K(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ijg} - l_1, \frac{\eta}{2}) \leq 1 - \varepsilon, \mathcal{T}(x_{ijg} - l_1, \frac{\eta}{2}) \geq \varepsilon \\ \textit{and}\, \mathcal{W}(x_{ijg} - l_1, \frac{\eta}{2}) \geq \varepsilon\}$$

and

$$K'(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(y_{ijg} - l_2, \frac{\eta}{2}) \leq 1 - \varepsilon,\ \mathcal{T}(y_{ijg} - l_2, \frac{\eta}{2}) \geq \varepsilon \\ \textit{and}\, \mathcal{W}(y_{ijg} - l_2, \frac{\eta}{2}) \geq \varepsilon\}.$$

Now as the inclusion

$$((\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \setminus K(\varepsilon)) \cap ((\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \setminus K'(\varepsilon)) \\ \subseteq \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) > 1 - \varepsilon, \\ \mathcal{T}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) < \varepsilon, \mathcal{W}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) < \varepsilon\}$$

holds, so we must have

$$K''(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) \leq 1 - \varepsilon, \\ \mathcal{T}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) \geq \varepsilon\, \textit{and}\, \mathcal{W}(x_{ijg} + y_{ijg} - l_1 - l_2, \eta) \geq \varepsilon\} \\ \subseteq K(\varepsilon) \cup K'(\varepsilon)$$

and consequently, $K''(\varepsilon) \in \mathcal{I}_3$ i.e., $x_{ijg} + y_{ijg} \to l_1 + l_2(\mathcal{I}_3 - \mathcal{N})$.

(ii) If $\alpha = 0$, then there is nothing to prove. So we assume $\alpha \neq 0$. Since, $x_{ijg} \to l_1(\mathcal{I}_3 - \mathcal{N})$, so for any $0 < \varepsilon < 1$, $K(\varepsilon) \in \mathcal{I}_3$, where

$$K(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ijg} - l, \frac{\eta}{|\alpha|}) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \frac{\eta}{|\alpha|}) \geq \varepsilon \\ \textit{and}\, \mathcal{W}(x_{ijg} - l, \frac{\eta}{|\alpha|}) \geq \varepsilon\}.$$

Now let $K'(\varepsilon)$ denote the set

$$\{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(\alpha x_{ijg} - \alpha l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(\alpha x_{ijg} - \alpha l, \eta) \geq \varepsilon \\ \textit{and}\, \mathcal{W}(\alpha x_{ijg} - \alpha l, \eta) \geq \varepsilon\}.$$

Then, the inclusion $K'(\varepsilon) \subseteq K(\varepsilon)$ holds and eventually, $K'(\varepsilon) \in \mathcal{I}_3$. Hence, $\alpha x_{ijg} \to \alpha l_1(\mathcal{I}_3 - \mathcal{N})$. □

Theorem 3.3. *Let (x_{ijg}) and (y_{ijg}) be two triple sequences in the NNS V such that $y_{ijg} \to l(\mathcal{N})$ and $\{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{ijg} \neq y_{ijg}\} \in \mathcal{I}_3$. Then, $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$.*

Proof. Suppose $\{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{ijg} \neq y_{ijg}\} \in \mathcal{I}_3$ holds and $y_{ijg} \to l(\mathcal{N})$. Then, by definition for every $0 < \varepsilon < 1$, the set $K(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(y_{ijg} - l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(y_{ijg} - l, \eta) \geq \varepsilon\, \textit{and}\, \mathcal{W}(y_{ijg} - l, \eta) \geq \varepsilon\}$ contains at most finite number of elements and consequently, $K(\varepsilon) \in \mathcal{I}_3$. Now, since the inclusion

$$K'(\varepsilon) = \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon \\ \textit{and}\, \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\} \subseteq K(\varepsilon) \cap \{(i,j,g) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{ijg} \neq y_{ijg}\}$$

holds, so we must have, $K'(\varepsilon) \in \mathcal{I}_3$. Hence, $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$. □

Definition 3.2. *Let V be a NNS. A triple sequence (x_{ijg}) is said to be $\mathcal{I}_3^*-$convergent to l with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I}_3)$ (i.e., $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus M \in \mathcal{I}_3$) such that $x_{ijg,(i,j,g)\in M} \to l(\mathcal{N})$.*

In this case, we write $\mathcal{I}_3^* - \mathcal{N} - \lim x_{ijg} = l$ or $x_{ijg} \to l(\mathcal{I}_3^* - \mathcal{N})$.

