

AN APPROXIMATE WAVE SOLUTION FOR PERTURBED KdV AND DISSIPATIVE NLS EQUATIONS: WEIGHTED RESIDUAL METHOD

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ABSTRACT. In the present work, we modified the conventional "weighted residual method" to some nonlinear evolution equations and tried to obtain the approximate progressive wave solutions for these evolution equations. For the illustration of the method we studied the approximate progressive wave solutions for the perturbed KdV and the dissipative NLS equations. The results obtained here are in complete agreement with the solutions of inverse scattering method. The present solutions are even valid when the dissipative effects are considerably large. The results obtained are encouraging and the method can be used to study the cylindrical and spherical evolution equations.

Keywords: Perturbed KdV equation; Dissipative NLS equation; Weighted residual method.

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1. INTRODUCTION

Quite often the exact solution of many nonlinear partial differential equations encountered in physics and engineering cannot be found by direct approaches. However, if the mathematical models described by these nonlinear differential equations allow us to introduce a smallness parameter, these differential equations may be approximated by some differential equation of which the explicit solution may be obtained. For instance, when the nature of the problem supports the long wave assumption and the linearized wave is dispersive, from the balance of nonlinearity and dispersion, the nonlinear differential equation under investigation may reduce to the Korteweg-deVries(KdV) equation, which is integrable. Similarly, if the waves whose wavelengths are close to each other are superposed, the resulting modulated harmonic waves lead to the nonlinear Schrödinger(NLS) equation, which is also integrable. However, in addition to the nonlinearity and dispersion, if the medium has dissipative effects, in such a case the resulting equations will be the perturbed KdV or dissipative NLS equations, which are not integrable. To obtain some approximate progressive wave solutions to these latter type of evolution equations some methods had been presented in the existing literature(most notably the inverse scattering methods[1-3]). These solutions are valid for small dissipative effects.

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In the present work, we modified the conventional "weighted residual method" and try to obtain approximate progressive wave solutions to some nonlinear evolution equations. Motivated with the solutions of conventional KdV and the nonlinear Schrödinger equations, for the illustration of the method, we studied the approximate progressive wave solutions for the perturbed KdV and the dissipative NLS equations. The results obtained here are in complete agreement with the solutions of inverse scattering method [1-3] and the solutions given in [4-6]. This method can be used in the analysis of the cylindrical and spherical evolution equations of KdV or NLS type.

2. WEIGHTED RESIDUAL METHOD AND SOME APPLICATIONS

Weighted residual method is a generic class of methods developed to obtain approximate solutions to the differential equation of the form

$$L(\phi) + f = 0, \quad \text{in } D, \quad (1)$$

where $\phi(x)$ is the unknown dependent variable, $f(x)$ is a known function and L denotes the differential operator involving the spatial derivatives. For convenience, the boundary conditions are assumed to be homogeneous.

Weighted residual method involves two major steps. In the first step, based on the general behavior of the dependent variable, an approximate solution is assumed. The assumed solution is often selected as the linear combination of some set of functions each of which satisfies the given boundary conditions, but not the differential equation. When this solution is substituted into the differential equation, in general, it does not satisfy the differential equation and the resulting error is called the *residual*. This residual is then made to vanish in some average sense over the domain of definition, which produces a system of algebraic equations. From the solution of these algebraic equations the approximate solution is completely determined.

If we have an evolution equation of the form

$$\frac{\partial \phi}{\partial t} + L(\phi) + f = 0 \quad \text{in } D, \quad (2)$$

where t is the time parameter, the variables ϕ and f are functions of x and t and the differential operator L involves the spatial derivatives. The same approach can be applied to such evolutionary problems and the resulting system of algebraic equations will be a system of ordinary differential equations in terms of the parameter t . The solution of these differential equations give the time evolution of the approximate solution.

As some applications, in what follows we shall study the approximate progressive wave solutions for two evolution equations which are not completely integrable. The solution of these evolution equations had been given before through the use tangent hyperbolic method [4, 6].

