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ON TRACKING OF SOLUTIONS OF PARABOLIC VARIATIONAL INEQUALITIES

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Abstract. The problem of constructing a feedback control algorithm for a parabolic variational inequality is considered. This algorithm should provide tracking a prescribed trajectory by a solution of the given inequality. Two solving algorithms, which are stable with respect to informational noises, are designed. The algorithms are based on the method of extremal shift, which is known in the theory of guaranteed control.

Keywords: Tracking, parabolic inequalities

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1. Introduction

In the present work, the problem of tracking a trajectory of a system with distributed parameters is discussed. The essence of this problem may be formulated in the following way. A motion of a system described by a parabolic variational inequality proceeds on a given time interval $T = [t_0, \vartheta]$, $\vartheta < +\infty$. The trajectory of the system $w(\cdot)$ (the solution of the inequality), depends on a time-varying control $v = v(\cdot)$. The phase states of the system are inaccurately measured at frequent enough time moments. It is required to organize a feedback control process for the variational inequality in such a way that it is possible to preserve given properties of the trajectory. The quality of the trajectory constructed is estimated by the distance from a given (prescribed, standard) trajectory $x(\cdot)$. The latter solution of a parabolic inequality is generated by some unknown input $u = u(\cdot)$. The basic work in this question can be treated as the problem of construction of a control $v = v(\cdot)$ providing the retention of the trajectory $w(\cdot)$ nearly $x(\cdot)$. This is the conceptual statement of the control problem under consideration.

In this paper, solving algorithms, which are stable with respect to informational noises and computational errors, are presented. The algorithms are based on the method of feedback control. They adaptively take into account inaccurate measurements of phase trajectories and are regularizing in the following sense: the more precise is incoming information, the better is the algorithm's output.

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2. Problem Statement

Let H and V be real Hilbert spaces, let V be a dense subspace of H and $V \subset H \subset V^*$ algebraically and topologically. We assume that $(\cdot,\cdot)$ stands for the inner product in H and $\langle\cdot,\cdot\rangle$ stands for the duality relation between V and V^*. We consider a system described by the parabolic variational inequality [1–3]

$$\langle \dot{w}(t) + Aw(t), x(t) - z\rangle + \varphi(w(t)) - \varphi(z) \leq (Bv(t) + f(t), x(t) - z) \tag{1}$$

$$\text{for a. a.} \quad t \in T \quad \text{and all} \quad z \in V, \quad w(t_0) = w_0.$$

Here $A : V \to V^*$ is a linear continuous ($A \in \mathcal{L}(V; V^*)$) and symmetrical operator satisfying (for some $c > 0$ and real ω_0) the coercitivity condition

$$\langle Aw, w\rangle + \omega_0 |w|_H^2 \geq c|w|_V^2 \quad \forall y \in V, \tag{2}$$

U is a Hilbert space, $f \in L_2(T; H)$ is a given function, $|\cdot|_H$, $|\cdot|_U$ and $|\cdot|_V$ stand for the norms in H, U and V, respectively, $B : U \to H$ is a linear continuous operator ($B \in \mathcal{L}(U; H)$), and $\varphi : V \to \overline{\mathbb{R}} = \{r \in \mathbb{R} : -\infty < r \leq +\infty\}$ is a lower semicontinuous convex function. Furthermore, without loss of generality, we assume that $\varphi(x) \geq 0 \ \ \forall x \in V$.

