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ON THE AXIOMATIC STRUCTURE OF HERTZIAN ELECTRODYNAMICS

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ABSTRACT. The mathematical foundation, axiomatic structure and principles of Hertzian Electrodynamics for moving bodies are reviewed. The feature of the present investigation is the introduction of a commutative property of the comoving time derivative operator which provides the Hertzian wave equations for material bodies in rotational motion.

Keywords: Maxwell Equations, Moving Media, Hertz Equations, Continuum Mechanics, Frame Indifference, Progressive Derivatives.

AMS Subject Classification: 78A25, 35Q60, 74A05.

1. INTRODUCTION

The present work is an attempt to review and extend certain aspects of the mathematical foundation, axiomatic structure and principles of the classical electrodynamic theory of moving bodies, which is established in literature as Hertzian Electrodynamics (HE).

Following the postulation of electromagnetic field equations of stationary media by Maxwell [1] in 1865, based on the teaching of his mentor Helmholtz, Hertz has introduced his celebrated differential field equations of moving media in 1890 [2]. As an exceptionally gifted professor of mechanics and electromagnetics and the genious experimentalist of his time (cf.[3]-[5]), Hertz obtained his field equations rigorously from the (differential) Maxwell equations of stationary media by a direct incorporation of the convective derivative operator, which is essentially a feature of the well established Material Frame Indifference Principle[1] (MFIP) from continuum mechanics. However, his representation of field equations of moving media was suffering from the widely accepted different interpretations of matter and aether velocities around the time and Hertz never found the opportunity to correct and improve them theoretically any further due to a jaw infection which kept him from studying effectively and eventually led to his death in 1894 at age 36. On the other hand, Heaviside had been working on the same topic separately since 1885 and it has been him who developed Hertz's equations into the form that is used ever since

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[1]MFIP and the debates around its alternative interpretations and correct mathematical formulations throughout the history of continuum mechanics have been reviewed comprehensively in a recent treatise by Frewer [9]. The common motive behind alternative applications of MFIP is that "the structural form and physical content of any law of continuum physics when subject to arbitrary coordinate transformations do not depend on any mathematical quantity which defines the geometrical structure of the underlying space-time manifold".

1893 by replacing the convective derivative with the comoving time derivative, correctly resolving the matter vs. aether velocity conflict and representing the field equations in modern vector algebraic notation in his treatise [6]. Hertz is known to have remarked (cf.[7]) that the priority in the derivation of the classical field equations of moving media actually belongs to Heaviside.

On the other hand, following the interferometry experiments of Fizeau, Mascart, Michelson and Morley in the period 1850-1887, alternative descriptions of electrodynamics of moving bodies were being shaped mainly in the works of Voigt, Larmor, Cohn, Lorentz, Wiechert and Poincaré which eventually led to the Special Relativity Theory (SRT) of Einstein [8] in 1905.[2] With the huge impact of SRT, the traditional concepts "general invariance", "frame indifference", "Newtonian space time", "Euclidean transformations" of classical continuum physics have been substituted with "general covariance", "form invariance", "Minkowski space-time" and "Lorentz transformations", whose reflection onto classical electromagnetism was first sketched with the so called Maxwell-Minkowski Electrodynamics (MME) in 1908 [12]. Since the Lorentz transformations cannot be generalized directly for nonuniform velocities while keeping the Maxwell Equations form invariant, MME was restricted to uniform velocities[3] and the early relativistic formulations of electrodynamics of bodies with arbitrary velocities had to wait until Einstein introduced the principle of general space-time covariance in 1916 [13] in the context of General Relativity Theory (GRT), which asserts that physical laws are covariant not only under Lorentz transformations but also under arbitrary space-time transformations. Following the wide acceptance of SRT and GRT as experimental facts, the scientific efforts in electrodynamics of moving bodies were mostly concentrated on developing covariant theories (cf.[14]-[22]).

It is obvious that frame indifferent and form invariant field theories belong to two different worldviews based on the reason that Lorentz transformations are not covering generalizations of Galilean transformations conceptually[4]. They incidate different formulas and numerical results for any given physical mechanism and therefore they cannot be valid simultaneously. However, it should be seen that the rise of SRT was not related to a disproof of the Heaviside version[5] of frame indifferent Hertz equations. The argument in the Introduction part of the 1905 paper [8] regarding an asymmetry in electromagnetic induction between a magnet and a conductor, which also served as the motivation of introducing SRT, was only addressing a shortcoming of Maxwell equations, which is directly removed when one considers the material frame indifferent Hertz equations. Based on a letter[6] of Einstein to his lover Mileva Maric in 1899 ([10], p.15) in which he expresses his

[2] We refer the interested reader to [10] for a concise history of this period with proper references.

[3] Various heuristic approaches - such as the "instantaneous rest frame" hypothesis of Van Bladel [20] - have been developed in literature to generalize the field equations of MME for nonuniform velocities to predict the results of scattering problems without involving GRT.

[4] The mathematical fact that Lorentz transformations do not exactly coincide with Galilean transformations on a first order approximation is sometimes overlooked in literature. We refer the reader to [29] for a detailed examination of this issue.

[5] While the volumes of works by Heaviside summarizing his contributions to applied mathematics and electric science and technology were not properly recognized and grasped by his contemporaries (cf.[33])due to their high scientific level and language, it is a big disappointment to observe that the situation has continued till date on many occasions where his corrected form of Hertz equations as implemented in all branches of electrical technology for over a century are still "reinvented" and even authenticity may be claimed inattentively by certain authors outside electrical engineering community.

[6] In that letter Einstein criticizes the paper [11] for two specific reasons: the aether assumption (rejected based on the interpretations of the famous interferometry experiments around the time), and (the intuition) that the actual equations should be possible to be presented in a simpler way! However, neither of these critics can be addressed at Heaviside's 1893 version of improved HE.

conviction on the invalidity of Hertz's theories of moving bodies presented in [11], we infer that he was either unaware of Heaviside's 1893 version of improved HE or preferred to ignore frame indifferent formulations categorically while preparing his 1905 paper.

While frame indifferent and form invariant electromagnetic field theories, as two alternative worldviews and mathematical formulations in describing physical phenomena, were meant to be equally respected on a theoretical ground, 20^{th} century has experienced an ideological consensus in adopting form invariance as the mainstream physics, where theoretical works that contradict the accepted "physical facts" are largely avoided by "respected" journals and contrary experimental evidences are disregarded. This understanding is still dominant since "dissident works" –even today– are accepted only by a few indexed journals and rather heavily by nonindexed but openminded ones as Galilean Electrodynamics and Apeiron[7].

The sole purpose of the paper is to revive interest in the Heaviside version of frame indifferent Hertz equations. While we are not interested in questioning the logical consistency, mathematical correctness or outlining the continual critism against SRT, GRT or associated covariant theories in literature in the first place, still some statements should be done in defense of HE. As a first, considering the never ending discussions till date on the *correct* and *accurate* measurements and interpretations of interferometry and GPS experiments in scientific communities regarding the aether drift and the constancy of the speed of light, the declared results of any experiment, by themselves, cannot be accepted as a disproof of HE, and a proof of SRT and GRT (or opposite) until sufficiently accurate and objective experimental set ups that both sides would consent on the distinctive ability of their results are established collectively. It should be reminded that the result of a physical experiment is usually interpreted as a proof by both opposite claims in literature. Regarding a comparison of theoretical aspects, it should be stated that the followers of SRT could actually never manage to present a fully developed (and nonheuristic) alternative theory[8] with the same analytical capabilities as HE since this requires relativistic versions of the classical convective derivative and transport theorems in the first place. Secondly, the application of the principle of general space-time covariance as an alternative to MFIP in practice requires an extensive background in tensor algebra, group theory, projective geometry and so forth with unavoidable sophistication even in the most basic applications[9] and puzzling concepts and interpretations of time and clock synchronization as opposed to the sound, practical and effective analytical tools of HE adapted from classical continuum mechanics which cannot be avoided or replaced with alternatives in undergraduate education.

Our present investigation starts with a short review of the basic analytical tools from kinematics of deformable bodies, which can be met in many textbooks on the topic (cf.[25]-[28]), and their incorporation in certain purely mathematical conservation relations in Sections 2 and 3. A detailed exposition of the comoving time derivative operator and

[7] In that regard the works by Phipps as mostly collected in his books ([23],[24]) serve as very critical eye-opener references in defense of HE and have also stimulated the present author enormously.

[8] It is worthy of commendation that after a century following the "miracle year" 1905, physicists are still paying huge effort to develop relativistic versions of Ampere's and Faraday's induction laws (cf [30],[31]), while they are trivial features of HE. In that context if a relativistic theory that is exempt from the deficiencies of SRT and at the same time applies for arbitrary velocity fields is really required, in the author's opinion the most practical and reasonable way is directly sketching a covering theory of HE as deviced in [24, 32].

[9] As oppposed to the sophistication in the applications of SRT and GRT, the practical solutions of a group of 2-D canonical scattering problems in the context of HE are demonstrated in the sequel paper [45].

its various differential and commutative properties are carried out in Appendix A for fluency. Hertz equations are derived in a straightforward manner in Section 5 by identifying the purely mathematic fields and conservation relations of Section 3 with electromagnetic field quantities and laws in virtue of MFIP. Section 6 is devoted to the demonstration of Hertzian wave equations and Lorentz potentials in simple media for the special cases of translational and rotational motions utilizing the commutative properties of the comoving time derivative operator. At this point the specific contribution of the present investigation to literature can be considered as the introduction of commutative properties of the comoving time derivative operator in case of rotational motion (Theorems A.3 and A.4), which are employed in the derivation of the corresponding Hertzian wave equations. We finally remark on the invariance of wavenumber and reduced field equations for monochromatic waves in Section 7. Throughout the text R_n represents n-dimensional Euclidean space.

2. The Basics of Motion of Material Bodies

Let us consider a material body filling a domain whose material points (or matter particles) are in arbitrary motion characterized by instantaneous velocity vector field $\vec{\mathrm{v}}(\vec{r};t)$, as observed in Cartesian reference configuration $Ox_1x_2x_3t$ as depicted in Fig.1.

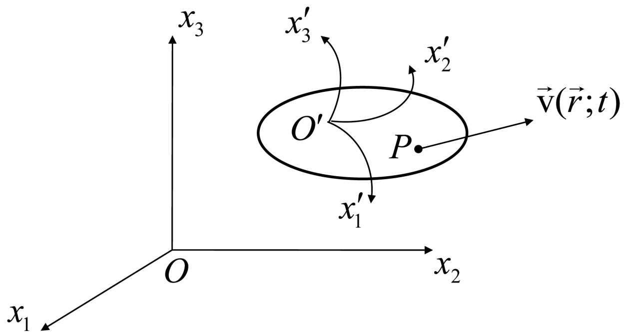

Figure 1. A material body in arbitrary motion with instantaneous velocity

Axiom of Continuity: *Let the general coordinate transformations between (primed) current and (unprimed) referential coordinates of the points constituting the material medium be given by the sets*

$$x_i' = f_i'(x_j;t),\; x_i = f_i(x_j';t) \;\;,\; i,j = 1,2,3, \tag{2.1}$$

where the maps $f_i, f_i' \in C^2(R_3)$ are assumed bijective, not necessarily linear and provide an admissible change of coordinates locally in the moving material medium.