2.1. Perturbed Korteweg-deVries equation: In this sub-section, as an application of the modified weighted residual method we shall try to present an approximate progressive wave solution for the perturbed KdV equation given by

$$\frac{\partial \phi}{\partial t} + \gamma_1 \phi \frac{\partial \phi}{\partial x} + \gamma_2 \frac{\partial^3 \phi}{\partial x^3} + \gamma_3 \phi = 0, \quad (3)$$

where the coefficients γ_1 , γ_2 and γ_3 describes the nonlinearity, dispersion and dissipation of the medium, respectively. When the dissipative term vanishes the equation (3) reduces

to the following conventional KdV equation

$$\frac{\partial \phi_0}{\partial t} + \gamma_1 \phi_0 \frac{\partial \phi_0}{\partial x} + \gamma_2 \frac{\partial^3 \phi_0}{\partial x^3} = 0. \quad (4)$$

This equation admits the localized travelling wave solution of the form

$$\phi_0 = a_0 \operatorname{sech}^2 \zeta_0, \quad \zeta_0 = \alpha_0(x - c_0 t), \quad c_0 = \frac{\gamma_1 a_0}{3}, \quad \alpha_0 = \left(\frac{\gamma_1 a_0}{12\gamma_2}\right)^{1/2}, \quad (5)$$

where a_0 is the constant wave amplitude. Motivated with the solution given in (5) we shall propose a progressive wave solution to the equation (3) of the form

$$\phi = a(\tau) V(\zeta), \quad \zeta = \alpha(t)[x - c(t)], \quad (6)$$

where $a(t)$, $\alpha(t)$, $c(t)$ and $V(\zeta)$ are some unknown functions of their arguments. Introducing (6) into (3) we obtain

$$\left[\frac{a'(t)}{a(t)} + \gamma_3 \right] V(\zeta) + \frac{\alpha'(t)}{\alpha(t)} \zeta V' + \alpha(t) \left[-c'(t)V + \gamma_1 a(t) V V' + \gamma_2 \alpha^2(t) V''' \right] = 0, \quad (7)$$

where a prime denotes the differentiation of the corresponding quantity with respect to its argument. The part of equation (7) in the big bracket assumes the localized travelling wave solution of the form

$$V(\zeta) = \operatorname{sech}^2 \zeta, \quad \alpha(t) = \left[\frac{\gamma_1 a(t)}{12\gamma_2} \right]^{1/2}, \quad c'(t) = \frac{\gamma_1}{3} a(t). \quad (8)$$

This solution is formally the same with that of given in (5), except that $a(t)$ is an unknown function. To determine $a(t)$ we need an evolution equation for this variable.

When the solution given in (8) is inserted into the equation (7) the residual term $R(t, \zeta) \neq 0$ becomes

$$R(t, \zeta) = \left[\frac{a'(t)}{a(t)} + \gamma_3 \right] V(\zeta) + \frac{\alpha'(t)}{\alpha(t)} \zeta V'. \quad (9)$$

According to the requirement of the weighted residual method, when the equation (9) is multiplied by a suitable weighing function and integrated over ζ , from $\zeta = -\infty$ to $\zeta = \infty$, the result should vanish. For the present work we shall select the function $V(\zeta)$ as the weighing function. Multiplying (9) by $V(\zeta)$ and integrating the result from $\zeta = -\infty$ to $\zeta = \infty$ and utilizing the localization conditions, i. e., $V(\zeta)$ and its derivatives vanish as $\zeta \rightarrow \pm\infty$, we obtain

$$\left[\frac{a'(t)}{a(t)} - \frac{\alpha'(t)}{2\alpha(t)} + \gamma_3 \right] \langle V \rangle^2 = 0, \quad \langle V \rangle^2 = \int_{-\infty}^{\infty} V^2(\zeta) d\zeta. \quad (10)$$

Here, it is to noted that, in this problem the interval is selected to be $(-\infty, \infty)$ and the boundary conditions as the localization conditions at $\pm\infty$.