Let $w(t_0) = w_0 \in D(\varphi)$, where $D(\varphi) = \{w \in V\colon \varphi(w) < +\infty\}$. It is known that under such conditions, for any $v(\cdot) \in L_2(T; U)$, there exists a unique solution $w(\cdot) = w(\cdot; t_0, w_0, v(\cdot))$ of inequality (1) with the following properties [1, 2]: $w(\cdot) = w(\cdot; t_0, w_0, v(\cdot)) \in W(T) = W^{1,2}(T; H) \cap L_2(T; V)$, $w(t) \in D(\varphi_\alpha)$ $\forall t \in T$, $t \to \varphi_\alpha(w(t)) \in AC(T)$. Here the function $\varphi_\alpha(y) : H \to \overline{\mathbb{R}}$ is defined by

$$\varphi_\alpha(y) = \begin{cases} 1/2\langle Ay, y\rangle + \omega_0/2|y|_H^2 + \varphi(y), & \text{if} \quad y \in D(\varphi) \\ +\infty, & \text{otherwise,} \end{cases}$$

$W^{1,2}(T; H) = \{w(\cdot) \in L_2(T; H) : \dot{w}(\cdot) \in L_2(T; H)\}$, the derivative $\dot{w}(\cdot)$ is understood in the sense of distributions, $AC(T)$ is the set of absolutely continuous functions $x(\cdot) : T \to \mathbb{R}$.

Assume that along with inequality (1) we have another inequality of the same form:

$$\langle \dot{x}(t) + Ax(t), x(t) - z\rangle + \varphi(x(t)) - \varphi(z) \leq (Bu(t) + f(t), x(t) - z), \tag{3}$$

$$\text{a.a.} \quad t \in T \quad \text{and all} \quad z \in V$$

with an initial state $x(t_0) = x_0 \in D(\varphi)$. This inequality (in what follows, we call it etalon) is subject to the action of some etalon control $u(\cdot) \in L_2(T; U)$. The etalon control as well as the corresponding solution $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ of inequality (3) are a priori unknown. At discrete, frequent enough, time moments

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m \quad (\tau_0 = t_0,\ \tau_m = \vartheta,\ \tau_{i+1} = \tau_i + \delta),$$

the states $w(\tau_i) = w(\tau_i; t_0, w_0, v(\cdot))$ of inequality (1) as well as the states $x(\tau_i) = x(\tau_i; t_0, x_0, u(\cdot))$ of etalon inequality (3) are measured. The states $w(\tau_i)$ are measured with an error. The results of measurements are elements $\xi_i^h \in H$ satisfying the inequalities

$$|w(\tau_i) - \xi_i^h|_H \leq h, \quad i \in [1 : m-1]. \tag{4}$$

By virtue of the embedding of the space $W^{1,2}(T; H)$ into the space $C(T; H)$, inequalities (4) make sense. Here, the value $h \in (0, 1)$ is the measurement accuracy. It is required to design an algorithm for forming the control $v = v^h(\cdot)$ in the inequality (1) allowing us to track the solution $x(\cdot)$ of inequality (3) by the solution $w(\cdot)$ of inequality (1). Thus, we consider the problem consisting in constructing an algorithm, which, using the current measurements of the values $w(\tau_i)$ and $x(\tau_i)$, forms in real time mode (by the feedback principle) the control $v = v^h(\cdot)$ in the right-hand part of inequality (1) such that the deviation of $w(\cdot) = w(\cdot; t_0, w_0, v^h(\cdot))$ from $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ in metric of the space $C(T; H) \cap L_2(T; V)$ is small if the measurement accuracy h is small enough. In the case when the etalon control u as well as the control v in inequality (1) are subject to instantaneous constraints ($u \in P$, $v \in P$, where $P \subset U$ is a given bounded and closed set), the problem above can be solved by means of the method of extremal shift [4]. Namely, if the control $v = v^h(\cdot)$ in the right-hand part of inequality (1) is calculated by the formula

$$v^h(t) = v(\tau_i, \xi_i^h, x(\tau_i)) = \arg\min\{(\xi_i^h - x(\tau_i), Bv) : v \in P\} \text{ for } t \in [\tau_i, \tau_{i+1}), \tag{5}$$

then, as it follows from [5, 6], for any $\varepsilon > 0$ one can find numbers $h_1 > 0$ and $\delta_1 > 0$ such that the inequality

$$\sup_{t \in T} |w(t; t_0, w_0, v^h(\cdot)) - x(t; t_0, x_0, u(\cdot))|_H \le \varepsilon$$

is fulfilled if $h \in (0, h_1)$ and $\delta \in (0, \delta_1)$. The last inequality is valid for any etalon control, i.e., for any Lebesgue measurable function $u(t) \in P$ for almost all $t \in T$.

