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APPLICATION OF THE GENERALIZED CLIFFORD-DIRAC ALGEBRA TO THE PROOF OF THE DIRAC EQUATION FERMI-BOSE DUALITY

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Abstract. The consideration of the bosonic properties of the Dirac equation with arbitrary mass has been continued. As the necessary mathematical tool the structure and different representations of the 29-dimensional extended real Clifford-Dirac algebra (Phys. Lett. A., 2011, v.375, p.2479) are considered briefly. As a next step we use the start from the Foldy-Wouthuysen representation. On the basis of these two ideas the property of Fermi-Bose duality of the Dirac equation with nonzero mass is proved. The proof is given on the three maim examples: bosonic symmetries, bosonic solutions and bosonic conservation laws. It means that Dirac equation can describe not only the fermionic but also the bosonic states.

Keywords: Dirac equation, Foldy-Wouthuysen representation, spinor field, fermions, bosons, Clifford algebra, new supersymmetry.

AMS Subject Classification: 11.30-z.; 11.30.Cp.;11.30.j.

1. Introduction

This paper is an extended version of brief article [1], which was reported at the 14-th International Conference on Mathematical Methods in Electromagnetic Theory and published in the Proceedings of this conference. The property of Fermi-Bose duality of the Dirac equation and the generalized real Clifford-Dirac (CD) algebra are under consideration. The concept of Fermi-Bose duality of the Dirac equation means that this equation in addition to the well-known fermionic properties has also the bosonic properties. We are able to prove the existence of bosonic symmetries, solutions and conservation laws for the free Dirac equation with nonzero mass. The 64-dimensional extended real Clifford-Dirac (ERCD) algebra and especially the 29-dimensional proper ERCD algebra, which were put into consideration in [2]-[5], are shown to be the mathematical tools for that proof.

Starting from the first steps of quantum mechanics and during the period of its growth many authors have been investigated the Fermi-Bose duality of the massless spinor field and of the related electromagnetic field in the terms of field strengths [6]-24]. Such investigations were called the relations between the Dirac and Maxwell equations, the Maxwell – Dirac isomorphism [22], etc.

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The concept Fermi-Bose (FB) duality of the spinor field has been mentioned first by L. Foldy [25]. The extended consideration has been given in [26]. P. Garbaczewski proved [26] that the Fock space $\mathcal{H}^{\mathrm{F}}(\mathrm{H}^{3,\mathrm{M}})$ over the quantum mechanical space $\mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes \mathrm{M}}$ of the particle, which is described by the field $\phi : \mathrm{M}(1,\mathrm{N}) \rightarrow \mathrm{C}^{\otimes \mathrm{N}}$, allows one to fulfill the dual FB quantization of the field ϕ in $\mathcal{H}^{\mathrm{F}}$. And both the canonical commutation relations (CCR) and anticommutation relations (CAR) were used to realize the above mentioned quantization. Moreover, for the both types of quantization the uniqueness of the vacuum in $\mathcal{H}^{\mathrm{F}}$ was proved. The dual FB quantization was illustrated for different examples and in the spaces $\mathrm{M}(1,\mathrm{N})$ of arbitrary dimensions. The massless spinor field was considered in details [26].

In our publications, consideration of the FB duality concept of the field was extended by applying the group-theoretical approach for the problem (FB duality was often called by us as the relationship between the fields of integer and half-integer spins, see e. g. [27]-[33]). As a first step we have considered in details the case of massless Dirac equation. Both Fermi and Bose local representations of the universal covering $\mathcal{P} \supset \mathcal{L} = \mathrm{SL(2,C)}$ of proper ortochronous Poincaré group $\mathrm{P}_+^\uparrow = \mathrm{T}(4){\times})\mathrm{L}_+^\uparrow \supset \mathrm{L}_+^\uparrow = \mathrm{SO(1,3)}$, with respect to which the Dirac equation is invariant, were found. The same was realized [30], [31] for the slightly generalized original Maxwell equations, in which the complex valued 4-object $\mathcal{E}(x) = E(x) - iH(x)$ of field strengths is the tensor-scalar (s=1,0) $\mathcal{P}$-covariant. In general, we have proved the existence of bosonic symmetries, solutions and conservation laws for the massless Dirac equation ([27]-[33] and references therein). Thus, the systematic investigation of the bosonic properties of Dirac equation with $m = 0$ was given.