Since $V(\zeta)$ is square integrable and $\langle V \rangle \neq 0$, from (10) one obtains

$$\frac{a'(t)}{a(t)} - \frac{\alpha'(t)}{2\alpha(t)} + \gamma_3 = 0. \quad (11)$$

Eliminating $\alpha(t)$ between the equations (8) and (11), the following differential equation is obtained

$$\frac{3}{4} \frac{a'(t)}{a(t)} + \gamma_3 = 0. \quad (12)$$

The solution of the equation (12) gives

$$a(t) = a_0 \exp\left(-\frac{4}{3}\gamma_3 t\right), \quad (13)$$

where a_0 is the initial value of the amplitude. Inserting (13) into (8) the other unknown quantities are given by

$$\alpha(t) = \left(\frac{\gamma_1 a_0}{12\gamma_2}\right)^{1/2} \exp\left(-\frac{2}{3}\gamma_3 t\right), \quad c(t) = \frac{\gamma_1 a_0}{4\gamma_3} [1 - \exp\left(-\frac{4}{3}\gamma_3 t\right)]. \tag{14}$$

Thus, the general approximate solution for the perturbed KdV equation (3) may be given by

$$\begin{aligned} \phi &= a_0 \exp\left(-\frac{4}{3}\gamma_3 t\right) \operatorname{sech}^2 \zeta, \\ \zeta &= \left(\frac{\gamma_1 a_0}{12\gamma_2}\right)^{1/2} \exp\left(-\frac{2}{3}\gamma_3 t\right) \left\{x - \frac{\gamma_1 a_0}{4\gamma_3} [1 - \exp\left(-\frac{4}{3}\gamma_3 t\right)]\right\}. \end{aligned} \tag{15}$$

This solution is exactly the same with that of presented in [2,3] who employed the perturbed inverse scattering method and the one given in [1]. In the limiting case as $\gamma_3 \rightarrow 0$ the solution reduces to the solution of the conventional KdV equation given in (5).

2.2. Dissipative nonlinear Schrödinger equation. As a second example, in this subsection we shall try to apply the weighted residual method for the progressive wave solution to the dissipative nonlinear Schrödinger equation given by

$$i \frac{\partial \phi}{\partial t} + \mu_1 \frac{\partial^2 \phi}{\partial x^2} + \mu_2 |\phi|^2 \phi + i\mu_3 \phi = 0 \tag{16}$$

Here the coefficient μ_3 characterizes the dissipation. Now, we shall introduce the following transformation

$$\phi = U(x, t) \exp(-\mu_3 t) \tag{17}$$

Inserting (17) into (16) we obtain the following conventional nonlinear Schrödinger equation with variable coefficient

$$i \frac{\partial U}{\partial t} + \mu_1 \frac{\partial^2 U}{\partial x^2} + \mu_2 \exp(-2\mu_3 t) |U|^2 U = 0. \tag{18}$$

It is well understood that the conventional nonlinear Schrödinger equation with constant coefficient ($\mu_3 = 0$) has the following solitary wave solution

$$U_0(x, t) = a_0 \operatorname{sech} \zeta_0 \times \exp\{i[\Omega t - Kx]\}, \tag{19}$$

where a_0 is the constant wave amplitude, Ω is the frequency, K is the wave number of the carrier wave and the other quantities are defined by

$$\zeta_0 = \omega_0(x + 2\mu_1 Kt), \quad \omega_0 = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2} a_0, \quad \Omega = \frac{\mu_2}{2} a_0^2 - \mu_1 K^2. \tag{20}$$

Motivated with the solution given in (19) and (20), we shall propose a solution to the equation (18) of the form

$$U = a(t) V(\zeta) \times \exp\{i[\Omega(t) - Kx]\}, \quad \zeta = \alpha(t)[x + 2\mu_1 Kt + x_0], \tag{21}$$

where $V(\zeta)$, $\alpha(t)$, $\Omega(t)$ are some unknown real functions of their arguments, and x_0 is the initial phase. Introducing (21) into (18) and setting the real and imaginary parts equal to zero we obtain

$$-(\Omega' + \mu_1 K^2)V + \mu_1 \omega^2 V'' + \mu_2 a^2 \exp(-2\mu_3 t) V^3 = 0, \tag{22}$$

$$R(t, \zeta) = \frac{a'}{a} V + \frac{\omega'}{\omega} \zeta V' = 0, \tag{23}$$

where a prime denotes the differentiation of the corresponding quantity with respect to its argument.