Through out the paper, we assume that $w_0 \in D \subset V$, where D is a bounded set,

$$|w_0 - x_0|_H \le h. \tag{6}$$

Thus, the method of extremal shift allows us to solve the problem of tracking the solution of the etalon inequality under instantaneous constraints on the controls ($v, u \in P$). In the present paper, we consider the case when similar constraints are missing, i.e., any function from the space $L_2(T; U)$ can be the admissible control (both etalon, $u(\cdot)$, and real, $v(\cdot)$). No additional information on the functions $v(\cdot)$ and $u(\cdot)$ is required. We construct a corresponding modification of the method of extremal shift, using, according to [7, 8], the idea of its local regularization. Along with measuring the phase states at discrete time moments (see (4)), we also consider the case of "continuous" measuring of the states $x(t)$ and $w(t)$. Namely, it is assumed that, at every time $t \in T$, the phase states of inequalities (1) and (3) are measured; as a result, we have the functions $x(t)$ and $\xi^h(t) \in H$ with the properties

$$|\xi^h(t) - w(t)|_H \le h, \quad t \in T. \tag{7}$$

The functions $\xi^h(t)$, $t \in T$ are Lebesgue measurable.

3. Solving algorithm. The case of continuous measuring of phase states

First, we consider the case of "continuous" measurement of solutions of inequalities (1) and (3). In this case, inequalities (7) are valid. The problem consists in the following: it is necessary to design a rule forming (by the feedback principle) a control $v = v(\cdot, \xi^h(\cdot), w(\cdot))$ in the right-hand part of inequality (1) such that the deviation of $w(\cdot) = w(\cdot; t_0, w_0, v^h(\cdot))$ from $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$ in metric of the space $C(T; H) \cap L_2(T; V)$ is small if measurement accuracy h is sufficiently small.

Let a function $\alpha = \alpha(h) : (0, 1) \to (0, 1)$ be fixed. Let the control $v^{\alpha,h}(t)$ in inequality (1) be defined by the formula

$$v = v^{\alpha,h}(t) = \alpha^{-1} B^*(x(t) - \xi^h(t)). \tag{8}$$

Here B^* denotes the adjoint operator. Thus, we obtain the following pair of inequalities corresponding to system (1), (3):

$$\langle \dot{w}^{\alpha,h}(t) + Aw^{\alpha,h}(t), w^{\alpha,h}(t) - z\rangle + \varphi(w^{\alpha,h}(t)) - \varphi(z) \le \tag{9}$$

$$\le (Bv^{\alpha,h}(t) + f(t), w^{\alpha,h}(t) - z), \quad \text{a.a.} \quad t \in T \quad \text{and all} \quad z \in V;$$

and

$$\langle \dot{x}(t) + Ax(t), x(t) - z\rangle + \varphi(x(t)) - \varphi(z) \leq (Bu(t) + f(t), x(t) - z),$$
$$\text{a.a.} \quad t \in T \quad \text{and all} \quad z \in V,$$

subject to the initial conditions

$$x(t_0) = x_0, \quad w^{\alpha,h}(t_0) = w_0.$$

Here, we denote by $w^{\alpha,h}(\cdot)$ the solution of inequality (1) corresponding to function $v = v^{\alpha,h}(\cdot)$ of the form (8), i.e., the solution of inequality (9).

In what follows, we assume that $\omega > 0$. For simplicity, we set $\xi^h(t_0) = w_0$.