Then the Cartesian components of velocity and acceleration fields are expressed by $\mathrm{v}^i = \frac{\partial}{\partial t} f_i(x_j';t)$, $\mathrm{a}^i = \frac{\partial^2}{\partial t^2} f_i(x_j';t)$; the deformation gradient is defined by $\bar{\bar{F}}(\vec{r}';t) = grad'\vec{r} = F_{ij}\hat{x}_i \otimes \hat{x}_j'$ with $F_{ij} = \frac{\partial x_i}{\partial x_j'}$ in dyadic notation, and under Axiom of Continuity it has an inverse $\bar{\bar{F}}^{-1}(\vec{r};t) = grad\vec{r}' = F_{ij}^{-1}\hat{x}_i' \otimes \hat{x}_j$ with $F_{ij}^{-1} = \frac{\partial x_i'}{\partial x_j}$ and a nonzero Jacobian $J = det\left[\frac{\partial x_i}{\partial x_j'}\right]$. Then the differential arc, surface and volume elements in current configuration are mapped into reference configuration as

$$dx_i = \frac{\partial x_i}{\partial x_j'} dx_j' \text{ or } d\vec{c} = \bar{\bar{F}} \cdot d\vec{c}';\, n_i dS = J\frac{\partial x_j'}{\partial x_i} n_j' dS' \text{ or } d\vec{S} = J d\vec{S}' \cdot \left(\bar{\bar{F}}^{-1}\right)^T;\, d\vartheta = J\, d\vartheta' \tag{2.2}$$

In (2.2) and the rest of the paper the superscript T represents the transpose of the dyad when written in matrix form. These standard kinematical tools also yield the well known Reynolds and Helmholtz transport theorems

$$\frac{d}{dt}\int_{\vartheta}\varphi d\vartheta=\int_{\vartheta}\frac{\Diamond\varphi}{\Diamond t}d\vartheta,\ \frac{d}{dt}\int_{S}\vec{A}\cdot d\vec{S}=\int_{S}\frac{\Diamond\vec{A}}{\Diamond t}\cdot d\vec{S} \tag{2.3}$$

for sufficiently smooth scalar and vector fields. In (2.3) $\frac{\Diamond}{\Diamond t}$represents the comoving time derivative operator which is expressed for scalar and vector fields by

$$\frac{\Diamond\varphi}{\Diamond t}=\frac{\partial\varphi}{\partial t}+\vec{\mathrm{v}}\cdot grad\varphi+\varphi\ div\vec{\mathrm{v}}=\frac{D\varphi}{Dt}+\varphi\, div\vec{\mathrm{v}}=\frac{\partial\varphi}{\partial t}+div\left(\vec{\mathrm{v}}\varphi\right) \tag{2.4}$$

$$\frac{\Diamond\vec{A}}{\Diamond t}=\frac{\partial\vec{A}}{\partial t}+div\left(\vec{\mathrm{v}}\vec{A}\right)-\vec{A}\cdot grad\vec{\mathrm{v}}=\frac{D\vec{A}}{Dt}-\vec{A}\cdot grad\vec{\mathrm{v}}+\vec{A}\,div\vec{\mathrm{v}}=\frac{\partial\vec{A}}{\partial t}+\vec{\mathrm{v}}div\vec{A}-curl\left(\vec{\mathrm{v}}\times\vec{A}\right) \tag{2.5}$$

with $\frac{D}{Dt}=\frac{\partial}{\partial t}+\vec{\mathrm{v}}\cdot grad$ standing for the standard convective derivative operator. The limit description of the comoving time derivative operator in (2.4) and (2.5) and its various differential and commutative properties employed in Section 6 are provided in Appendix A.

In the rest of the paper we will adopt the terms "E-frame" and "L-frame" as abbreviations of Eulerian and Lagrangian frames from fluid mechanics for denoting reference (spatial) and current (material) configurations for brevity.

3. Conservation Relations

Let arbitrary regular open surface and volume regions be denoted by S', ϑ' and S, ϑ in L- and E-frames, respectively. Next, let us describe arbitrary (smooth enough) scalar and vector valued fields $\vec{E}'(\vec{r}';t)$, $\vec{B}'(\vec{r}';t)$, $\vec{H}'(\vec{r}';t)$, $\vec{D}'(\vec{r}';t)$, $\vec{J}'_C(\vec{r}';t)$, $\rho'_f(\vec{r}';t)$ which are assumed *purely mathematical* and satisfy the set of integral relations

$$\oint_{\partial S'}\vec{E}'\cdot d\vec{c}'+\frac{d}{dt}\int_{S'}\vec{B}'\cdot d\vec{S}'=\vec{0},\ \oint_{\partial S'}\vec{H}'\cdot d\vec{c}'+\frac{d}{dt}\int_{S'}\vec{D}'\cdot d\vec{S}'=\int_{S'}\vec{J}'_C\cdot d\vec{S}' \tag{3.1a,b}$$

$$\oint_{\partial\vartheta'}\vec{D}'\cdot d\vec{S}'=\int_{\vartheta'}\rho'_f\,d\vartheta',\ \oint_{\partial\vartheta'}\vec{B}'\cdot d\vec{S}'=0 \tag{3.1c,d}$$

$$\oint_{\partial\vartheta'}\vec{J}'_C\cdot d\vec{S}'+\frac{d}{dt}\int_{\vartheta'}\rho'_f d\vartheta'=0, \tag{3.2}$$

or alternatively[10], the differential set

$$curl'\vec{E}'(\vec{r}';t)+\frac{\partial}{\partial t}\vec{B}'(\vec{r}';t)=\vec{0},\ curl'\vec{H}'(\vec{r}';t)-\frac{\partial}{\partial t}\vec{D}'(\vec{r}';t)=\vec{J}'_C(\vec{r}';t) \tag{3.3a,b}$$

$$div'\vec{D}'(\vec{r}';t)=\rho'_f(\vec{r}';t),\ div'\vec{B}'(\vec{r}';t)=0 \tag{3.3c,d}$$

$$div'\vec{J}'_C(\vec{r}';t)+\frac{\partial}{\partial t}\rho'_f(\vec{r}';t)=0 \tag{3.4}$$

Let us *assume* the map of (3.1)-(3.4) into E-frame under (2.1) are given by

$$\oint_{\partial S}\vec{E}\cdot d\vec{c}+\frac{d}{dt}\int_{S}\vec{B}\cdot d\vec{S}=\vec{0},\ \oint_{\partial S}\vec{H}\cdot d\vec{c}-\frac{d}{dt}\int_{S}\vec{D}\cdot d\vec{S}=\int_{S}\vec{J}_C\cdot d\vec{S} \tag{3.5a,b}$$

$$\oint_{\partial\vartheta}\vec{D}\cdot d\vec{S}=\int_{\vartheta}\rho_f\,d\vartheta,\ \oint_{\partial\vartheta}\vec{B}\cdot d\vec{S}=0 \tag{3.5c,d}$$

[10]A proof for formal equivalences between such differential and integral representations of the same relation can be seen in many basic textbooks such as [28], p.42 and [34], p.3.

$$\oint_{\partial\vartheta} \vec{J}_C \cdot d\vec{S} + \frac{d}{dt}\int_{\vartheta} \rho_f \, d\vartheta = 0 \tag{3.6}$$

and

$$curl\vec{E}(\vec{r};t) + \frac{\Diamond}{\Diamond t}\vec{B}(\vec{r};t) = \vec{0}, \; curl\vec{H}(\vec{r};t) - \frac{\Diamond}{\Diamond t}\vec{D}(\vec{r};t) = \vec{J}_C(\vec{r};t) \tag{3.7a,b}$$

$$div\vec{D}(\vec{r};t) = \rho_f(\vec{r};t), \; div\vec{B}(\vec{r};t) = 0 \tag{3.7c,d}$$

$$div\vec{J}_C(\vec{r};t) + \frac{\Diamond}{\Diamond t}\rho_f(\vec{r};t) = 0 \tag{3.8}$$

via Reynolds and Helmholtz transport theorems, while the fields in L-frame are mapped into E-frame as $\rho_f(\vec{r};t)$, $\vec{E}(\vec{r};t)$, $\vec{B}(\vec{r};t)$, $\vec{H}(\vec{r};t)$, $\vec{D}(\vec{r};t)$, $\vec{J}_C(\vec{r};t)$ under the "conservation relations"

$$\oint_{\partial S'} \vec{E}' \cdot d\vec{c}' = \oint_{\partial S} \vec{E} \cdot d\vec{c}, \; \oint_{\partial S'} \vec{H}' \cdot d\vec{c}' = \oint_{\partial S} \vec{H} \cdot d\vec{c} \tag{3.9a,b}$$

$$\int_{S'} \vec{B}' \cdot d\vec{S}' = \int_S \vec{B} \cdot d\vec{S}, \; \int_{S'} \vec{D}' \cdot d\vec{S}' = \int_S \vec{D} \cdot d\vec{S} \tag{3.9c,d}$$

$$\int_{S'} \vec{J}'_C \cdot d\vec{S}' = \int_S \vec{J}_C \cdot d\vec{S} \, , \int_{\vartheta'} \rho'_f d\vartheta' = \int_{\vartheta} \rho_f \, d\vartheta \tag{3.9e,f}$$

In virtue of (2.2) the conservation relations require the field transformations

$$\vec{E}' = \vec{E} \cdot \bar{\bar{F}}, \vec{H}' = \vec{H} \cdot \bar{\bar{F}}, \vec{B}' = J\bar{\bar{F}}^{-1} \cdot \vec{B}, \vec{D}' = J\bar{\bar{F}}^{-1} \cdot \vec{D}, \vec{J}'_C = J\bar{\bar{F}}^{-1} \cdot \vec{J}_C, \rho'_f = J\rho_f . \tag{3.10}$$

Regarding the map of the spatial and temporal derivatives of the fields one may invoke

$$\int_{S'} curl'\vec{E}' \cdot d\vec{S}' = \int_S curl\vec{E} \cdot d\vec{S}, \int_{S'} curl'\vec{H}' \cdot d\vec{S}' = \int_S curl\vec{H} \cdot d\vec{S} \tag{3.11a,b}$$

$$\int_{S'} \frac{\partial \vec{B}'}{\partial t} \cdot d\vec{S}' = \int_S \frac{\Diamond \vec{B}}{\Diamond t} \cdot d\vec{S}, \; \int_{S'} \frac{\partial \vec{D}'}{\partial t} \cdot d\vec{S}' = \int_S \frac{\Diamond \vec{D}}{\Diamond t} \cdot d\vec{S} \tag{3.11c,d}$$

$$\int_{\vartheta'} div'\vec{D}' d\vartheta' = \int_{\vartheta} div\vec{D} \, d\vartheta, \; \int_{\vartheta'} div'\vec{B}' \, d\vartheta' = \int_{\vartheta} div\vec{B} \, d\vartheta \tag{3.11e,f}$$

$$\int_{\vartheta'} div'\vec{J}'_C d\vartheta' = \int_{\vartheta} div\vec{J}_C \, d\vartheta, \; \int_{\vartheta'} \frac{\partial \rho'_f}{\partial t} d\vartheta' = \int_{\vartheta} \frac{\Diamond \rho_f}{\Diamond t} \, d\vartheta \tag{3.11g,h}$$

which necessiate

$$curl'\vec{E}' = J\bar{\bar{F}}^{-1} \cdot curl\vec{E}, \; curl'\vec{H}' = J\bar{\bar{F}}^{-1} \cdot curl\vec{H} \tag{3.12a,b}$$

$$div'\vec{D}' = Jdiv\vec{D}, \; div'\vec{B}' = Jdiv\vec{B} \tag{3.12c,d}$$

$$\frac{\partial \vec{B}'}{\partial t} = J\bar{\bar{F}}^{-1} \cdot \frac{\Diamond \vec{B}}{\Diamond t}, \; \frac{\partial \vec{D}'}{\partial t} = J\bar{\bar{F}}^{-1} \cdot \frac{\Diamond \vec{D}}{\Diamond t} \tag{3.12e,f}$$

$$div'\vec{J}'_C = Jdiv\vec{J}_C \, , \; \frac{\partial \rho'_f}{\partial t} = J\frac{\Diamond \rho_f}{\Diamond t} \tag{3.12g,h}$$