The solution of the equation (22) is given by

$$V(\zeta) = \operatorname{sech}\zeta, \quad (24)$$

with

$$\alpha(t) = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2} \exp(-2\mu_3 t) a(t),$$

$$\Omega'(t) = -\mu_1 K^2 + \frac{\mu_2}{2} \exp(-2\mu_3 t) a^2(t). \quad (25)$$

In order to complete the solution we need an additional equation between the unknowns $a(t)$, $\alpha(t)$ and $\Omega(t)$. For that purpose we shall use the equation (23) and treat it as the residue of the problem. Multiplying the equation (23) by $V(\zeta)$ and integrating the result from $\zeta = -\infty$ to $\zeta = \infty$ and utilizing the localization condition for $V(\zeta)$, we obtain

$$\left[\frac{a'(t)}{a(t)} - \frac{\alpha'(t)}{2\alpha(t)}\right] \langle V \rangle^2 = 0, \quad \langle V \rangle^2 = \int_{-\infty}^{\infty} V^2 d\zeta. \quad (26)$$

Since $V(\zeta)$ is square integrable and $\langle V \rangle \neq 0$, from equation (26) we obtain

$$\frac{a'(t)}{a(t)} - \frac{\alpha'(t)}{2\alpha(t)} = 0. \quad (27)$$

Combining the equations (25) and (27) the solutions for $a(t)$, $\alpha(t)$ and $\Omega(t)$ are given by

$$a(t) = a_0 \exp(-\mu_3 t), \quad \alpha(t) = \left(\frac{\mu_2}{2\mu_1}\right)^{1/2} a_0 \exp(-2\mu_3 t),$$

$$\Omega(t) = \Omega_0 - \mu_1 K^2 t + \frac{\mu_2}{8\mu_3} a_0^2 [1 - \exp(-4\mu_3 t)], \quad (28)$$

where a_0 is the initial value of the amplitude of the carrier wave and Ω_0 is the initial phase of the harmonic wave. Thus, the general solution of the dissipative nonlinear Schrödinger equation (16) is given by

$$\phi = a_0 \exp(-2\mu_3 t) \operatorname{sech}\left[\left(\frac{\mu_2}{2\mu_1}\right)^{1/2} a_0 \exp(-2\mu_3 t)(x + 2\mu_1 K t)\right]$$

$$\times \exp\left\{i\left[\Omega_0 - \mu_1 K^2 t + \frac{\mu_2}{8\mu_3} a_0^2 (1 - \exp(-4\mu_3 t)) - Kx\right]\right\}. \quad (29)$$

Setting $x_0 = 0$, $\Omega_0 = \pi/2$, $a_0 = 2\phi_0$, $\mu_1 = P$, $\mu_2 = Q$, $\mu_3 = \Gamma$, $K = \sqrt{\frac{Q}{2P}} c$ the equation (29) reduces to

$$\phi = 2i\phi_0 \exp(-2\Gamma t) \operatorname{sech}\left[\sqrt{\frac{Q}{2P}} (\xi + \sqrt{2PQ} ct)\right]$$

$$\times \exp\left\{-i\sqrt{\frac{Q}{2P}} [cx + c^2 \sqrt{\frac{PQ}{2}} t - \frac{\phi_0^2}{\Gamma} \sqrt{\frac{PQ}{2}} (\exp(-4\Gamma t) - 1)]\right\}. \quad (30)$$

This result is exactly the same with that of found by Xue [5], who employed the result of inverse scattering method [1]. In the equation (29), if one sets $\Omega_0 = 0$, the solution will be reduced to the one given in [6], in which the hyperbolic tangent method is employed.

3. CONCLUSION

In the present work, by slightly modifying the conventional " weighted residual method " to some nonlinear evolution equations, which are not integrable, we obtained approximate progressive wave solutions. Particularly, we applied this method to give progressive wave solutions for the perturbed KdV and the dissipative nonlinear Schrödinger equations. The obtained results show that they are in quite good agreement with the results of perturbed inverse scattering method [1-3]. It is hoped that this method can be applied to obtain approximate analytical solutions for the progressive waves in the cylindrical and the spherical KdV and NLS equations, which are not integrable.

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Hilmi Demiray for the photograph and biography, see TWMS Journal of Applied and Engineering Mathematics, Volume 1, No.1, 2011.