When set P coincides with the control space U, solving problem (5) becomes "pointless". If we "slightly correct" the functional to be minimized in (5) and, instead of the problem of minimizing the functional $l(s_i, v)$, where $l(s, v) = (s, Bv)$, $s_i = x(\tau_i) - \xi_i^h$, consider the problem of minimizing the quadratic functional $L_\alpha(s_i, v) = l(s_i, v) + \alpha|v|_U^2$ ($\alpha > 0$), then the last problem has a unique solution. It is easily seen that if in the functional L_α instead of s_i the value $x(t) - \xi^h(t)$ is used, then the solution of the last problem is exactly $v^{\alpha,h}(t)$. Then we have the following theorem.

Theorem 3.1. *Let $\alpha = \alpha(h) = h^{2/3}$. Then the following inequality*

$$|x(t) - w^{\alpha,h}(t)|_H^2 + 2c\int_{t_0}^{t} |x(\tau) - w^{\alpha,h}(\tau)|_V^2\, d\tau \leq d_0 h^{2/3}, \quad t \in T,$$

is fulfilled. Here $d_0 > 0$ is a constant that does not depend on $h \in (0, 1)$.

Proof. Due to (8), it holds that

$$|v^{\alpha,h}(t)|_U^2 \leq 2b^2\alpha^{-2}(h^2 + |\mu_{\alpha,h}(t)|_H^2), \quad t \in T,$$

where $\mu_{\alpha,h}(t) = x(t) - w^{\alpha,h}(t)$, $b = |B^*|_{L(H;U)}$ is the norm of the linear operator $B^* \in L(H; U)$. In this case, we have

$$\int_{t_0}^{t} |v^{\alpha,h}(\tau)|_U^2 d\tau \leq 2b^2\alpha^{-2}\int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_H^2 d\tau + c_1 h^2\alpha^{-2}. \tag{10}$$

It can be easily seen that, due to (7), the following inequality is fulfilled:

$$(B(u(t) - v^{\alpha,h}(t)), \mu_{\alpha,h}(t))_H \leq (B(u(t) - v^{\alpha,h}(t)), x(t) - \xi^h(t))_H +$$
$$+ bh\{|u(t)|_U + |v^{\alpha,h}(t)|_U\} \quad \text{for a.a. } t \in T.$$

Then, set in (3) $z = w^{\alpha,h}(t)$, and in (9), $z = x(t)$. Summing the expressions, we derive

$$\langle \dot{\mu}_{\alpha,h}(t) + A\mu_{\alpha,h}(t), \mu_{\alpha,h}(t)\rangle \leq (B(u(t) - v^{\alpha,h}(t), \mu_{\alpha,h}(t)) \quad \text{for a.a.} \quad t \in T. \tag{11}$$

By virtue of coercivity condition (2), from (11) we obtain

$$\frac{1}{2}\frac{d|\mu_{\alpha,h}(t)|_H^2}{dt} + c|\mu_{\alpha,h}(t)|_V^2 \leq (B(u(t) - v^{\alpha,h}(t)), \mu_{\alpha,h}(t))_H + \omega|\mu_{\alpha,h}(t)|_H^2 \leq$$
$$\leq (B(u(t) - v^{\alpha,h}(t)), x(t) - \xi^h(t))_H + bh\{|u(t)|_U + |v^{\alpha,h}(t)|_U\} + \omega|\mu_{\alpha,h}(t)|_H^2.$$

Therefore,

$$\frac{d|\mu_{\alpha,h}(t)|_H^2}{dt} + 2c|\mu_{\alpha,h}(t)|_V^2 + \alpha\{|v^{\alpha,h}(t)|_U^2 - |u(t)|_U^2\} \leq$$
$$\leq -2(v^{\alpha,h}(t), B^*(x(t) - \xi^h(t)))_U + \alpha|v^{\alpha,h}(t)|_U^2 + 2(u(t), B^*(x(t) - \xi^h(t)))_U -$$
$$- \alpha|u(t)|_U^2 + 2bh\{|u(t)|_U + |v^{\alpha,h}(t)|_U\} + 2\omega|\mu_{\alpha,h}(t)|_H^2. \tag{12}$$