Theorem 3.4. *Let V be a NNS and $\mathcal{I}_3$ be a strong admissible ideal. If $x_{ijg} \to l(\mathcal{I}_3^* - \mathcal{N})$ then $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$.*

Proof. Since $x_{ijg} \to l(\mathcal{I}_3^* - \mathcal{N})$, so by definition there exists a set $M \in \mathcal{F}(\mathcal{I}_3)$ (i.e., $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus M \in \mathcal{I}_3$) such that $x_{ijg,(i,j,g)\in M} \to l(\mathcal{N})$. This means that for any $0 < \varepsilon < 1$, there exists $k_0 \in \mathbb{N}$ such that for all $i, j, g \geq k_0$,

$$\mathcal{S}(x_{ijg} - l, \eta) > 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) < \varepsilon \text{ and } \mathcal{W}(x_{ijg} - l, \eta) < \varepsilon.$$

Then we have,

$$K(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon \text{ and } \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\}$$
$$\subseteq ((\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus M) \cup (M \cap ((\{1,2,...,(k_0-1)\} \times \mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1,2,...,(k_0-1)\})$$
$$\cup (\mathbb{N}\times\{1,2,...,(k_0-1)\}\times\mathbb{N}))) \in \mathcal{I}_3$$

Thus by hereditary property, $K(\varepsilon) \in \mathcal{I}_3$. Hence, $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$. □

Remark 3.1. *The converse of the Theorem 3.4 is not necessarily true. The following example illustrates the fact.*

Example 3.2. *Let $(\mathbb{R}, ||\cdot||)$ be a NS. For all $s, t \in [0,1]$, define the continuous t-norm $s \odot t = st$ and the continuous t-conorm $s \circledast t = \min\{s+t, 1\}$. We take $\mathcal{S}, \mathcal{T}, \mathcal{W}$ from Example 2.1, for all $\eta > 0$. Then, $(\mathbb{R}, \mathcal{N}, \odot, \circledast)$ is a NNS. Let $\mathbb{N} = \bigcup_{m=1}^{\infty} D_m$ be a disjoint decomposition of $\mathbb{N}$ such that each D_p is an infinite set. Then it is obvious that $\mathbb{N}\times\mathbb{N}\times\mathbb{N} = \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{f=1}^{\infty}(D_m \times D_n \times D_f)$ is a disjoint decomposition of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Let $\mathcal{I}_3 = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : A \subseteq ((\bigcup_{m=1}^{p} D_m)\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times(\bigcup_{n=1}^{q} D_n)\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times(\bigcup_{f=1}^{z} D_f))$, for some $p, q, z \in \mathbb{N}\}$ be an admissible ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Clearly, for any set $H(\subseteq \mathbb{N})$ containing a finite number of elements we have, $H\times\mathbb{N}\times\mathbb{N}$, $\mathbb{N}\times H\times\mathbb{N}$, $\mathbb{N}\times\mathbb{N}\times H \in \mathcal{I}_3$. Define a triple sequence (x_{ijg}) as follows:*

$$x_{ijg} = \tfrac{1}{m} + \tfrac{1}{n} + \tfrac{1}{f}, \text{ whenever } (i,j,g) \in D_m \times D_n \times D_f.$$