Equations (3.7a,b) can also be written in the alternative form

$$curl\left(\vec{E} - \vec{v} \times \vec{B}\right) + \frac{\partial}{\partial t}\vec{B} = \vec{0}, \; curl\left(\vec{H} + \vec{v} \times \vec{D}\right) - \frac{\partial}{\partial t}\vec{D} = \vec{J}_f \tag{3.13a,b}$$

upon inserting

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{D} &= \frac{\partial}{\partial t}\vec{D} + \vec{\mathrm{v}} div\vec{D} - curl\left(\vec{\mathrm{v}}\times\vec{D}\right)\\ &= \frac{\partial}{\partial t}\vec{D} + \vec{\mathrm{v}}\rho_f - curl\left(\vec{\mathrm{v}}\times\vec{D}\right) = \frac{\partial}{\partial t}\vec{D} + \vec{J}_V - curl\left(\vec{\mathrm{v}}\times\vec{D}\right)\end{aligned} \tag{3.14}$$

$$\frac{\Diamond}{\Diamond t}\vec{B} = \frac{\partial}{\partial t}\vec{B} + \vec{\mathrm{v}} div\vec{B} - curl\left(\vec{\mathrm{v}}\times\vec{B}\right) = \frac{\partial}{\partial t}\vec{B} - curl\left(\vec{\mathrm{v}}\times\vec{B}\right) \tag{3.15}$$

where we describe

$$\vec{J}_V = \vec{\mathrm{v}}\rho_f,\ \vec{J}_f = \vec{J}_C + \vec{\mathrm{v}}\rho_f. \tag{3.16}$$

The relations (3.9a,b) require

$$\begin{aligned}\oint_{\partial S'}\vec{E}'\cdot d\vec{c}' &= \oint_{\partial S}\vec{E}\cdot d\vec{c} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}\\ &= -\int_S \frac{\Diamond}{\Diamond t}\vec{B}\cdot d\vec{S} = -\int_S \frac{\partial}{\partial t}\vec{B}\cdot d\vec{S} + \oint_{\partial S}\left(\vec{\mathrm{v}}\times\vec{B}\right)\cdot d\vec{c}\end{aligned} \tag{3.17a}$$

$$\begin{aligned}\oint_{\partial S'}\vec{H}'\cdot d\vec{c}' &= \oint_{\partial S}\vec{H}\cdot d\vec{c} = \frac{d}{dt}\int_S \vec{D}\cdot d\vec{S} + \int_S \vec{J}_C\cdot d\vec{S} = \int_S \frac{\Diamond}{\Diamond t}\vec{D}\cdot d\vec{S} + \int_S \vec{J}_C\cdot d\vec{S}\\ &= \int_S \frac{\partial}{\partial t}\vec{D}\cdot d\vec{S} + \int_S \left(\vec{J}_C + \vec{\mathrm{v}}\rho_f\right)\cdot d\vec{S} - \oint_{\partial S}\left(\vec{\mathrm{v}}\times\vec{D}\right)\cdot d\vec{c}\\ &= \int_S \frac{\partial}{\partial t}\vec{D}\cdot d\vec{S} + \int_S \vec{J}_f\cdot d\vec{S} - \oint_{\partial S}\left(\vec{\mathrm{v}}\times\vec{D}\right)\cdot d\vec{c}\end{aligned} \tag{3.17b}$$

In the special case of Euclidean (aka observer) transformations for rigid bodies in the form

$$\vec{r}' = \vec{c}(t) + \bar{\bar{Q}}(t)\cdot\vec{r},\ \vec{r} = \bar{\bar{Q}}^T(t)\cdot\left[\vec{r}' - \vec{c}(t)\right] \tag{3.18}$$

where $\bar{\bar{Q}}$ is an arbitrary time dependent tensor characterizing rotation ($det\left[\bar{\bar{Q}}_{ij}\right] = 1$, $\bar{\bar{Q}}^{-1} = \bar{\bar{Q}}^T$) and $\vec{c}(t)$ represents the translation vector $\vec{c}(t)$. The orthonormal Cartesian bases $\hat{r} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{r}' = (\hat{x}_1', \hat{x}_2', \hat{x}_3')$ are transformed as $\hat{r}' = \bar{\bar{Q}}\cdot\hat{r}$, $\hat{r} = \bar{\bar{Q}}^T\cdot\hat{r}'$ and one has $div\vec{\mathrm{v}} = 0$, $\bar{\bar{F}} = \bar{\bar{Q}}^{-1}$, $J = 1$, $J\bar{\bar{F}}^{-1} = \bar{\bar{Q}}$, which read

$$\vec{E}' = \bar{\bar{Q}}\cdot\vec{E},\ \vec{H}' = \bar{\bar{Q}}\cdot\vec{H},\ \vec{D}' = \bar{\bar{Q}}\cdot\vec{D},\ \vec{B}' = \bar{\bar{Q}}\cdot\vec{B},\ \vec{J}_C' = \bar{\bar{Q}}\cdot\vec{J}_C,\ \rho_f' = \rho_f \tag{3.19a}$$

$$curl'\vec{E}' = \bar{\bar{Q}}\cdot curl\vec{E},\ curl'\vec{H}' = \bar{\bar{Q}}\cdot curl\vec{H},\ div'\vec{D}' = div\vec{D},\ div'\vec{B}' = div\vec{B} \tag{3.19b}$$

$$\frac{\partial\vec{B}'}{\partial t} = \bar{\bar{Q}}\cdot\frac{\Diamond\vec{B}}{\Diamond t},\ \frac{\partial\vec{D}'}{\partial t} = \bar{\bar{Q}}\cdot\frac{\Diamond\vec{D}}{\Diamond t},\ div'\vec{J}_C' = div\vec{J}_C,\ \frac{\partial\rho_f'}{\partial t} = \frac{\Diamond\rho_f}{\Diamond t} \tag{3.19c}$$

Certain of the important results employed in the present paper for the two special cases of Euclidean motion are depicted in Table 1.

TABLE 1. Certain analytical results for two special types of Euclidean motion.

Type of Motion	Translational	Rotational
Coordinate Maps	$x_i = x_i' + \int_{-\infty}^{t} \mathrm{v}^i(\xi)\, d\xi$	$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_3' = x_3,\ \phi(t) = \int_{-\infty}^{t} \omega(\xi) d\xi$
Velocity	$\vec{\mathrm{v}}(\vec{r};t) = \vec{\mathrm{v}}(t)$	$\vec{\mathrm{v}}(\rho,\phi,z;t) = \omega(t)\rho\,\hat{\phi}(t)$, $\hat{\phi}(t) = -\hat{x}_1\sin\phi(t) + \hat{x}_2\cos\phi(t)$
Acceleration	$\vec{\mathrm{a}} = \frac{d\vec{\mathrm{v}}}{dt}$	$\vec{\mathrm{a}} = -\omega^2(t)\rho\hat{\rho}(t) + \frac{d\omega}{dt}\rho\hat{\phi}(t)$, $\hat{\rho}(t) = \hat{x}_1\cos\phi(t) + \hat{x}_2\sin\phi(t)$
Differential Properties of Velocity	$div\vec{\mathrm{v}} = 0$ $curl\vec{\mathrm{v}} = \vec{0}$	$div\vec{\mathrm{v}} = 0$ $curl\vec{\mathrm{v}} = 2\omega(t)\hat{z}$
Velocity Gradient	$\bar{\bar{L}} = grad\vec{\mathrm{v}} = \bar{\bar{0}}$	$\bar{\bar{L}} = grad\vec{\mathrm{v}} = \omega(t)\left(\hat{\rho}(t)\hat{\phi}(t) - \hat{\phi}(t)\hat{\rho}(t)\right)$
Deformation Gradient	$\bar{\bar{F}} = \left[\frac{\partial x_i}{\partial x_j'}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \bar{\bar{I}}$	$\bar{\bar{F}} = \left[\frac{\partial x_i}{\partial x_j'}\right] = \begin{bmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{bmatrix}$ $= \bar{\bar{Q}}^T$
Jacobian of Deformation Gradient	$J = det\left[\frac{\partial x_i}{\partial x_j'}\right] = 1$	$J = det\left[\frac{\partial x_i}{\partial x_j'}\right] = 1$
Convective Derivative	$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathrm{v}}(t)\cdot grad$	$\frac{D}{Dt} = \frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}$
Comoving Time Derivatives	$\frac{\diamond}{\diamond t}f = \frac{D}{Dt}f$ $\frac{\diamond}{\diamond t}\vec{A} = \frac{D}{Dt}\vec{A}$	$\frac{\diamond}{\diamond t}f = \frac{D}{Dt}f$ $\frac{\diamond}{\diamond t}\vec{A} = \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)\vec{A}$
Certain Commutative Properties	$div\left(\frac{\diamond}{\diamond t}\vec{A}\right) = \frac{\diamond}{\diamond t}\left(div\vec{A}\right)$ $\frac{\diamond}{\diamond t}\left(grad\vec{A}\right) = grad\left(\frac{\diamond}{\diamond t}\vec{A}\right)$ $= grad\left(\frac{D}{Dt}\vec{A}\right)$ $\frac{\diamond}{\diamond t}\left(curl\vec{A}\right) = curl\left(\frac{\diamond}{\diamond t}\vec{A}\right)$ $= curl\left(\frac{D}{Dt}\vec{A}\right)$	$div\left(\frac{\diamond}{\diamond t}\vec{A}\right) = \frac{\diamond}{\diamond t}\left(div\vec{A}\right)$ $\frac{\diamond}{\diamond t}\left(grad\vec{A}\right) = grad\left(\frac{D}{Dt}\vec{A}\right)$ $\frac{\diamond}{\diamond t}\left(curl\vec{A}\right) = curl\left(\frac{\diamond}{\diamond t}\vec{A}\right)$

4. Maxwell Equations of Stationary Media

Let us associate the L-frame *mathematical* fields in Section 3 with the *material* electromagnetic field quantities of stationary media in MKSA units as follows:

$\vec{E}'(\vec{r}';t)$: Electrical field vector [V/m]
$\vec{B}'(\vec{r}';t)$: Magnetic induction density field [Wb/m^2] (or [T])
$\vec{H}'(\vec{r}';t)$: Magnetic field [A/m]
$\vec{D}'(\vec{r}';t)$: Displacement density field [C/m]
$\vec{J}_C'(\vec{r}';t)$: Conduction current density [A/ m^2]
$\rho_f'(\vec{r}';t)$: Free charge density [C/m^3]

Postulate 1: *Macroscopic electromagnetic phenomena of stationary continuous material media are governed by the Maxwell equations* (3.1)*, or equivalently* (3.3)[11].

The postulate inherently involves the principle of superposition for sources and fields. When the Maxwell equations are considered as the fundamental laws of stationary continuous media, then the continuity relations (3.2) and (3.4) follow as corollaries. It is also possible to go backwards by postulating the four continuity relations for free and polarized electrical/magnetic currents to derive the four Maxwell equations in symmetric form ([35]).

For a complete description of material media we also involve the closed form constitutive relations

$$\vec{D}' = \vec{f}_d\left(\vec{E}';\vec{H}'\right) = \varepsilon_0\vec{E}' + \vec{P^e}', \vec{B}' = \vec{f}_b\left(\vec{E}';\vec{H}'\right) = \mu_0\vec{H}' + \vec{P^m}', \vec{J}_C' = \vec{f}_C\left(\vec{E}';\vec{H}'\right) \quad (4.1)$$

with the additional field quantities

$\varepsilon_0 = (1/36\pi)\times 10^{-9}$: Dielectric permittivity of free space [F/m]

$\mu_0 = 4\pi\times 10^{-7}$: Magnetic permittivity of free space [H/m]

$\vec{P^e}'(\vec{r}';t)$: Electrical polarization density field [C/m]

$\vec{P^m}'(\vec{r}';t)$: Magnetic polarization density field [Wb/m^2]

and $\vec{f}_d$, $\vec{f}_b$, $\vec{f}_C$ being suitable arbitrary functions that may characterize material media.

The corresponding Lorentz potentials are represented by

$$\vec{B}'(\vec{r}';t) = curl'\vec{A}(\vec{r}';t),\ \vec{E}'(\vec{r}';t) = -\frac{\partial}{\partial t}\vec{A}'(\vec{r}';t) - grad'V'(\vec{r}';t) \quad (4.2\text{a,b})$$

and the Poynting theorem in point form is obtained as

$$\mathrm{div}\vec{P}' + \vec{E}'\cdot\vec{J}_d^{e\prime} + \vec{H}'\cdot\vec{J}_d^{m\prime} + \vec{E}'\cdot\vec{J}_C' = 0 \quad (4.3\text{a})$$

where

$$\vec{P}'(\vec{r}';t) = \vec{E}'(\vec{r}';t)\times\vec{H}'(\vec{r}';t) \quad (4.3\text{b})$$

$$\vec{J}_d^{e\prime}(\vec{r}';t) = \frac{\partial}{\partial t}\vec{D}'(\vec{r}';t),\ \vec{J}_d^{m\prime}(\vec{r}';t) = \frac{\partial}{\partial t}\vec{B}'(\vec{r}';t) \quad (4.3\text{c,d})$$

stand for the electric and magnetic displacement current densities regardless of the constitutive parameters of the medium involved.