Note that the control $v^{\alpha,h}(t)$ of form (8) satisfies the equation

$$v^{\alpha,h}(t) = \arg\min\{\alpha|v|_U^2 - 2(B^*(x(t) - \xi^h(t)), v)_U : v \in U\}. \tag{13}$$

By (13), it follows from inequality (12) that

$$\varepsilon_h(t) \le \varepsilon_h(t_0) + \int_{t_0}^{t} 2bh\{|u(\tau)|_U + |v^{\alpha,h}(\tau)|_U\}d\tau + 2\omega \int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_H^2 d\tau, \tag{14}$$

where

$$\varepsilon_h(t) = |\mu_{\alpha,h}(t)|_H^2 + 2c \int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_V^2 \, d\tau + \alpha \int_{t_0}^{t} \{|v^{\alpha,h}(\tau)|_U^2 - |u(\tau)|_U^2\} \, d\tau.$$

By the inclusion $u(\cdot) \in L_2(T; U)$, we obtain

$$\int_{t_0}^{\vartheta} 2bh|u(\tau)|_U \, d\tau \le c_2 h.$$

This and (14) imply the inequality

$$\varepsilon_h(t) \le \varepsilon_h(t_0) + c_3(h + h^\beta + h^{2-\beta} \int_{t_0}^{t} |v^{\alpha,h}(\tau)|_U^2 d\tau) + 2\omega \int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_H^2 d\tau, \quad \beta \in (0,1). \tag{15}$$

Using the relation (10), inequality $\varepsilon_h(t_0) \le h^2$ and (6), we have

$$\varepsilon_h(t) \le c_4(h^\beta + h^{4-\beta}\alpha^{-2}) + c_5(h^{2-\beta}\alpha^{-2} + 1) \int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_H^2 d\tau. \tag{16}$$

Consequently we get the bound

$$|\mu_{\alpha,h}(t)|_H^2 \le c_6(h^\beta + \alpha + h^{4-\beta}\alpha^{-2}) + c_5(h^{2-\beta}\alpha^{-2} + 1) \int_{t_0}^{t} |\mu_{\alpha,h}(\tau)|_H^2 d\tau.$$

Using the Gronwall lemma, for $t \in T$, we obtain

$$|\mu_{\alpha,h}(t)|_H^2 \le c_6(h^\beta + \alpha + h^{4-\beta}\alpha^{-2}) \exp\{c_5(t - t_0)(h^{2-\beta}\alpha^{-2} + 1)\}. \tag{17}$$

Let $\beta \in (0,1)$ be a constant such that

$$h^{2-\beta}\alpha^{-2} \le \text{const}. \tag{18}$$

Then we have

$$|\mu_{\alpha,h}(t)|_H^2 \le c_7(h^\beta + \alpha). \tag{19}$$

Inequalities (16), (18) and (19) imply

$$\varepsilon_h(t) \le c_4(h^\beta + h^{4-\beta}\alpha^{-2}) + c_8(h^{2-\beta}\alpha^{-2} + 1)(h^\beta + \alpha) \le c_9(h^\beta + \alpha). \tag{20}$$

The validity of the theorem follows from (20), if $\beta = 2/3$. This completes the proof of the theorem. □

4. Solving algorithm. The case of discrete measuring of phase states

Let us describe the algorithm for solving the problem in the case of discrete measuring of phase states of the inequalities. In this case, inequalities (4) are fulfilled.

Let $l(\cdot) : W^{1,2}(T;H) \cap L_2(T;V) \to \mathbb{R}^+$,

$$l(y(\cdot)) = |y(\cdot)|_{C(T;H)} + |\dot{y}(\cdot)|_{L_2(T;H)} + |y(\cdot)|_{L_2(T;V)}.$$

In a standard way (see, for example, [1, 2, 8]), we establish the validity of the following lemma.