However, in the papers [6]-[17], [19]-[21], [23], [24], [27], [29] the simplest example of a free, massless Dirac equation and its relation to the Maxwell equations was considered. Interest to such problems has grown after the investigations [18], [22], [34], [35] of the physically meaningful case (mass is nonzero and the interaction potential is nonzero too) and our own research steps [28], [30]-[33] in the same direction. Unfortunately, only the stationary Dirac and Maxwell equations were considered.

In another approach [36]-[40], the quadratic relations between the fermionic and bosonic amplitudes were found and used. In our papers [1]-[5], [27]-[33] and here we discuss linear relations between fermionic and bosonic amplitudes.

Our results were continued and used by the authors of [41]-[48], where the references to our above mentioned papers on the case $m = 0$ were given. Nevertheless, the general case, when the mass in the Dirac equation is not equal to zero, was still open for investigations and considerations.

Only recently [2]-[5] we were able to extend our consideration for the Dirac equation with nonzero mass. The important step was as follows. We started from the Foldy-Wouthuysen (FW) [49] representation of the Dirac equation and the results for the standard Dirac equation were found as a consequences of the FW transformation. The reasons of our start from FW representation were explained in [4], [5]. In our papers [2]-[5], bosonic representations of universal covering $\mathcal{L} = \mathrm{SL(2,C)}$ of proper ortochronous Lorentz group $\mathrm{L}_+^\uparrow = \mathrm{SO(1,3)}$ were found, with respect to which the Dirac and FW equations with nonzero mass are invariant. And the main results are the bosonic spin (1,0) representations of Poincaré group $\mathcal{P}$, with respect to which these equations are invariant. These results were proved on the basis of 64 dimensional extended real Clifford-Dirac (ERCD) algebra and 29 dimensional proper ERCD algebra, which were put into consideration in [2]-[5] and essentially generalized the standard 16 dimensional Clifford-Dirac (CD) algebra.

Here we consider (i) the dual (fermionic and bosonic) symmetries [2]-[5] of the FW [43] and Dirac equations with nonzero mass, (ii) continue the construction of bosonic solutions [50] of these equations and (iii) demonstrate the existence of both Fermi and Bose conservation laws for the spinor field. Thus, we present the third level proof of the FB duality of the Dirac equation.

In order to give the adequate derivation of the conservation laws we appeal to the Noether theorem and the Lagrange approach. Therefore, the Lagrangian, which Euler-Lagrange equation coincide with the FW equation, is also presented here and briefly discussed.

In section 2, the necessary notations and definitions are given.

In section 3, the 29-dimensional proper extended real Clifford - Dirac algebra [3]-[5], which is the mathematical basis of our consideration, is presented briefly.

In section 4 the 64-dimensional ERCD algebra is reviewed.

In section 5 the main useful representations of the proper extended real Clifford - Dirac algebra are considered.

In section 6, the bosonic spin s=(1,0) symmetry [3]-[5] of the FW and Dirac equations with nonzero mass is considered briefly as the first step in our proof of the Dirac equation FB duality.

In section 7, we continue to construct the bosonic [50] spin s=(1,0) multiplet solutions of the FW and Dirac equations with nonzero mass. It is the second step in our proof of the Dirac equation FB duality.

In section 8, the brief formulation of the Lagrange approach for the FW field is given.

In section 9, the FB duality of the spinor field is demonstrated on the example of the existence of both Fermi and Bose series of conservation laws for this field (the third step of our proof).

In section 10, the relationship between fermionic and bosonic solutions together with some interpretation is discussed.

In section 11, the brief general conclusions are formulated.