Then, $\mathcal{S}(x_{ijg} - 0, \eta) = \frac{\eta}{\eta + ||x_{ijg}||} \to 1$, $\mathcal{T}(x_{ijg} - 0, \eta) = \frac{||x_{ijg}||}{\eta + ||x_{ijg}||} \to 0$ and $\mathcal{W}(x_{ijg} - 0, \eta) = \frac{||x_{ijg}||}{\eta} \to 0$ as $i, j, g \to \infty$. This shows that $x_{ijg} \to 0(\mathcal{I}_3 - \mathcal{N})$. But we claim that $x_{ijg} \nrightarrow 0(\mathcal{I}_3^* - \mathcal{N})$. To prove this, let us assume the contrary. Then, there exists a set $M \in \mathcal{F}(\mathcal{I}_3)$ (i.e., $(\mathbb{N}\times\mathbb{N}\times\mathbb{N})\setminus M \in \mathcal{I}_3$) such that $x_{ijg,(i,j,g)\in M} \to 0(\mathcal{N})$. Now by definition of $\mathcal{I}_3$, we can say that there exists $p, q, z \in \mathbb{N}$ such that $(\mathbb{N}\times\mathbb{N}\times\mathbb{N})\setminus M \subseteq ((\bigcup_{m=1}^{p} D_m)\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times(\bigcup_{n=1}^{q} D_n)\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times(\bigcup_{f=1}^{z} D_f))$. But then, $D_{p+1}\times D_{q+1}\times D_{z+1} \subseteq M$ holds and as a consequence $x_{ijg} = \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{z+1}$ for infinitely many $(i,j,g) \in D_{p+1}\times D_{q+1}\times D_{z+1} \subseteq M$, which is a contradiction to the fact that $x_{ijg,(i,j,g)\in M} \to 0(\mathcal{N})$.

Theorem 3.5. *Let V be a NNS and $\mathcal{I}_3$ be an ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with the property $(AP3)$. Then, if $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$ then $x_{ijg} \to l(\mathcal{I}_3^* - \mathcal{N})$.*

Proof. Suppose $\mathcal{I}_3$ satisfies the condition $(AP)_3$ and $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$. Then, by definition, for every $0 < \varepsilon < 1$,

$$K(\varepsilon) \in \mathcal{I}_3, \text{ where } K(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \varepsilon,\ \mathcal{T}(x_{ijg} - l, \eta) \geq \varepsilon \ \textit{and}\ \mathcal{W}(x_{ijg} - l, \eta) \geq \varepsilon\}.$$

Suppose for $m \in \mathbb{N}$, D_m denotes the set

$$\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : 1 - \frac{1}{m} \leq \mathcal{S}(x_{ijg} - l, \eta) < 1 - \frac{1}{m+1}, \frac{1}{m+1} < \mathcal{T}(x_{ijg} - l, \eta) \leq \frac{1}{m} \ \textit{and}\ \frac{1}{m+1} < \mathcal{W}(x_{ijg} - l, \eta) \leq \frac{1}{m}\}.$$

Then, it is clear that for all $m \in \mathbb{N}$, $D_m \in \mathcal{I}_3$ and for $m \neq n$, $D_m \cap D_n = \emptyset$. By virtue of $(AP3)$, there exists a sequence of sets (E_m) such that for each $m \in \mathbb{N}$, the symmetric differences $D_m \triangle E_m$ is contained in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $E = \bigcup_{m=1}^{\infty} E_m \in \mathcal{I}_3$.

Now we will prove that for $M = (\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus E \in \mathcal{F}(\mathcal{I}_3)$, $x_{ijg,(i,j,g)\in M} \to l(\mathcal{N})$. Let $\kappa > 0$ be given. By Archimedean property, choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \kappa$. Then the following inclusion

$$\{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - l, \eta) \leq 1 - \kappa, \mathcal{T}(x_{ijg} - l, \eta) \geq \kappa \ \textit{and}\ \mathcal{W}(x_{ijg} - l, \eta) \geq \kappa\} \subseteq \bigcup_{m=1}^{m_0} D_m \quad (4)$$

holds. Since, $D_m \triangle E_m, m = 1, 2, ..., m_0$ are contained in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, so there exists $k_0 \in \mathbb{N}$ such that

$$(\bigcup_{m=1}^{m_0} E_m) \cap \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : i \geq k_0, j \geq k_0, g \geq k_0\} = (\bigcup_{m=1}^{m_0} D_m) \cap \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : i \geq k_0, j \geq k_0, g \geq k_0\}. \quad (5)$$