The integral form of Poynting theorem (4.3) in ϑ' is written as

$$P_{in}'(\vec{r}';t) = P_d^{e\prime}(\vec{r}';t) + P_d^{m\prime}(\vec{r}';t) + P_C'(\vec{r}';t) \quad (4.4\text{a})$$

where

$$P_{in}' = -\oint_{\partial\vartheta'}\vec{P}'\cdot d\vec{S}',\ P_d^{e\prime} = \int_{\vartheta'}\vec{E}'\cdot\vec{J}_d^{e\prime}\,d\vartheta' = \int_{\vartheta'}\vec{E}'\cdot\frac{\partial\vec{D}'}{\partial t}\,d\vartheta' \quad (4.4\text{b,c})$$

$$P_d^{m\prime} = \int_{\vartheta'}\vec{H}'\cdot\vec{J}_d^{m\prime}\,d\vartheta' = \int_{\vartheta'}\vec{H}'\cdot\frac{\partial\vec{B}'}{\partial t}\,d\vartheta',\ P_C' = \int_{\vartheta'}\vec{E}'\cdot\vec{J}_C'\,d\vartheta' \quad (4.4\text{d,e})$$

and can be interpreted as follows:

The total electromagnetic power P_{in}' entering (or pumped by external sources to) an arbitrary stationary material medium ϑ' is equal to the sum of

(1) the total electrical power $P_d^{e\prime}$ stored in that medium;

[11]There is a direct analogy between the equivalence of integral and differential forms of any law in electromagnetics and the well known postulate of localization from continuum mechanics, which tells that the integral balance laws are valid, not only for the full body, but also for every arbitrary region of the body.

(2) the total magnetic power $P_d^{m\prime}$ stored in that medium;
(3) the total electrical power $P_C^{\prime}$ dissipated as heat in that medium.

In a simple medium described by the constitutive relations

$$\vec{D}' = \varepsilon\vec{E}',\ \vec{B}' = \mu\vec{H}',\ \vec{J}_C' = \sigma\vec{E}' \tag{4.5}$$

the well known wave equations read

$$L'\vec{E}'(\vec{r}';t) = (1/\varepsilon)\, grad'\rho_f'(\vec{r}';t),\ L'\vec{H}'(\vec{r}';t) = \vec{0} \tag{4.6}$$

$$L'\vec{A}'(\vec{r}';t) = \vec{0},\ L'V'(\vec{r}';t) = -\,(1/\varepsilon)\,\rho_f'(\vec{r}';t) \tag{4.7}$$

under the Lorentz gauge relation

$$div'\vec{A}'(\vec{r}';t) + \varepsilon\mu\frac{\partial}{\partial t}V'(\vec{r}';t) + \sigma\mu V'(\vec{r}';t) = 0, \tag{4.8}$$

where $L' = lap' - \varepsilon\mu\frac{\partial^2}{\partial t^2} - \sigma\mu\frac{\partial}{\partial t}$ is the stationary wave operator.
The Lorentz force law in L-frame is an additional (external) postulate to Maxwell's field theory of stationary media given by the following:

Postulate 2: *The mechanical force acting on free charges and conduction currents at rest in Maxwell's field theory of stationary continuous media is described by the Lorentz force law, which we express for the force volume density field as*

$$\vec{f}'(\vec{r}';t) = \frac{d\vec{F}'}{d\vartheta'}(\vec{r}';t) = \rho_f'(\vec{r}';t)\,\vec{E}'(\vec{r}';t) + \vec{J}_C'(\vec{r}';t)\times\vec{B}'(\vec{r}';t) \tag{4.9}$$

The Lorentz force law is our unique bridge connecting the disciplines of electromagnetism and mechanics.

We shall also outline the special cases of electrostatic and magnetostatic field equations in L-frame as

$$curl'\vec{E}'(\vec{r}') = \vec{0},\ div'\vec{D}'(\vec{r}') = \rho_f'(\vec{r}'),\ \vec{E}'(\vec{r}') = -grad'V'(\vec{r}') \tag{4.10a,b}$$

$$lap'V'(\vec{r}') = -\,(1/\varepsilon)\,\rho_f'(\vec{r}') \quad \text{(in a simple medium)} \tag{4.10c}$$

$$\vec{f}'(\vec{r}') = \frac{d\vec{F}'}{d\vartheta'}(\vec{r}') = \rho_f'(\vec{r}')\,\vec{E}'(\vec{r}') \tag{4.10d}$$

and

$$curl'\vec{H}'(\vec{r}') = \vec{J}_C'(\vec{r}'),\ div'\vec{B}'(\vec{r}') = 0, div'\vec{J}_C'(\vec{r}') = 0,\ \vec{B}'(\vec{r}') = curl'\vec{A}(\vec{r}') \tag{4.11a-d}$$

$$div'\vec{A}'(\vec{r}') = 0 \quad \text{(Coulomb gauge)} \tag{4.11e}$$

$$lap'\vec{A}'(\vec{r}') = -\mu\vec{J}_C'(\vec{r}') \quad \text{(in a simple medium)} \tag{4.11f}$$

Postulate 3: *The electromagnetic field equations of stationary continuous media and the Lorentz force law relation are also valid in the sense of Schwartz-Sobolev distributions.*

For a mathematical description of physical phenomena we generally refer to idealized models. An example for a geometrical idealization could be the assumption that certain sources or fields are localized in a compact region or concentrated in non-volumetric domains such a surface, a curve or a point. While a field theory in continuum is usually derived in the space of continuous functions, the presence of (idealized) singularities pose a much harder problem, the solution of which is usually investigated in the space of generalized functions first introduced by Sobolev (cf.[36]) and later developed extensively by mid 20^{th} century as documented in the reference works by many great mathematicians led by the pioneers Schwartz [37], Gel'fand and Shilov [38]. The requirement for generalized functions in a field theory can be realized immediately in an attempt to express analytically the volume density function of a source quantity concentrated in a non-volumetric domain. Since the point form field equations in mathematical physics are always given through density functions in space and time, such equations would otherwise not permit algebraic operations in non-volumetric domains.

The linear structure of electromagnetic field theory provides a postulate in a Schwartz-Sobolev space setting possible in terms of the most studied class of generalized functions called the Dirac delta distributions, which have been an indispensable tool in theoretical physics ever since the beginning of 20^{th} century. Today we observe (cf.[39]) that distributional techniques yield physically valid results not only in classical electromagnetism but in almost all disciplines of natural sciences.

Although the distributional results of Maxwell equations were derived and utilized much earlier in literature[12], to the best of the author's knowledge, this fact was introduced as a *postulate* and treated systematically first by İdemen [40] in 1973 (see also [41]-[43]). Along with other types of complementary conditions such as radiation condition, periodicity, boundedness, *etc.*, we can consider the description of the boundary value problem under investigation formally completed.

5. Hertz Equations of Moving Media

The reflection of MFIP onto the field equations of moving media in the works of Hertz and Heaviside can be culminated in the following postulate:

Postulate 4: *The laws of macroscopic electromagnetism of stationary continuous media are frame indifferent.*

This is another way to saying that E- and L-frame observers are in full agreement with

(1) the nature (or state) of any physical quantity
(2) the structural form and content of any physical law, and
(3) the result of any measurement taken

in the two frames. To open it up, the first item requires that the electromagnetic field quantities in the laws of stationary media are observed as the same quantities (denoted without primes) in E-frame linked by the passive transformations (2.1). Accordingly, as a further requirement by the second item, the Hertzian field equations can be expressed directly by (3.5)-(3.8). Finally, the third item requires the conservation (balance) relations (3.9), which impose *the invariance of the electromotive force, magnetomotive force, total electric/magnetic charge and flux, and total conduction current* as measured by ideal devices in the two frames.

These relations also introduce the field transformations (3.10) and (3.12); the alternative representation of the Hertzian equations in (3.13) where $\vec{J}_V$ and $\vec{J}_f$ denote the convective current and the total free current as observed in E-frame; and the explicit representations of electromotive and magnetomotive forces in (3.17) as observed in E-frame.

In the context of HE, Postulates 1 to 4 are sufficient in constructing a material description of macroscopic electromagnetism for arbitrary continuous media in arbitrary motion.

[12]As one of the comprehensive papers on the distributional investigation of Maxwell equations the reader may refer to [44]. Many works on the distributional investigation of the balance laws of nonlinear electromagnetic continua were also available at the time.

One does not require an additional postulate on the distributional behavior of Hertz equations as observed in E-frame since they coincide with the passive transformations of the corresponding distributional relations obtained via Postulate 3 in L-frame.

The constitutive relations in E-frame can always be expressed in closed form as

$$\vec{D} = \vec{f}_d\left(\vec{E};\vec{H}\right) = \varepsilon_0\vec{E} + \vec{P}^e,\ \vec{B} = \vec{f}_b\left(\vec{E};\vec{H}\right) = \mu_0\vec{H} + \vec{P}^m,\ \vec{J}_C = \vec{f}_C\left(\vec{E};\vec{H}\right) \tag{5.1}$$

Regarding the Lorentz potentials, from (3.7d) and (3.13a) one can directly write

$$\vec{B}(\vec{r};t) = curl\vec{A}(\vec{r};t) \tag{5.2}$$

$$\begin{aligned}\vec{E}(\vec{r};t) &= \vec{\mathrm{v}}(\vec{r};t) \times \vec{B}(\vec{r};t) - \tfrac{\partial}{\partial t}\vec{A}(\vec{r};t) - gradV(\vec{r};t)\\ &= \vec{\mathrm{v}}(\vec{r};t) \times curl\vec{A}(\vec{r};t) - \tfrac{\partial}{\partial t}\vec{A}(\vec{r};t) - gradV(\vec{r};t)\end{aligned} \tag{5.3}$$

regardless of the constitutive parameters of the medium involved. In his book ([24], Ch.5) Phipps defines the last two terms at the r.h.s. of (5.3) as 'the Maxwell $\vec{E}$- field'

$$\vec{E}_{Max}(\vec{r};t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r};t) - gradV(\vec{r};t) \tag{5.4}$$

in the context of HE based on its structural similarity with (4.2b).