Lemma 4.1. *There exists a number $K = K(\omega, D, c, |B|_{L(U;H)})$ such that the inequality*

$$l(x(\cdot; t_0, x, u(\cdot))) \le K(1 + |u(\cdot)|_{L_2(T;U)})$$

is fulfilled for any $x \in D$ and $u(\cdot) \in L_2(T;U)$.

Let a family of partitions

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = t_0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h),$$

$\delta(h) \in (0,1)$, and a function $\alpha(h) : (0,1) \to (0,1)$ be fixed. First, before the moment t_0, a value h and a partition Δ_h of the interval T is chosen and fixed. The work of the algorithm is decomposed into $m-1$ ($m = m_h$) identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following sequence of actions is performed. First, in the moment τ_i the element

$$v_i^h = \alpha^{-1} B'(x(\tau_i) - \xi_i^h) \tag{21}$$

is calculated. Then, the control determined by the formula

$$v(t) = v^h(t) = v_i^h, \quad t \in \delta_i \tag{22}$$

is fed onto the input of inequality (1). Then, instead of the phase state $w^h(\tau_i) = w^h(\tau_i; \tau_{i-1}, w^h(\tau_{i-1}), v_{i-1}^h)$, the phase state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), v_i^h)$ is realized. The work of the algorithm stops at the time moment ϑ.

Let a family of partitions Δ_h of the time interval T and a function $\alpha(h)$ have the following property

$$h\delta^{-1}(h) \le C_1, \ \delta(h)\alpha^{-2}(h) \le C_2, \ \alpha(h) \to 0, \ \delta(h) \to 0 \text{ as } h \to 0+. \tag{23}$$

Here C_1 and $C_2 > 0$ are constants, which do not depend on h.

Theorem 4.1. *Uniformly with respect to $h \in (0,1)$, the inequality*

$$\lambda_h(t) \equiv |x(t) - w^h(t)|_H^2 + 2c \int_{t_0}^{t} |x(\tau) - w^h(\tau)|_V^2 \, d\tau \le$$
$$\le d_1(h + \alpha(h) + \delta(h)) \ \forall t \in T, \tag{24}$$

is true. Here $d_1 > 0$ is a constant, which does not depend on h, $\alpha(h)$ and $\delta(h)$.

Proof. Taking into account coercivity condition (2), we derive for a.a. $t \in \delta_i$

$$\frac{1}{2}\frac{d|\mu^h(t)|_H^2}{dt} + c|\mu^h(t)|_V^2 - \omega|\mu^h(t)|_H^2 \le (B(u(t) - v^h(t)), \mu^h(t))_U,$$

where $\mu^h(t) = x(t) - w^h(t)$. Furthermore, it is easily seen that, for a.a. $t \in \delta_i$, the inequality

$$(B(u(t) - v^h(t), \mu^h(t))_U \le (B(u(t) - v^h(t)), x(\tau_i) - \xi_i^h)_U + \varrho_i(t, h)$$

is fulfilled. Here,

$$\varrho_i(t,h) = b(|u(t)|_U + |v^h(t)|_U)(h + \int\limits_{\tau_i}^{t} \{|\dot{w}^h(\tau)|_H + |\dot{x}(\tau)|_H\}\, d\tau),\ t \in \delta_i.$$

For a.a. $t \in \delta_i$, we deduce that

$$\frac{1}{2}\frac{d|\mu^h(t)|_H^2}{dt} + c|\mu^h(t)|_V^2 \le (B(u(t) - v^h(t)), x(\tau_i) - \xi_i^h)_U + \omega|\mu^h(t)|_H^2 + \varrho_i(t,h), \qquad (25)$$

Consequently, (25) implies the inequality

$$\frac{d|\mu^h(t)|_H^2}{dt} + 2c|\mu^h(t)|_V^2 + \alpha\{|v^h(t)|_U^2 - |u(t)|_U^2\} \le$$
$$\le -2(v^h(t), B^*(x(\tau_i) - \xi_i^h))_U + \alpha|v^h(t)|_U^2 +$$
$$+ 2(u(t), B^*(x(\tau_i) - \xi_i^h))_U - \alpha|u(t)|_U^2 + 2\varrho_i(t,h) + 2\omega|\mu^h(t)|_H^2, \quad t \in \delta_i. \qquad (26)$$