If $i, j, g \geq k_0$, and $(i,j,g) \in M$, then $(i,j,g) \notin \bigcup_{m=1}^{m_0} E_m$ and consequently from Equation (5), $(i,j,g) \notin \bigcup_{m=1}^{m_0} D_m$. From (4), it is clear that for any $i, j, g \geq k_0$ and $(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$,

$$\mathcal{S}(x_{ijg} - l, \eta) > 1 - \kappa, \mathcal{T}(x_{ijg} - l, \eta) < \kappa \ \textit{and}\ \mathcal{W}(x_{ijg} - l, \eta) < \kappa.$$

This means that $x_{ijg} \to l(\mathcal{I}_3^* - \mathcal{N})$. Hence the theorem. □

Definition 3.3. *Let (x_{ijg}) be a triple sequence in a NNS V. Then (x_{ijg}) is said to be $\mathcal{I}_3-$Cauchy if for any $0 < \varepsilon < 1$ and $\eta > 0$, there exists three positive integers $M = M(\varepsilon), N = N(\varepsilon), P = P(\varepsilon)$ such that $KC(\varepsilon) \in \mathcal{I}_3$, where*

$$KC(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg} - x_{MNP}, \eta) \leq 1 - \varepsilon, \mathcal{T}(x_{ijg} - x_{MNP}, \eta) \geq \varepsilon, \ \textit{and}\ \mathcal{W}(x_{ijg} - x_{MNP}, \eta) \geq \varepsilon\}.$$

In particular, if we take $\mathcal{I}_3 = \mathcal{I}_{\delta^3} = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \delta^3(A) = 0\}$, then the above definition coincides with the definition of statistical Cauchy sequences for triple sequences in NNS, which was recently investigated by Granados and Das [8].

Theorem 3.6. *Let (x_{ijg}) be a triple sequence in a NNS V. Then, (x_{ijg}) is $\mathcal{I}_3-$convergent sequence if and only if it is $\mathcal{I}_3-$Cauchy sequence.*

Proof. Suppose $x_{ijg} \to l(\mathcal{I}_3 - \mathcal{N})$. For a given $0 < \varepsilon < 1$, we choose $\nu > 0$ such that $(1-\varepsilon) \odot (1-\varepsilon) > 1-\nu$ and $\varepsilon \circledast \varepsilon < \nu$. Then by definition, for any $0 < \varepsilon < 1$, $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$, where

$$K(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg}-l, \frac{\eta}{2}) \le 1-\varepsilon,\ \mathcal{T}(x_{ijg}-l, \frac{\eta}{2}) \ge \varepsilon \quad and\, \mathcal{W}(x_{ijg}-l, \frac{\eta}{2}) \ge \varepsilon\}.$$

Thus, the set $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon)$ is non-empty. Let $(M,N,P) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon)$. Then we have,

$$\mathcal{S}(x_{MNP}-l, \frac{\eta}{2}) > 1-\varepsilon,\ \mathcal{T}(x_{MNP}-l, \frac{\eta}{2}) < \varepsilon\, and\, \mathcal{W}(x_{MNP}-l, \frac{\eta}{2}) < \varepsilon.$$

Now suppose $KC(\varepsilon) = \{(i,j,g) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ijg}-x_{MNP}, \eta) \le 1-\nu,\ \mathcal{T}(x_{ijg} - x_{MNP}, \eta) \ge \nu\, and\, \mathcal{W}(x_{ijg}-x_{MNP}, \eta) \ge \nu\}$. We claim that $KC(\varepsilon) \subseteq K(\varepsilon)$ because if the inclusion does not hold then we must have some $(M_0,N_0,P_0) \in KC(\varepsilon) \setminus K(\varepsilon)$ which immediately yields $\mathcal{S}(x_{M_0N_0P_0} - x_{MNP}, \eta) \le 1-\nu$ and $\mathcal{S}(x_{M_0N_0P_0} - l, \frac{\eta}{2}) > 1-\varepsilon$. In particular, $\mathcal{S}(x_{MNP} - l, \frac{\eta}{2}) > 1-\varepsilon$. But then,