The introduction of the comoving time derivative requires us to describe the electric/magnetic displacement current density of the medium in E-frame as

$$\vec{J}^e_d(\vec{r};t) = \frac{\Diamond}{\Diamond t}\vec{D}(\vec{r};t),\ \vec{J}^m_d(\vec{r};t) = \frac{\Diamond}{\Diamond t}\vec{B}(\vec{r};t). \tag{5.5}$$

Then the Poynting theorem in the moving medium can be written as

$$div\vec{P} + \vec{E}\cdot\frac{\Diamond}{\Diamond t}\vec{D} + \vec{H}\cdot\frac{\Diamond}{\Diamond t}\vec{B} + \vec{E}\cdot\vec{J}_C + \vec{E}\cdot\vec{J}_V = 0 \tag{5.6}$$

while the Poynting vector in E-frame is defined in the usual form

$$\vec{P}(\vec{r};t) = \vec{E}(\vec{r};t) \times \vec{H}(\vec{r};t). \tag{5.7}$$

The integral form of Poynting theorem in E-frame is expressed by

$$P_{in}(\vec{r};t) = P^e_d(\vec{r};t) + P^m_d(\vec{r};t) + P_C(\vec{r};t) \tag{5.8a}$$

$$P_{in} = -\oint_{\partial\vartheta}\vec{P}\cdot d\vec{S} \tag{5.8b}$$

$$\begin{aligned}P^e_d &= \int_\vartheta \vec{E}\cdot\vec{J}^e_d\,d\vartheta = \int_\vartheta \vec{E}\cdot\frac{\Diamond}{\Diamond t}\vec{D}\,d\vartheta\\ &= \int_\vartheta \vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\,d\vartheta + \int_\vartheta \vec{E}_{Max}\cdot\vec{J}_V\,d\vartheta - \int_\vartheta \vec{E}\cdot curl\left(\vec{\mathrm{v}}\times\vec{D}\right)\,d\vartheta\end{aligned} \tag{5.8c}$$

$$\begin{aligned}P^m_d &= \int_\vartheta \vec{H}\cdot\vec{J}^m_d\,d\vartheta = \int_\vartheta \vec{H}\cdot\frac{\Diamond}{\Diamond t}\vec{B}\,d\vartheta\\ &= \int_\vartheta \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\,d\vartheta - \int_\vartheta \vec{H}\cdot curl\left(\vec{\mathrm{v}}\times\vec{B}\right)\,d\vartheta\end{aligned} \tag{5.8d}$$

$$P_C = \int_\vartheta \vec{E}\cdot\vec{J}_C\,d\vartheta \tag{5.8e}$$

In a simple medium the first integrals at the r.h.s. of (5.8c,d) can also be written as

$$\begin{aligned}\int_{\vartheta} \vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\,d\vartheta &= \int_{\vartheta} \varepsilon\vec{E}\cdot\frac{\partial}{\partial t}\vec{E}\,d\vartheta = \int_{\vartheta}\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\vec{E}^2\right)d\vartheta \\ &= \frac{d}{dt}\int_{\vartheta}\frac{1}{2}\varepsilon\vec{E}^2\,d\vartheta - \oint_{\partial\vartheta}\frac{1}{2}\varepsilon\vec{E}^2\,\vec{\mathrm{v}}\cdot d\vec{S}\end{aligned} \tag{5.9a}$$

$$\begin{aligned}\int_{\vartheta} \vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\,d\vartheta &= \int_{\vartheta} \mu\vec{H}\cdot\frac{\partial}{\partial t}\vec{H}\,d\vartheta = \int_{\vartheta}\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\vec{H}^2\right)d\vartheta \\ &= \frac{d}{dt}\int_{\vartheta}\frac{1}{2}\mu\vec{H}^2\,d\vartheta - \oint_{\partial\vartheta}\frac{1}{2}\mu\vec{H}^2\,\vec{\mathrm{v}}\cdot d\vec{S}\end{aligned} \tag{5.9b}$$

In the final steps of calculation in (5.9) we employed the scalar Reynolds theorem.

The Lorentz force law in E-frame takes the form

$$\vec{f}(\vec{r};t) = \frac{d\vec{F}}{d\vartheta}(\vec{r};t) = \rho_f(\vec{r};t)\,\vec{E}(\vec{r};t) + \vec{J}_C(\vec{r};t)\times\vec{B}(\vec{r};t). \tag{5.10}$$

Substituting (5.3) and (5.4) into (5.10) gives

$$\vec{f}(\vec{r};t) = \frac{d\vec{F}}{d\vartheta}(\vec{r};t) = \rho_f(\vec{r};t)\,\vec{E}_{Max}(\vec{r};t) + \vec{J}_f(\vec{r};t)\times\vec{B}(\vec{r};t). \tag{5.11}$$

The resultant expression (5.11) is the map of the Lorentz force law (4.9) of stationary media, regardless of the choice of Lorentz gauge. Detailed discussion around (5.11) can be found at [24], Ch.5).

The map of the electrostatic and magnetostatic field equations of stationary media in (4.10) and (4.11) into E-frame can be written respectively as

$$curl\vec{E}(\vec{r};t) = \vec{0},\ \frac{\diamondsuit}{\diamondsuit t}\vec{D}(\vec{r};t) = \vec{0},\ div\vec{D}(\vec{r};t) = \rho_f(\vec{r};t) \tag{5.12a-c}$$

$$\vec{E}(\vec{r};t) = -gradV(\vec{r};t),\ \frac{\diamondsuit}{\diamondsuit t}\rho_f(\vec{r};t) = 0 \tag{5.12d,e}$$

$$lapV(\vec{r};t) = -\left(1/\varepsilon\right)\rho_f(\vec{r};t) \qquad \text{(in a simple medium)} \tag{5.12f}$$

$$\vec{f}(\vec{r};t) = \frac{d\vec{F}}{d\vartheta}(\vec{r};t) = \rho_f(\vec{r};t)\,\vec{E}(\vec{r};t) \tag{5.12g}$$

and

$$\frac{\diamondsuit}{\diamondsuit t}\vec{B}(\vec{r};t) = \vec{0},\ curl\vec{H}(\vec{r};t) = \vec{J}_C(\vec{r};t), div\vec{B}(\vec{r};t) = 0 \tag{5.13a-c}$$

$$div\vec{J}_C(\vec{r};t) = 0,\ \vec{B}(\vec{r};t) = curl\vec{A}(\vec{r};t) \tag{5.13d,e}$$

$$lap\vec{A}(\vec{r};t) = -\mu\vec{J}_C(\vec{r};t) \qquad \text{(in a simple medium)} \tag{5.13f}$$

$$\vec{f}(\vec{r};t) = \frac{d\vec{F}}{d\vartheta}(\vec{r};t) = \vec{J}_C(\vec{r};t)\times\vec{B}(\vec{r};t) \tag{5.13g}$$

It should be emphasized that while the electrostatic field quantities in L-frame are observed as time dependent in E-frame as in (5.12), this does not imply a presence of an additional magnetic field. What happens is that the field lines follow the arbitrary motion of the source as a whole, *without* any deformation in shape. Therefore it should not be mixed with any type of radiation mechanism specific to time varying sources where the field lines actually change their shape in L-frame. In that regard HE puts it very clearly that "*stationary (time independent) sources with arbitrary velocity do not radiate*". Similar considerations hold for magnetostatic media in (5.13).

Based on the balance laws, the transformations for Lorentz potentials of a rigid medium read

$$V' = V,\ \frac{\partial V'}{\partial t} = \frac{\Diamond V}{\Diamond t},\ grad' V' = \bar{\bar{Q}} \cdot gradV,\ lap' V' = lapV \tag{5.14a}$$

$$\vec{A}' = \bar{\bar{Q}} \cdot \vec{A},\ \frac{\partial \vec{A}'}{\partial t} = \bar{\bar{Q}} \cdot \frac{\Diamond \vec{A}}{\Diamond t},\ div' \vec{A}' = div\vec{A}, lap' \vec{A}' = \bar{\bar{Q}} \cdot lap\vec{A} \tag{5.14b}$$

Next we shall seek the wave equations and Lorentz potentials in simple media for the two special cases of Euclidean motion summarized in Table 1.

6. Hertzian Wave Equations and Lorentz Potentials in Simple Media

6.1. **Special Case of Translational Motion.** In this case it is sufficient to replace the partial time derivative $\frac{\partial}{\partial t}$ in the Maxwell equations in L-frame with convective derivative as $\frac{\Diamond}{\Diamond t} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathrm{v}}(t) \cdot grad$, which shapes (3.7a,b) and (3.8) into

$$curl\vec{E}(\vec{r};t) + \frac{D}{Dt}\vec{B}(\vec{r};t) = \vec{0},\ curl\vec{H}(\vec{r};t) - \frac{D}{Dt}\vec{D}(\vec{r};t) = \vec{J}_C(\vec{r};t) \tag{6.1a,b}$$

$$div\vec{J}_C(\vec{r};t) + \frac{D}{Dt}\rho_f(\vec{r};t) = 0 \tag{6.2}$$

In virtue of the commutative properties of the convective derivative, the wave equations for fields and Lorentz potentials satisfy

$$L_D\vec{E}(\vec{r};t) = (1/\varepsilon)\, grad\rho_f(\vec{r};t),\ L_D\vec{H}(\vec{r};t) = \vec{0} \tag{6.3a,b}$$

$$L_D\vec{A}(\vec{r};t) = \vec{0},\ L_D V(\vec{r};t) = -\,(1/\varepsilon)\,\rho_f(\vec{r};t) \tag{6.3c,d}$$

$$div\vec{A}(\vec{r};t) + \varepsilon\mu\frac{D}{Dt}V(\vec{r};t) + \sigma\mu V(\vec{r};t) = 0 \tag{6.3e}$$

$$\vec{E} = \vec{\mathrm{v}}(t) \times curl\vec{A} - \frac{\partial}{\partial t}\vec{A} - gradV = grad\left(\vec{\mathrm{v}}(t)\cdot\vec{A}\right) - \frac{D}{Dt}\vec{A} - gradV \tag{6.3f}$$

where

$$L_D = lap - \varepsilon\mu\frac{D^2}{Dt^2} - \sigma\mu\frac{D}{Dt} \tag{6.3g}$$

is the "convective wave operator". To understand the nature of (6.3g) let us consider the special case of R_1 where

$$\vec{\mathrm{v}}(t) = \hat{x}_1\mathrm{v}(t),\ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathrm{v}(t)\frac{\partial}{\partial x_1} \tag{6.4a,b}$$

In this case each field component satisfies the scalar convective wave operator

$$L_D = \left[1 - \varepsilon\mu v^2(t)\right]\frac{\partial^2}{\partial x_1^2} + 2\varepsilon\mu v(t)\frac{\partial^2}{\partial x_1\partial t} - \varepsilon\mu\frac{\partial^2}{\partial t^2} + \left[\sigma\mu\mathrm{v}(t) - \varepsilon\mu\mathrm{a}(t)\right]\frac{\partial}{\partial x_1} - \sigma\mu\frac{\partial}{\partial t} \tag{6.4c}$$

where $\mathrm{a}(t) = \frac{d\mathrm{v}}{dt}$ is the acceleration.

The discriminant of the partial differential operator in (6.4c) reads

$$\Delta = 4\varepsilon^2\mu^2\mathrm{v}^2(t) - 4\left[1-\varepsilon\mu\mathrm{v}^2(t)\right](-\varepsilon\mu) = 4\varepsilon\mu > 0 \tag{6.4d}$$

which altogether provide the following evidences:

(1) The discriminant values of the wave operators in L- and E-frames are the same (invariant). Therefore the vector operator L_D in E-frame is of hyperbolic type regardless of the instantaneous value of the velocity of the material points.
(2) $1-\varepsilon\mu\mathrm{v}^2(t)=0$, which also describes the speed of light in a simple medium, is a critical value for the velocity of material points in a simple medium, for which the wave propagation phenomenon *breaks down*.

It should be noticed that the investigation so far does not introduce any upper limit for the speeds of material points; meaning that $1-\varepsilon\mu\mathrm{v}^2(t)$ can also take negative values in (6.4c), which might address a possibility of speeds of material points faster than the speed of light in the same simple medium.

6.2. **Special Case of Rotational Motion.** Via the commutative properties of the co-moving time derivative provided in Theorem A.4, the wave equations and Lorentz potentials for the fields in a simple medium in rotational motion can be written directly as

$$L_\diamondsuit\vec{E}(\vec{r};t) = (1/\varepsilon)\, grad\rho_f(\vec{r};t),\ L_\diamondsuit\vec{H}(\vec{r};t) = \vec{0} \tag{6.5a,b}$$

$$L_\diamondsuit\vec{A}(\vec{r};t) = \vec{0},\ L_\diamondsuit V(\vec{r};t) = L_D V(\vec{r};t) = -\,(1/\varepsilon)\,\rho_f(\vec{r};t) \tag{6.5c,d}$$

$$div\vec{A}(\vec{r};t) + \varepsilon\mu\frac{D}{Dt}V(\vec{r};t) + \sigma\mu V(\vec{r};t) = 0 \tag{6.5e}$$

$$\vec{E} = \vec{\mathrm{v}}\times curl\vec{A} - \frac{\partial}{\partial t}\vec{A} - gradV = grad\left(\vec{\mathrm{v}}\cdot\vec{A}\right) - \frac{\diamondsuit}{\diamondsuit t}\vec{A} - gradV \tag{6.5f}$$

where

$$L_\diamondsuit = lap - \varepsilon\mu\frac{\diamondsuit^2}{\diamondsuit t^2} - \sigma\mu\frac{\diamondsuit}{\diamondsuit t} \tag{6.5g}$$

is the corresponding "progressive wave operator".