Therefore, by fact of the rule of forming the control $v^h(\cdot)$ and using the relations (21), (22), we conclude from (26) that, for a.a. $t \in \delta_i$,

$$\varepsilon^h(t) \le \varepsilon^h(\tau_i) + b\int\limits_{\tau_i}^{t} \{|u(\tau)|_U + |v^h(\tau)|_U\}\, d\tau \times \qquad (27)$$

$$\times\, (h + \int\limits_{\tau_i}^{t} \{|\dot{w}^h(\tau)|_H + |\dot{x}(\tau)|_H\}\, d\tau) + 2\omega\int\limits_{\tau_i}^{t} |\mu^h(\tau)|_H^2\, d\tau \le$$

$$\le \varepsilon^h(\tau_i) + c_1h^2 + c_2\delta\int\limits_{\tau_i}^{t} \{|u(\tau)|_U^2 + |v^h(\tau)|_U^2\}\, d\tau +$$

$$+ c_3\delta\int\limits_{\tau_i}^{t} \{|\dot{w}^h(\tau)|_H^2 + |\dot{x}(\tau)|_H^2\} d\tau + 2\omega\int\limits_{\tau_i}^{t} |\mu^h(\tau)|_H^2\, d\tau,$$

where

$$\varepsilon^h(t) = |\mu^h(\tau)|_H^2 + 2c\int\limits_{t_0}^{t} |\mu^h(\tau)|_V^2\, d\tau + \alpha\int\limits_{t_0}^{t} \{|v^h(\tau)|_U^2 - |u(\tau)|_U^2\}\, d\tau.$$

Summing the right-hand and left-hand parts of (27) over i and taking into account Lemma 1, for $t \in T$, we obtain

$$\varepsilon^h(t) \le \varepsilon^h(t_0) + c_4h^2\delta^{-1} +$$

$$+ c_5\delta\{1 + \int\limits_{t_0}^{t} \{|u(\tau)|_U^2 + |v^h(\tau)|_U^2\} d\tau\} + 2\omega\int\limits_{t_0}^{t} |\mu^h(\tau)|_H^2\, d\tau. \qquad (28)$$

Now, we use the inclusion $u(\cdot) \in L_2(T;U)$ and the inequality

$$\int\limits_{t_0}^{t} |v^h(\tau)|_U^2\, d\tau = \sum_{j=0}^{i(t)-1} \int\limits_{\tau_j}^{\tau_{j+1}} |v^h(\tau)|_U^2\, d\tau + \int\limits_{\tau_i(t)}^{t} |v^h(\tau)|_U^2\, d\tau \le \delta\sum_{j=0}^{i(t)} |v_j^h|_U^2.$$

Here, the symbol $i(t)$ stands for the integer part of the value $(t-t_0)\delta^{-1}$. By virtue of (28),

$$\varepsilon^h(t) \le \varepsilon^h(t_0) + c_4 h^2 \delta^{-1} + c_6 \delta + c_7 \gamma_{h,\delta}(t) + 2\omega \int\limits_{\tau_i}^{t} |\mu^h(\tau)|_H^2 \, d\tau, \tag{29}$$

where,

$$\gamma_{h,\delta}(t) = \delta^2 \sum_{j=0}^{i(t)} |v_j^h|_U^2.$$

Using (4) and the rule of forming v_i^h and with the relation (21), we get

$$|v_i^h|_U^2 \le 2b^2(\varrho_i^h + h^2)\alpha^{-2} \le c_8(\varrho_i^h + h^2)\alpha^{-2}, \tag{30}$$

where $\varrho_i^h = |x(\tau_i) - w^h(\tau_i)|_H^2$. Due to (6), we have

$$\varepsilon^h(t_0) \le h^2. \tag{31}$$

Therefore, we derive from (29)–(31) the estimate

$$\lambda_h(t) \le c_9(\delta + h^2\delta^{-1} + \alpha + \gamma_{h,\delta}(t)) + 2\omega \int\limits_{t_0}^{t} \lambda_h(\tau)\, d\tau.$$