$$1-\nu \ge \mathcal{S}(x_{M_0N_0P_0}-x_{MNP}, \eta) \ge \mathcal{S}(x_{M_0N_0P_0}-l, \frac{\eta}{2}) \odot \mathcal{S}(x_{MNP}-l, \frac{\eta}{2}) > (1-\varepsilon)\odot(1-\varepsilon) > 1-\nu,$$

which is a contradiction. Further, we have, $\mathcal{T}(x_{M_0N_0P_0} - x_{MNP}, \eta) \ge \nu$ and $\mathcal{T}(x_{M_0N_0P_0} - l, \frac{\eta}{2}) < \varepsilon$. In particular, $\mathcal{T}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon$. But then,

$$\nu \le \mathcal{T}(x_{M_0N_0P_0} - x_{MNP}, \eta) \le \mathcal{T}(x_{M_0N_0P_0} - l, \frac{\eta}{2}) \circledast \mathcal{T}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu,$$

which is a contradiction. Finally, we have, $\mathcal{W}(x_{M_0N_0P_0} - x_{MNP}, \eta) \ge \nu$ and $\mathcal{W}(x_{M_0N_0P_0} - l, \frac{\eta}{2}) < \varepsilon$. In particular, $\mathcal{W}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon$. But then,

$$\nu \le \mathcal{W}(x_{M_0N_0P_0} - x_{MNP}, \eta) \le \mathcal{W}(x_{M_0N_0P_0} - l, \frac{\eta}{2}) \circledast \mathcal{W}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu,$$

which is a contradiction. Thus all possibilities contradict the existence of an element $(M_0, N_0, P_0) \in KC(\varepsilon) \setminus K(\varepsilon)$. Therefore, we must have $KC(\varepsilon) \subseteq K(\varepsilon)$ and as a consequence $KC(\varepsilon) \in \mathcal{I}_3$. Hence (x_{ijg}) is $\mathcal{I}_3-$Cauchy.

To prove the converse part, we assume that (x_{ijg}) is a $\mathcal{I}_3-$Cauchy sequence but not $\mathcal{I}_3-$convergent. For a given $0 < \varepsilon < 1$, we choose $\nu > 0$ such that $(1-\varepsilon)\odot(1-\varepsilon) > 1-\nu$ and $\varepsilon \circledast \varepsilon < \nu$. Then, since (x_{ijg}) is not $\mathcal{I}_3-$convergent, so

$$\mathcal{S}(x_{ijg} - x_{MNP}, \eta) \ge \mathcal{S}(x_{ijg} - l, \frac{\eta}{2}) \odot \mathcal{S}(x_{MNP} - l, \frac{\eta}{2}) > (1-\varepsilon)\odot(1-\varepsilon) > 1-\nu,$$

$$\mathcal{T}(x_{ijg} - x_{MNP}, \eta) \le \mathcal{T}(x_{ijg} - l, \frac{\eta}{2}) \circledast \mathcal{T}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu,$$

$$\mathcal{W}(x_{ijg} - x_{MNP}, \eta) \le \mathcal{W}(x_{ijg} - l, \frac{\eta}{2}) \circledast \mathcal{W}(x_{MNP} - l, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu,$$

holds for $P(\varepsilon,\nu) = \{i \le M, j \le N, g \le P : \mathcal{T}(x_{ijg} - x_{MNP}, \eta) \le 1-\nu\}$. Therefore, $P(\varepsilon,\nu) \in \mathcal{F}(\mathcal{I}_3)$, which is a contradiction to the fact that (x_{ijg}) is $\mathcal{I}_3-$Cauchy. Hence, (x_{ijg}) must be a $\mathcal{I}_3-$convergent sequence. This completes the proof. □

Definition 3.4. *Let (x_{ijg}) be a triple sequence in a NNS V. Then (x_{ijg}) is said to be $\mathcal{I}_3^*-$Cauchy with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I}_3)$ (i.e., $(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \setminus M \in \mathcal{I}_3$) such that $x_{ijg,(i,j,g)\in M}$ is an ordinary Cauchy sequence with respect to neutrosophic norm (NN).*