In the special case of R_1 where $\frac{\partial}{\partial\rho}\equiv 0$, $\frac{\partial}{\partial z}\equiv 0$, each field component satisfies the reduced form

$$L_\diamondsuit = \left(\frac{1}{\rho^2} - \varepsilon\mu\omega^2(t)\right)\frac{\partial^2}{\partial\phi^2} + 2\varepsilon\mu\omega(t)\frac{\partial^2}{\partial\phi\partial t} - \varepsilon\mu\frac{\partial^2}{\partial t^2} + \text{lower order terms} \tag{6.6a}$$

of the progressive wave operator for which the discriminant reads

$$\Delta = 4\varepsilon\mu/\rho^2 > 0 \tag{6.6b}$$

and therefore similar physical arguments as for translational motion hold.

7. Invariance of Wavenumber and Reduced Field Equations

Let us consider the field equations of an arbitrarily moving medium given by (3.3) and (3.7). In particular let us confine our interest to monochromatic waves, where the solution (or representation) of any scalar/vector field or source quantity is given by $u'(\vec{r}';t) = g'(\vec{k}'\cdot\vec{r}'-\omega' t)$ and $u(\vec{r};t) = g(\vec{k}\cdot\vec{r}-\omega t)$ in L- and E-frames, respectively. These physical quantities are connected by the field transformations (3.10), which necessitate phase invariance expressed by

$$\vec{k}'\cdot d\vec{r}' - \omega' dt = \vec{k}\cdot d\vec{r} - \omega dt = 0 \tag{7.1}$$

for arbitrary observation point and time. When rigid bodies are involved, this condition can be met *iff*

(1) $\vec{k}=\vec{k}',\ \omega=\omega'+\vec{k}\cdot\vec{\mathrm{v}}$ for uniform translational motion with $d\vec{r}=d\vec{r}'+\vec{\mathrm{v}}dt$

(2) $\vec{k}=\bar{\bar{Q}}^T\cdot\vec{k}',\ \omega=\omega'$ for uniform rotational motion with $d\vec{r}=\bar{\bar{Q}}^T\cdot d\vec{r}'$.

In each case it is seen that one observes the *same* wavenumber ($k=k'$) in L- and E-frames, while for uniform translational motion one also observes a first order Doppler effect.

Let us express the scalar/vector field quantities in E-frame in phasor form as

$$u(\vec{r};t) = Re\left\{u(\vec{r})e^{-i\omega t}\right\},\ u(\vec{r}) = U_0 e^{i\vec{k}\cdot\vec{r}}. \tag{7.2}$$

One can easily check the relation

$$\frac{\Diamond}{\Diamond t}u(\vec{r};t) = Re\left\{\frac{\Diamond}{\Diamond t}\left(u(\vec{r})e^{-i\omega t}\right)\right\} = Re\left\{-i\omega' u(\vec{r})e^{-i\omega t}\right\},$$

for uniform translational and rotational motions respectively as follows: Set $U(\vec{r};t)\equiv u(\vec{r})e^{-i\omega t}$ for brevity. For uniform translational motion one can write

$$\begin{aligned}\frac{\Diamond}{\Diamond t}U(\vec{r};t) &= \left(\frac{\partial}{\partial t}+\vec{\mathrm{v}}\cdot grad\right)U(\vec{r};t) = \left(-i\omega+\vec{\mathrm{v}}\cdot(i\vec{k})\right)U(\vec{r};t)\\ &= -i\left(\omega-\vec{\mathrm{v}}\cdot\vec{k}\right)U(\vec{r};t) = -i\omega' U(\vec{r};t),\end{aligned}$$

while for uniform rotational motion one has

$$\begin{aligned}\frac{\Diamond}{\Diamond t}U(\vec{r};t) &= \left(\frac{\partial}{\partial t}+\vec{\mathrm{v}}\cdot grad\right)U(\vec{r};t) = \left(-i\omega+i\frac{\partial\vec{k}}{\partial t}\cdot\vec{r}+i\vec{\mathrm{v}}\cdot\vec{k}\right)U(\vec{r};t)\\ &= \left(-i\omega+i\frac{\partial}{\partial t}(\vec{k}\cdot\vec{r})\right)U(\vec{r};t)\\ &= \left(-i\omega+i\frac{\partial}{\partial t}(\vec{k}'\cdot\vec{r}')\right)U(\vec{r};t) = -i\omega U(\vec{r};t) = -i\omega' U(\vec{r};t)\end{aligned}$$

where we incorporated $\vec{k}\cdot\vec{r} = (\bar{\bar{Q}}^T\cdot\vec{k}')\cdot(\bar{\bar{Q}}^T\cdot\vec{r}') = (\vec{k}'\cdot\bar{\bar{Q}})\cdot(\bar{\bar{Q}}^T\cdot\vec{r}') = \vec{k}'\cdot(\bar{\bar{Q}}\cdot\bar{\bar{Q}}^T)\cdot\vec{r}' = \vec{k}'\cdot\vec{r}'$. As a result one reaches the reduced field equations

$$curl\vec{E}(\vec{r}) - i\omega'\vec{B}(\vec{r}) = \vec{0},\ curl\vec{H}(\vec{r}) + i\omega'\vec{D}(\vec{r}) = \vec{J}_C(\vec{r}) \tag{7.3a,b}$$

$$div\vec{D}(\vec{r}) = \rho_f(\vec{r}),\ div\vec{B}(\vec{r}) = 0 \tag{7.3c,d}$$

$$div\vec{J}_C(\vec{r}) - i\omega'\rho_f(\vec{r}) = 0 \tag{7.4}$$

as well as Helmholtz equations of the form

$$\left(lap+k^2\right)u(\vec{r}) = f(\vec{r}) \tag{7.5}$$

in E-frame, which are similar to the corresponding reduced field equations L-frame. The field equations (7.3)-(7.5) also apply for *arbitrary* waveforms which can be expressed as superpositions of sinusoids via Fourier series or integral representations.

8. Concluding Remarks

In this work we reviewed the mathematical foundation, axiomatic structure and principles of Hertzian Electrodynamics for moving bodies, also introducing a commutative property of the comoving time derivative operator which provides the Hertzian wave equations for material bodies in rotational motion. A similar investigation of Hertzian wave equations for material media in non-Euclidean motion characterizing expansion or contraction mechanisms with specific applications in electromagnetic theory, are left as the subject of a separate work. The prediction of HE for canonical scattering problems of practical interest are examined in the sequel works [45], [46]. To conclude, recent research ([47]-[49]) has revealed that it is also possible to postulate a direct correspondence between the laws of HE and continuum mechanics, which might be of interest in the context of unification of the frame indifferent laws of continuum physics.

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Appendix A. The Comoving Time Derivative

In this appendix we provide a short review of the limit description and various differential properties of the comoving time derivative operator for scalar and vector fields as used in the text.

Theorem A.1. *Consider an arbitrary scalar or vector field quantity $g(\vec{r};t)$ in an arbitrary material medium traveling with an instantaneous velocity field $\vec{\mathrm{v}}(\vec{r};t)$ w.r.t. E-frame in R_3. The time rate of change experienced in L-frame is called the 'convective' (aka substantial, total time, Euler's material) derivative, and has the form*

$$\frac{D}{Dt}g(\vec{r};t) = \frac{\partial}{\partial t}g(\vec{r};t) + \vec{\mathrm{v}}(\vec{r};t)\cdot grad g(\vec{r};t) \tag{A.1}$$

In particular, setting $g(\vec{r};t) = \vec{\mathrm{v}}(\vec{r};t)$ yields the expression of the acceleration field $\vec{\mathrm{a}} = \frac{D\vec{\mathrm{v}}}{Dt}$.

Definition A.1. *The comoving time derivative of an arbitrary (smooth enough) scalar or vector density field $g(\vec{r};t)$ as observed in E-frame is described by*

$$\frac{\Diamond}{\Diamond t}g(\vec{r};t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[g'(\vec{r}';t+\Delta t) - g'(\vec{r}';t)\right] \tag{A.2}$$

with the assumption $g(\vec{r};t) = g'(\vec{r}';t)$ at time t.

The comoving time derivative is also known as 'the upper convected material derivative' or 'Oldroyd derivative' in continuum mechanics when tensor density fields are concerned and is the only member of a family of invariant time derivatives (cf.[50]) that correctly postulates field equations not only in continuum mechanics but also in HE of moving bodies. The Oldroyd derivative was introduced in [51] for establishing invariant forms of rheological equations of state for a homogeneous continuum, suitable for application to all conditions of motion and stress, particularly when the frame of reference is a coordinate system *convected with the material.* In that sense the comoving time derivative of scalar/vector density fields can be interpreted as the Oldroyd derivative of a tensor of rank zero/one. In the context of electrical engineering we prefer the terminology 'comoving time derivative' (cf.[52]) to 'Oldroyd derivative' since the latter is rather established in continuum mechanics and essentially related with tensor quantities. For more information and geometrical interpretations of the comoving time derivative one may refer to [53].

A.1. A Proof of the Representation of Scalar and Vector Comoving Time Derivative Operators.

A derivation of (2.5) based on Definition A.1 can be found in [54] and in sufficient detail in ([55], Sec.4.4). In this section we will provide the proofs of both (2.4) and (2.5) in a unified manner with our own notation.

Consider a scalar or vector density field whose volume integral in an arbitrarily moving material medium describes a field quantity (such as total charge or mass), the physical nature and quantity of which is assumed the same for all observers in Euclidean space. To be specific let $D(t)$, $D'(t)$ and g, g' be the representations of the same moving material medium and the density field in E- and L-frames at time t, respectively. Furthermore, let the Cartesian E-coordinates $\{x_i\}$ coincide with the general curvilinear L-coordinates $\{x'_i\}$ at time t, which requires the field quantities and the differential volume elements to be the same at that instant:

$$d\vartheta = d\vartheta',\ g(\vec{r};t) = g'(\vec{r}';t) \quad (at\ \ time\ \ t) \tag{A.3}$$

Then, after an infinitesimal period Δt, the medium and the density fields are denoted by $D(t+\Delta t)$, $D'(t+\Delta t)$ and $g(\vec{r};t+\Delta t)$, $g'(\vec{r}';t+\Delta t)$ as depicted in Figure 2.