Applying the Gronwall lemma to this inequality, we obtain

$$\lambda_h(t) \le c_9(\delta + h^2\delta^{-1} + \alpha + \gamma_{h,\delta}(t)) +$$
$$+ 2\omega \int\limits_{t_0}^{t} c_9(\delta + h^2\delta^{-1} + \alpha + \gamma_{h,\delta}(\tau)) \exp(2\omega(t-\tau))\, d\tau. \tag{32}$$

The function $t \to \gamma_{h,\delta}(t)$ is nondecreasing; therefore, using (32) we deduce that

$$\lambda_h(t) \le c_{10}(\delta + h^2\delta^{-1} + \alpha + \gamma_{h,\delta}(t)). \tag{33}$$

Furthermore,

$$\varrho_i^h \le \lambda_i^h, \tag{34}$$

where $\lambda_j^h = \lambda_h(\tau_j)$. For $t \in [\tau_i, \tau_{i+1}]$, using inequalities (30) and (34), we conclude that

$$\gamma_{h,\delta}(t) \le c_8 \delta^2 \sum_{j=0}^{i(t)} (\lambda_j^h + h^2)\alpha^{-2}.$$

Then,

$$\gamma_{h,\delta}(\tau_i) \le c_8 \delta^2 \sum_{j=0}^{i} (\lambda_j^h + h^2)\alpha^{-2}. \tag{35}$$

From inequalities (33) and (35), it follows that

$$\lambda_i^h \le c_{10}(\delta + h^2\delta^{-1} + \alpha) + c_{11}\delta h^2 \alpha^{-2} + c_{12}\delta^2\alpha^{-2} \sum_{j=0}^{i} \lambda_j^h. \tag{36}$$

By the discrete Gronwall lemma [9] and (36), we obtain

$$\lambda_i^h \le c_{13}(\alpha + \delta + h^2\delta^{-1} + \delta h^2 \alpha^{-2}) \exp\{c_{12}(\vartheta - t_0)\delta\alpha^{-2}\}. \tag{37}$$

From the inequalities

$$h\delta^{-1}(h) \le C_1, \quad \delta\alpha^{-2}(h) \le C_2 \quad \text{as} \quad h \to 0$$

and relations (23) and (37), we have

$$\lambda_i^h \leq c_{14}(h + \delta + \alpha), \quad i \in [0 : m].$$

This result and the inequality (35) imply that

$$\gamma_{h,\delta}(\tau_i) \leq c_{15}(h + \delta + \alpha), \quad i \in [0 : m]. \tag{38}$$

Moreover, due to (33) and (38), the following inequality is true:

$$\lambda_h(t) \leq c_{16}(\delta + h^2\delta^{-1} + \alpha + \gamma_{h,\delta}(\vartheta)) \leq c_{17}(h + \delta + \alpha).$$

Relation (24) follows from this inequality. This completes the proof of the theorem. □

As a result of Theorem 2 we have the following corollary.

Corollary 4.1. *Let $\delta(h) = h$ and $\alpha(h) = h^{1/2}$. Then the following inequality*

$$\lambda_h(t) \leq d_2 h^{1/2} \quad \forall t \in T,$$

is true. Here, $d_2 > 0$ is a constant that does not depend on h.

Remark 4.1. *We considered the case, when $\omega > 0$. Theorems 1 and 2 are also valid for non-positive ω. Moreover, the proofs of these theorems are simplified. For example, if $\omega \leq 0$, in the proof of Theorem 1, the terms containing the expressions with ω in inequalities (12), (14), and (15) can be omitted.*

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