Theorem 3.7. *Let (x_{ijg}) be a triple sequence in a NNS V. If (x_{ijg}) is $\mathcal{I}_3^*-$Cauchy with respect to neutrosophic norm (NN), then it is $\mathcal{I}_3-$Cauchy with respect to neutrosophic norm (NN).*

Proof. We omit the proof as it follows from the same argument for convergence. □

Theorem 3.8. *Let V be a NNS and $\mathcal{I}_3$ be an ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with the property $(AP3)$. If (x_{ijg}) is $\mathcal{I}_3-$Cauchy with respect to neutrosophic norm (NN), then it is $\mathcal{I}_3^*-$Cauchy with respect to neutrosophic norm (NN).*

Proof. We omit the proof as it essentially follows from the same argument for convergence. □

4. Conclusions

In this paper, we mainly investigated various fundamental properties of $\mathcal{I}_3$ and $\mathcal{I}_3^*-$convergence of triple sequences in neutrosophic normed spaces. Theorem 3.4 and Theorem 3.5 establishes the connection between $\mathcal{I}_3$ and $\mathcal{I}_3^*-$convergence of triple sequences. Theorem 3.7 and Theorem 3.8 are presented to study the implication relationship between $\mathcal{I}_3$ and $\mathcal{I}_3^*-$Cauchy triple sequences. Theorem 3.5 and Theorem 3.8 also reveals the essence of the condition $(AP3)$ in the study. As a continuation of this work, one may study various properties such as solidity, symmetricity, monotonicity of the sequence spaces formed by the collection of all $\mathcal{I}_3-$convergent triple sequences in the neutrosophic normed spaces.

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References

[1] Atanassov, K. T., (1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1), pp. 87-96.

[2] Das, P., Kostyrko, P., Wilczyński, W. and Malik, P., (2008), I and I^*-convergence of double sequences, Math. Slovaca, 58(5), pp. 605-620.

[3] Dems, K., (2004/2005), On I-Cauchy sequences, Real Anal. Exchange, 30(1), pp. 123-128.

[4] Fast, H., (1951), Sur la convergence statistique, Cooloq. Math., 2, pp. 241-244.

[5] Felbin, C., (1992), Finite dimensional fuzzy normed linear space, Fuzzy Sets and Systems, 48(2), pp. 239-248.

[6] Fridy, J. A., (1985), On statistical convergence, Analysis, 5(4), pp. 301-313.

[7] Fridy, J. A., (1993), Statistical limit points, Proc. Amer. Math. Soc., 118(4), pp. 1187-1192.

[8] Granados, C. and Das, S., Statistical Convergence of triple Sequences in Neutrosophic Normed Spaces, Revista Colombiana de Matemáticas (Submitted).

[9] Karakus, S., Demirci, K. and Duman, O., (2008), Statistical convergence on intuitionistic fuzzy normed spaces, Chaos Solitons Fractals., 35(4), pp. 763-169.

[10] Khan, V. A. and Ahamd, M., (2020), On $(\lambda,\mu)-$zweier ideal convergence in intuitionistic fuzzy normed space, Yugosl. J. Oper. Res., 4, pp. 413-427.

[11] Khan, V. A., Fatima, H., Khan, M. D. and Ahamd, A., (2021), Spaces of neutrosophic $\lambda-$statistical convergence sequences and their properties, J. Math. Comput. Sci., 23(1), pp. 1-9.

[12] Khan, V. A., Idrisi, M. I., and Ahamd, M., (2021), On $I-$convergent triple sequence spaces defined by a compact operator and an Orlicz function, TWMS J. App. and Eng. Math., 11(4), pp. 1012-1022.

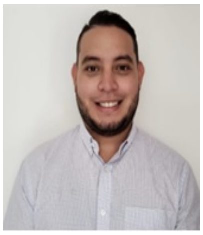
[13] Khan, V. A., Khan, M. D. and Ahmad, M., (2021), Some new type of lacunary statistically convergent sequences in neutrosophic normed space, Neutrosophic Sets Syst., 42, pp. 240-252.