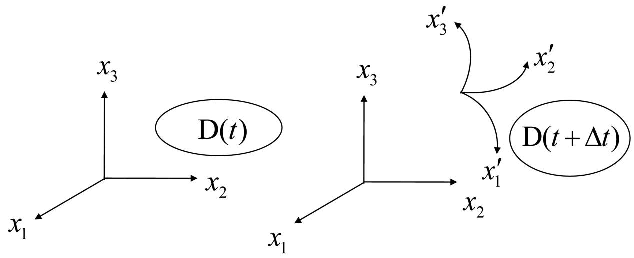

Figure 2. E- and L- frames of the moving material medium at times t and $t+\Delta t$

At time $t+\Delta t$ the coordinate transformations between the two systems can be given by first order as

$$x_j = x'_j{+}\mathrm{v}^j(x_i;t)\,\Delta t,\ x'_i = x_i - \mathrm{v}^i(x_j;t)\,\Delta t,\ i,j = 1,2,3 \tag{A.4}$$

Spatial partial differentiations in (A.4) read

$$\frac{\partial x_j}{\partial x'_i} = \delta_i^j + \Delta t\frac{\partial \mathrm{v}^j}{\partial x_i} + \mathrm{o}(\Delta t),\ \frac{\partial x'_i}{\partial x_j} = \delta_j^i - \Delta t\frac{\partial \mathrm{v}^i}{\partial x_j} + \mathrm{o}(\,\Delta t) \tag{A.5}$$

where $\delta_i^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ denotes the Kronecker delta. Since the medium is dynamic, we cannot talk about the validity of (A.3) also at time $t+\Delta t$. Instead, the only conclusive statement one can do at time $t+\Delta t$ is the invariance (or conservation) of the integral

$$\int_{D(t+\Delta t)} g(\vec{r};t+\Delta t)d\vartheta = \int_{D'(t+\Delta t)} g'(\vec{r}';t+\Delta t)d\vartheta' \tag{A.6}$$

An application of the property (A.6) can be found at ([56], Ch.2). At time $t+\Delta t$, the differential volume elements are connected by

$$d\vartheta = J\,d\vartheta' \tag{A.7}$$

where

$$J = det\left[\frac{\partial x_i}{\partial x'_j}\right] = 1 + \Delta t\frac{\partial \mathrm{v}^i}{\partial x_i} + \mathrm{o}(\Delta t) = 1 + (\Delta t)div\vec{\mathrm{v}} + \mathrm{o}(\Delta t) \tag{A.8}$$

denotes the nonzero Jacobian of the transformation matrix (aka the deformation gradient). Substituting (A.7) into (A.6) yields

$$\int_{D'(t+\Delta t)} g(\vec{r};t+\Delta t)J\,d\vartheta' = \int_{D'(t+\Delta t)} g'(\vec{r}';t+\Delta t)d\vartheta \tag{A.9}$$

which, due to postulate of localization, necessitates the transformation rule

$$g'(\vec{r}';t+\Delta t) = Jg(\vec{r};t+\Delta t) \tag{A.10}$$

We further consider the Taylor series expansion of $g\,(\vec{r};t+\Delta t)$ around time t as

$$g(\vec{r};t+\Delta t) = g(\vec{r};t) + \Delta t\frac{D}{Dt}g(\vec{r};t) + o(\Delta t) \tag{A.11}$$

When $g(\vec{r};t)$ is a scalar field, substituting (A.11) and (A.3) into (A.10) one gets

$$\begin{aligned} g'(\vec{r}';t+\Delta t)-g'(\vec{r}';t) &= (J-1)g'(\vec{r}';t)+J\,(\Delta t)\,\tfrac{D}{Dt}g(\vec{r};t)+\mathrm{o}(\Delta t) \\ &= (\Delta t)\left[(div\vec{\mathrm{v}})g(\vec{r};t)+\tfrac{D}{Dt}g(\vec{r};t)\right]+\mathrm{o}(\Delta t) \end{aligned} \tag{A.12}$$

and its comoving time derivative (A.2) can be obtained in virtue of (A.12) directly as (2.4).

When $g\,(\vec{r};t)$ is a (contravariant) vector field as $\vec{A}(\vec{r};t)$, the relation (A.10) can be written in terms of its contravariant components as

$$\begin{aligned} A^{i'}\,(\vec{r}';t+\Delta t) &= J\tfrac{\partial x'_i}{\partial x_j}A^j\,(\vec{r};t+\Delta t) \\ &= (1+(\Delta t)div\vec{\mathrm{v}}+o(\Delta t))\left(\delta^i_j-\Delta t\tfrac{\partial \mathrm{v}^i}{\partial x_j}+\mathrm{o}(\,\Delta t)\right)A^j\,(\vec{r};t+\Delta t) \\ &= A^i\,(\vec{r};t+\Delta t)+(\Delta t)\left[(div\vec{\mathrm{v}})\,A^i\,(\vec{r};t)-\tfrac{\partial \mathrm{v}^i}{\partial x_j}A^j\,(\vec{r};t)\right]+\mathrm{o}(\Delta t) \\ &= A^i\,(\vec{r};t)+(\Delta t)\left[\tfrac{D}{Dt}A^i(\vec{r};t)+(div\vec{\mathrm{v}})\,A^i(\vec{r};t)-\left(gradv^i\right)\cdot\vec{A}(\vec{r};t)\right]+\mathrm{o}(\Delta t) \end{aligned} \tag{A.13}$$

The relation (A.13) provides the connection between the contravariant components of $\vec{A}$ at times t and $t+\Delta t$. Multiplying each side by the unit vectors $\hat{x}'_i$ and $\hat{x}_i$ and using (A.3), it can be arranged as

$$\vec{A}'\left(\vec{r}';t+\Delta t\right)-\vec{A}'\left(\vec{r}';t\right)=(\Delta t)\left[\frac{D}{Dt}\vec{A}\,(\vec{r};t)+(div\vec{\mathrm{v}})\,\vec{A}\,(\vec{r};t)-\vec{A}\,(\vec{r};t)\cdot grad\vec{\mathrm{v}}\right]+\mathrm{o}(\Delta t) \tag{A.14}$$

Finally, (A.14) can be placed into (A.2) to get the desired relation (2.5).

A.2. **Certain Differential Properties of the Comoving Time Derivative.** For constant quantities c, $\vec{C}$ and the scalar/vector fields $f(\vec{r};t)$, $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$, $\vec{B}(\vec{r};t)$ of $C^1(R_3)$ one can observe the following properties.

Property 1: $\frac{\Diamond}{\Diamond t}c=c\,(div\vec{\mathrm{v}})$, $\frac{\Diamond}{\Diamond t}\vec{C}=-\vec{C}\cdot grad\vec{\mathrm{v}}+\vec{C}\,(div\vec{\mathrm{v}})=curl\left(\vec{C}\times\vec{\mathrm{v}}\right)$

Property 2: $\frac{\Diamond}{\Diamond t}(cf)=c\frac{\Diamond f}{\Diamond t}$, $\frac{\Diamond}{\Diamond t}\left(c\vec{A}\right)=c\frac{\Diamond\vec{A}}{\Diamond t}$

Property 3: $\frac{\Diamond}{\Diamond t}(f\pm g)=\frac{\Diamond f}{\Diamond t}\pm\frac{\Diamond g}{\Diamond t}$, $\frac{\Diamond}{\Diamond t}\left(\vec{A}\pm\vec{B}\right)=\frac{\Diamond\vec{A}}{\Diamond t}\pm\frac{\Diamond\vec{B}}{\Diamond t}$

Property 4: $\frac{\Diamond}{\Diamond t}(fg)=\frac{\Diamond f}{\Diamond t}g+f\frac{\Diamond g}{\Diamond t}-fg\,(div\vec{\mathrm{v}})$

Property 5:

$$\begin{aligned} \frac{\Diamond}{\Diamond t}\left(f\vec{A}\right) &= \frac{\partial}{\partial t}\left(f\vec{A}\right)+\vec{\mathrm{v}}\cdot grad\left(f\vec{A}\right)-f\vec{A}\cdot grad\vec{\mathrm{v}}+f\vec{A}\,(div\vec{\mathrm{v}}) \\ &= \left[\frac{\partial}{\partial t}f+\vec{\mathrm{v}}\cdot gradf+f\,(div\vec{\mathrm{v}})\right]\vec{A}+f\left[\frac{\partial}{\partial t}\vec{A}+\vec{\mathrm{v}}\cdot grad\vec{A}-\vec{A}\cdot grad\vec{\mathrm{v}}+\vec{A}\,(div\vec{\mathrm{v}})\right] \\ &\quad -f\vec{A}\,(div\vec{\mathrm{v}}) \\ &= \frac{\Diamond f}{\Diamond t}\vec{A}+f\frac{\Diamond\vec{A}}{\Diamond t}-f\vec{A}\,(div\vec{\mathrm{v}}) \end{aligned}$$

Property 6: $\frac{\Diamond}{\Diamond t}\left(\vec{A}\cdot\vec{B}\right)=\frac{D}{Dt}\left(\vec{A}\cdot\vec{B}\right)+\vec{A}\cdot\vec{B}\,(div\vec{\mathrm{v}})$

Property 7:

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\left(\vec{A}\times\vec{B}\right) &= \frac{D\vec{A}}{Dt}\times\vec{B}+\vec{A}\times\frac{D\vec{B}}{Dt}+\vec{A}\times\vec{B}\,(div\,\vec{\mathrm{v}})-\left(\vec{A}\times\vec{B}\right)\cdot grad\vec{\mathrm{v}}\\ &= \frac{\Diamond\vec{A}}{\Diamond t}\times\vec{B}+\vec{A}\times\frac{\Diamond\vec{B}}{\Diamond t}-\vec{A}\times\vec{B}\,(div\,\vec{\mathrm{v}})-\left(\vec{A}\times\vec{B}\right)\cdot grad\vec{\mathrm{v}}\\ &\quad+\vec{A}\times\left(\vec{B}\cdot grad\vec{\mathrm{v}}\right)\quad+\left(\vec{A}\cdot grad\vec{\mathrm{v}}\right)\times\vec{B}\end{aligned}$$

Property 8*:* For $n\in\mathbb{R}$

$$\frac{\Diamond}{\Diamond t}f^n = nf^{n-1}\frac{\Diamond f}{\Diamond t}-(n-1)f^n\,(div\,\vec{\mathrm{v}}) = nf^{n-1}\frac{Df}{Dt}+f^n\,(div\,\vec{\mathrm{v}}) = f^{n-1}\left[\frac{\Diamond f}{\Diamond t}+(n-1)\frac{Df}{Dt}\right]$$

A.3. **Certain Commutative Properties of the Comoving Time Derivative.** For the purpose of deriving the progressive wave equations in Section 6 we provide below three theorems investigating certain commutative properties between the comoving time and spatial (nabla) derivative operators

Theorem A.2. *In an arbitrarily moving material medium a density field vector $\vec{A}(\vec{r};t)$ of $C^2(R_3)$ provides the commutative property*

$$div\left(\frac{\Diamond}{\Diamond t}\vec{A}\right)=\frac{\Diamond}{\Diamond t}\left(div\vec{A}\right) \tag{A.15}$$

Proof. The proof requires demonstration of the equality

$$div\left(\vec{\mathrm{v}}\cdot grad\vec{A}-\vec{A}\cdot grad\vec{\mathrm{v}}+\vec{A}div\,\vec{\mathrm{v}}\right)=\vec{\mathrm{v}}\cdot grad\left(div\vec{A}\right)+\left(div\vec{A}\right)(div\,\vec{\mathrm{v}}) \tag{A.16}$$

For this purpose we shall introduce the following tensor identities cf.[57], Ch. 7

$$\vec{A}\cdot\bar{\bar{\Phi}}=\bar{\bar{\Phi}}^T\cdot\vec{A},\ div\left(\bar{\bar{\Phi}}\cdot\vec{A}\right)=\left(div\bar{\bar{\Phi}}\right)\cdot\vec{A}+\bar{\bar{\Phi}}:grad\vec{A} \tag{A.17}$$

where $\vec{A}$ is a vector, $\bar{\bar{\Phi}}$ is a tensor of rank two (a dyad), and ':' is the tensor inner product defined as $\vec{A}:\vec{B}=A_{ij}B_{ij}$. From (A.17) one can write

$$div\left(\vec{A}\cdot\bar{\bar{\Phi}}\right)=div\left(\bar{\bar{\Phi}}^T\cdot\vec{A}\right)=\left(div\bar{\bar{\Phi}}^T\right)\cdot\vec{A}+\bar{\bar{\Phi}}^T:grad\vec{A}$$

and use this property to calculate

$$\begin{aligned}div\left(\vec{\mathrm{v}}\cdot grad\vec{A}\right)&=div\left(\left(grad\vec{A}\right)^T\cdot\vec{\mathrm{v}}\right)=\left[div\left(grad\vec{A}\right)^T\right]\cdot\vec{\mathrm{v}}+\left(grad\vec{A}\right)^T:grad\vec{\mathrm{v}}\\&=\vec{\mathrm{v}}\cdot grad\left(div\vec{A}\right)+grad\vec{\mathrm{v}}:\left(grad\vec{A}\right)^T\end{aligned}$$

$$\begin{aligned}div\left(\vec{A}\cdot grad\vec{\mathrm{v}}\right)&=div\left((grad\vec{\mathrm{v}})^T\cdot\vec{A}\right)=\left[div\,(grad\vec{\mathrm{v}})^T\right]\cdot\vec{A}+(grad\vec{\mathrm{v}})^T:grad\vec{A}\\&=\vec{A}\cdot grad\,(div\vec{\mathrm{v}})+grad\vec{A}:(grad\vec{\mathrm{v}})^T\end{aligned}$$

We also have the tensor properties

$$grad\vec{\mathrm{v}}:\left(grad\vec{A}\right)^T=grad\vec{A}:(grad\vec{\mathrm{v}})^T$$

$$div\left(\vec{A}div\,\vec{\mathrm{v}}\right)=\left(div\vec{A}\right)(div\,\vec{\mathrm{v}})+[grad\,(div\,\vec{\mathrm{v}})]\cdot\vec{A}$$

which altogether verify the desired equality (A.16) upon a direct substitution. □

A similar proof is available in the investigation in [58], Sec. 3.1 for the Maxwell - Cattaneo wave equation in heat conduction.