[14] Khan, V. A., Khan, M. D. and Ahmad, M., (2021), Some results of neutrosophic normed spaces via Fibonacci matrix, U.P.B. Sci. Bull. Series A, 83(2), pp. 99-110.

[15] Kirisci, M., (2019), Fibonacci statistical convergence on intuitionistic fuzzy normed spaces, J. Intell. Fuzzy Syst., 36(1), pp. 1–8.

[16] Kirisci, M. and Simsek, N., (2020), Neutrosophic normed space and statistical convergence, J. Anal., 28(4), pp. 1059–1073.

[17] Kisi, O., (2021), Ideal convergence of sequences in neutrosophic normed spaces, J. Intell. Fuzzy Syst., 41(2), pp. 2581–2590.
[18] Kostyrko, P., Macaj, M., Salat, T. and Sleziak, M., (2005), I-convergence and extremal I-limit points, Math. Slovaca, 55(4), pp. 443-464.
[19] Kostyrko, P., Salat, T. and Wilczynski, W., (2000/2001), I-convergence, Real Anal. Exchange, 26(2), pp. 669-686.
[20] Kumar, V., (2007), On I and I^*-convergence of double sequences, Math. Commun., 12(2), pp. 171-181.
[21] Menger, K., (1942), Statistical metrics, Sigma J. Eng. Nat. Sci., 28(12), pp. 535-537.
[22] Mohiuddine, S. A. and Danish Lohani, Q. M., (2009), On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos Solitons Fractals., 42(3), pp. 1731-1737.
[23] Mursaleen, M. and Edely, O. H. H., (2003), Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1), pp. 223-231.
[24] Mursaleen, M. and Mohiuddine, S. A., (2009), Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals, 41(5), pp. 2414-2421.
[25] Pringsheim, A., (1900), Zur theorie der zweifach unendlichen zahlenfolgen, Math. Ann., 53, pp. 289-321.
[26] Saadati, R. and Park, J. H., (2006), ,On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals., 27(2), pp. 331-344.
[27] Sahiner, A., Gurdal, M. and Duden, K., (2007), Triple sequences and their statistical convergence, Selcuk. J. Appl. Math., 8(2), pp. 49-55.
[28] Sahiner, A. and Tripathy, B. C., (2008), Some $\mathcal{I}-$related properties of triple sequences, Selcuk. J. Appl. Math., 9(2), pp. 9-18.
[29] Salat, T., (1980), On statistically convergent sequences of real numbers, Math. Slovaca., 30(2), pp. 139-150.
[30] Savas, E. and Gurdal, M., (2015), A generalized statistical convergence in intuitionistic fuzzy normed spaces, Sci. Asia, 41, pp. 289-294.
[31] Smarandache, F., (2005), Neutrosophic set a generalization of the intuitionistic fuzzy sets, Int. J. Pure Appl. Math., 24, pp. 287-297.
[32] Steinhaus, H., (1951), Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2, pp. 73-74.
[33] Tripathy, B. C., (2003), Statistically convergent double sequences, Tamkang J. Math., 34(3), pp. 321-327.
[34] Tripathy, B. and Tripathy, B. C., (2005), On I-convergence of double sequences, Soochow J. Math., 31(4), pp. 549-560.
[35] Yamanci, U. and Gurdal, M., (2014), I -statistical convergence in 2-normed space, Arab J. Math. Sci., 20(1), pp. 41-47.
[36] Zadeh, L. A., (1965), Fuzzy sets, Inf. Control, 8(3), pp. 338-353.



Carlos Andrés Granados Ortiz got his master's degree from the Universidad del Atlántico in 2019, later in 2021 he started his doctorate in Mathematics at the Universidad de Antioquia. He has published many papers associated to general topology, sequences spaces, neutrosophic theory and mathematical ducation.

Chiranjib Choudhury has completed his B.Sc and M.Sc in Mathematics from Tripura University, Suryamaninagar, Agartala, India. Currently he is working as a research scholar (UGC-SRF) at the same department. His research interests includes the sequence spaces, summability theory, fuzzy set theory, neutrosophic set theory etc.