From Theorem A.2 we observe that the commutative property between the comoving time derivative and divergence operators applies regardless of the type of motion.

Theorem A.3.

$$\begin{aligned}
curl\left(\frac{\diamondsuit}{\diamondsuit t}\vec{A}\right) &= \frac{\partial}{\partial t}curl\vec{A} + \vec{\mathrm{v}}\cdot grad\left(curl\vec{A}\right) + \left(curl\vec{A}\right)(div\,\vec{\mathrm{v}}) + [grad\,(div\,\vec{\mathrm{v}})]\times\vec{A} \\
&\quad - \vec{A}\cdot grad\,(curl\vec{\mathrm{v}}) + (grad\vec{\mathrm{v}}) \underset{\cdot}{\times} \left(grad\vec{A}\right)^T - \left(grad\vec{A}\right) \underset{\cdot}{\times} (grad\vec{\mathrm{v}})^T \\
&= \frac{\diamondsuit}{\diamondsuit t}\left(curl\vec{A}\right) - \vec{A}\cdot grad\,(curl\vec{\mathrm{v}}) - \left(curl\vec{A}\right)\cdot(grad\vec{\mathrm{v}}) \\
&\quad + (grad\vec{\mathrm{v}}) \underset{\cdot}{\times} \left(grad\vec{A}\right)^T \quad - \left(grad\vec{A}\right) \underset{\cdot}{\times} (grad\vec{\mathrm{v}})^T
\end{aligned} \tag{A.18}$$

where $\underset{\cdot}{\times}$ *stands for the cross-dot product in dyadic algebra defined by*

$$\left(\vec{a}\vec{b}\right) \underset{\cdot}{\times} \left(\vec{c}\vec{d}\right) = (\vec{a}\times\vec{c})\left(\vec{b}\cdot\vec{d}\right),$$

which also satisfies

$$(grad\vec{a}) \underset{\cdot}{\times} \left(grad\vec{b}\right) = [grad\,(a_i)\times grad\,(b_j)]\,(\hat{x}_i\cdot\hat{x}_j)\,.$$

Proof. For our purpose let us consider the tensor identity

$$curl\left(\bar{\bar{\Phi}}\cdot\vec{A}\right) = \left(curl\bar{\bar{\Phi}}\right)\cdot\vec{A} + grad\vec{A} \underset{\cdot}{\times} \bar{\bar{\Phi}}.$$

Accordingly, one can write

$$\begin{aligned}
curl\left(\vec{\mathrm{v}}\cdot grad\vec{A}\right) &= curl\left(\left(grad\vec{A}\right)^T\cdot\vec{\mathrm{v}}\right) \\
&= \left[curl\left(grad\vec{A}\right)^T\right]\cdot\vec{\mathrm{v}} + (grad\vec{\mathrm{v}}) \underset{\cdot}{\times} \left(grad\vec{A}\right)^T \\
&= \vec{\mathrm{v}}\cdot grad\left(curl\vec{A}\right) + (grad\vec{\mathrm{v}}) \underset{\cdot}{\times} \left(grad\vec{A}\right)^T \\
curl\left(\vec{A}\cdot grad\vec{\mathrm{v}}\right) &= curl\left[(grad\vec{\mathrm{v}})^T\cdot\vec{A}\right] = \left[curl\,(grad\vec{\mathrm{v}})^T\right]\cdot\vec{A} + \left(grad\vec{A}\right) \underset{\cdot}{\times} (grad\vec{\mathrm{v}})^T \\
&= \vec{A}\cdot grad\,(curl\vec{\mathrm{v}}) + \left(grad\vec{A}\right) \underset{\cdot}{\times} (grad\vec{\mathrm{v}})^T
\end{aligned}$$

and also invoke the property

$$curl\left(\vec{A}div\,\vec{\mathrm{v}}\right) = \left(curl\vec{A}\right)(div\,\vec{\mathrm{v}}) + [grad\,(div\,\vec{\mathrm{v}})]\times\vec{A},$$

which altogether yield the desired relation (A.18). (A.18) also signifies that there does not exist a simple commutative property between curl and the comoving time derivative operators similar in form to the case in Theorem A.2 for arbitrary velocity fields. The sophisticated structure of (A.18) for arbitrary velocity fields renders it impractical in obtaining Hertzian wave equations in the most general (non-Euclidean) case. However, for any rigid body in arbitrary Euclidean motion as in (3.18) one can obtain the desired simple commutative property as follows: □

Theorem A.4. *In an arbitrary material medium in Euclidean motion a density field vector $\vec{A}(\vec{r};t)$ of $C^2(R_3)$ provides the commutative property*

$$curl\left(\frac{\diamondsuit}{\diamondsuit t}\vec{A}\right)=\frac{\diamondsuit}{\diamondsuit t}\left(curl\vec{A}\right) \tag{A.19}$$

Proof. It satisfies to look into the two special cases of Euclidean motion described in Table 1 separately.

Special Case 1: Translational Motion

In this special case the comoving time derivative reduces into the classical convective derivative directly as

$$\frac{\diamondsuit}{\diamondsuit t}\vec{A}=\frac{D}{Dt}\vec{A}=\left(\frac{\partial}{\partial t}+\vec{\mathrm{v}}(t)\cdot grad\right)\vec{A}=\frac{\partial\vec{A}}{\partial t}+grad\left(\vec{\mathrm{v}}(t)\cdot\vec{A}\right)-\vec{\mathrm{v}}(t)\times curl\vec{A}$$

and the desired result (A.19) is seen directly upon setting the spatial derivatives of the velocity vector to zero. If we invoke the special case $\vec{\mathrm{v}}=\vec{\mathrm{v}}(t)$ and substitute $grad\vec{A}$ in place of $\vec{A}$ in the general vector identity

$$grad\left(\vec{\mathrm{v}}\cdot\vec{A}\right)=\vec{\mathrm{v}}\cdot grad\vec{A}+\vec{A}\cdot grad\vec{\mathrm{v}}+\vec{A}\times curl\vec{\mathrm{v}}+\vec{\mathrm{v}}\times curl\vec{A}$$

one gets

$$grad\left(\vec{\mathrm{v}}(t)\cdot grad\vec{A}\right)=\vec{\mathrm{v}}(t)\cdot grad\left(grad\vec{A}\right),$$

through which one obtains an additional commutative property

$$grad\left(\frac{\diamondsuit}{\diamondsuit t}\vec{A}\right)=\frac{\diamondsuit}{\diamondsuit t}\left(grad\vec{A}\right) \tag{A.20}$$

required in deriving the potential wave operators.

Special Case 2: Rotational Motion

In this special case the velocity vector has the differential properties given in Table 1, which also introduce

$$\left(grad\vec{\mathrm{v}}\right)^T=-grad\vec{\mathrm{v}},\ grad\left(div\,\vec{\mathrm{v}}\right)=\vec{0},\ grad\left(curl\vec{\mathrm{v}}\right)=\bar{\bar{0}}.$$

Then, for a general vector field quantity in the form

$$\vec{A}(\rho,\phi,z;t)=\hat{\rho}A_\rho(\rho,\phi,z;t)+\hat{\phi}A_\phi(\rho,\phi,z;t)+\hat{z}A_z(\rho,\phi,z;t),$$

one obtains

$$\begin{aligned}
\left(curl\vec{A}\right)\cdot\left(grad\vec{\mathrm{v}}\right)&=\left(grad\vec{\mathrm{v}}\right)^T\cdot\left(curl\vec{A}\right)=-\left(grad\vec{\mathrm{v}}\right)\cdot\left(curl\vec{A}\right)\\
(1/\omega)\left(grad\vec{\mathrm{v}}\right)\underset{\cdot}{\times}\left(grad\vec{A}\right)^T&=-\hat{\rho}\frac{\partial A_z}{\partial\rho}-\frac{\hat{\phi}}{\rho}\frac{\partial A_z}{\partial\phi}+\hat{z}\left(\frac{\partial A_\rho}{\partial\rho}+\frac{1}{\rho}\left(A_\rho+\frac{\partial A_\phi}{\partial\phi}\right)\right)\\
(1/\omega)\left(grad\vec{A}\right)\underset{\cdot}{\times}\left(grad\vec{\mathrm{v}}\right)^T&=-(1/\omega)\left(grad\vec{A}\right)\underset{\cdot}{\times}\left(grad\vec{\mathrm{v}}\right)\\
&=-\hat{\rho}\frac{\partial A_\rho}{\partial z}-\hat{\phi}\frac{\partial A_\phi}{\partial z}+\hat{z}\left[\frac{\partial A_\rho}{\partial\rho}+\frac{1}{\rho}\left(A_\rho+\frac{\partial A_\phi}{\partial\phi}\right)\right]\left(grad\vec{\mathrm{v}}\right)\\
&\quad\underset{\cdot}{\times}\left(grad\vec{A}\right)^T-\left(grad\vec{A}\right)\underset{\cdot}{\times}\left(grad\vec{\mathrm{v}}\right)^T\\
&=-\omega(t)\hat{z}\times curl\vec{A},
\end{aligned}$$

which read the desired result

$$
\begin{aligned}
curl\left(\frac{\Diamond}{\Diamond t}\vec{A}\right) &= \frac{\Diamond}{\Diamond t}\left(curl\vec{A}\right) - \left(curl\vec{A}\right)\cdot(grad\vec{\mathrm{v}}) + (grad\vec{\mathrm{v}}) \underset{\cdot}{\times} \left(grad\vec{A}\right)^T \\
&\quad - \left(grad\vec{A}\right) \underset{\cdot}{\times} (grad\vec{\mathrm{v}})^T \\
&= \frac{\Diamond}{\Diamond t}\left(curl\vec{A}\right) + [grad\vec{\mathrm{v}}\cdot - \omega(t)\hat{z}\times]\left(curl\vec{A}\right) \\
&= \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)\left(curl\vec{A}\right) = \frac{\Diamond}{\Diamond t}\left(curl\vec{A}\right)
\end{aligned}
$$

since

$$
\begin{aligned}
\frac{D}{Dt}\left(curl\vec{A}\right) &= \left(\frac{\partial}{\partial t} + \vec{\mathrm{v}}\cdot grad\right)\left(curl\vec{A}\right) = \left(\frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}\right)\left(curl\vec{A}\right) \\
&= \left(\frac{\Diamond}{\Diamond t} + \omega(t)\hat{z}\times\right)\left(curl\vec{A}\right).
\end{aligned}
$$

□

One also observes

$$
\begin{aligned}
\frac{\Diamond}{\Diamond t}\left(grad\vec{A}\right) &= grad\left(\left(\frac{\Diamond}{\Diamond t} + \omega(t)\hat{z}\times\right)\vec{A}\right) \\
&= grad\left(\frac{\Diamond}{\Diamond t}\vec{A}\right) - \omega(t)\left(grad\vec{A}\right)\times\hat{z} = grad\left(\frac{D}{Dt}\vec{A}\right)
\end{aligned}
\tag{A.21}
$$

Similarly, for a general scalar field $f = f(\rho,\phi,z;t)$ one gets

$$
\begin{aligned}
\frac{\Diamond}{\Diamond t}f &= \frac{D}{Dt}f = \left(\frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}\right)f, \\
\frac{\Diamond}{\Diamond t}(gradf) &= \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)(gradf) = grad\left(\frac{D}{Dt}f\right)
\end{aligned}
\tag{A.22}
$$

B. Polat, for a photograph and biography, see TWMS Journal of Applied and Engineering Mathematics, Volume 1, No.2, 2011.