2. Notations and definitions

The system of units $\hbar = c = 1$ and metric $g = (g^{\mu\nu}) = (+ - - -)$, $a^{\mu} = g^{\mu\nu}a_{\nu}$, are taken. The Greek indices vary in the region $0, 1, 2, 3 \equiv \overline{0, 3}$, Latin – $\overline{1, 3}$, the summation over the twice repeated index is implied. The Dirac γ^{μ} matrices in the standard Pauli-Dirac (PD) representation are used. Our consideration is fulfilled in the rigged Hilbert space $\mathrm{S}^{3,4} \subset \mathrm{H}^{3,4} \subset \mathrm{S}^{3,4*}$ where $\mathrm{H}^{3,4}$ is given by

$$\mathrm{H}^{3,4} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathrm{C}^{\otimes 4} = \{\phi = (\phi^{\mu}) : \mathrm{R}^3 \to \mathrm{C}^{\otimes 4}; \int d^3x |\phi(t, \overrightarrow{x})|^2 < \infty\}\}, \tag{1}$$

and symbol "*" in $\mathrm{S}^{3,4*}$ means that the space of the Schwartz generalized functions $\mathrm{S}^{3,4*}$ is conjugated to the Schwartz test function space $\mathrm{S}^{3,4}$ by the corresponding topology. For more details see [4].

We consider the ordinary CD algebra as the algebra of 4×4 Dirac matrices in the standard PD representation in terms of the standard 2×2 Pauli matrices.

For the purposes related to physics it is useful to consider the corresponding groups an algebras with real parameters (e. q. the parameters $a = (a^{\mu}), \omega = (\omega^{\mu\nu})$ of the translations and rotations for the group $\mathrm{P}_{+}^{\uparrow}$). Therefore, corresponding generators are anti-Hermitian. The mathematical correctness of such choice of generators is verified in [51], [52].

3. Proper extended real Clifford-Dirac algebra and the Foldy-Wouthuysen representation

We consider the standard 16-dimensional CD algebra of the γ^μ matrices as a real one and add the imaginary unit $i = \sqrt{-1}$ together with the operator $\hat{C}$ of the complex conjugation (the involution operator in the space $\mathrm{H}^{3,4}$) into the set of the CD algebra possible generators. It enabled us to extend the standard CD algebra up to the 64-dimensional extended real CD algebra (ERCD algebra of [3]-[5]). Here the subalgebras of the ERCD algebra are considered briefly. The most important are the representations in $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$ of the 29-dimensional proper ERCD algebra SO(8) spanned over the orts

$$\gamma^1,\, \gamma^2,\, \gamma^3,\, \gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3,\, \gamma^5 = \gamma^1\gamma^3\hat{C},\, \gamma^6 = i\gamma^1\gamma^3\hat{C},\, \gamma^7 = i\gamma^0, \tag{2}$$

where $\gamma^0 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$, $\gamma^k = \begin{vmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{vmatrix}$ and σ^k are the standard Pauli matrices. The generators (2) satisfy the anticommutation relations [1]

$$\gamma^{\mathrm{A}}\gamma^{\mathrm{B}} + \gamma^{\mathrm{B}}\gamma^{\mathrm{A}} = -2\delta^{\mathrm{AB}},\ \mathrm{A},\mathrm{B} = \overline{1,7}, \tag{3}$$

and the generators of proper ERCD algebra $\alpha^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = 2s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}$ (together with unit ort, 4×4 matrix I_4, we have 29 independent orts I_4, $\alpha^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = 2s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}$)

$$s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}} = \{s^{\mathrm{AB}} = \frac{1}{4}[\gamma^{\mathrm{A}},\gamma^{\mathrm{B}}],\, s^{\mathrm{A}8} = -s^{8\mathrm{A}} = \frac{1}{2}\gamma^{\mathrm{A}}\},\ \widetilde{\mathrm{A}},\widetilde{\mathrm{B}} = \overline{1,8}, \tag{4}$$

satisfy the commutation relations of SO(8) algebra

$$[s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{B}}}, s^{\widetilde{\mathrm{C}}\widetilde{\mathrm{D}}}] = \delta^{\widetilde{\mathrm{A}}\widetilde{\mathrm{C}}}s^{\widetilde{\mathrm{B}}\widetilde{\mathrm{D}}} + \delta^{\widetilde{\mathrm{C}}\widetilde{\mathrm{B}}}s^{\widetilde{\mathrm{D}}\widetilde{\mathrm{A}}} + \delta^{\widetilde{\mathrm{B}}\widetilde{\mathrm{D}}}s^{\widetilde{\mathrm{A}}\widetilde{\mathrm{C}}} + \delta^{\widetilde{\mathrm{D}}\widetilde{\mathrm{A}}}s^{\widetilde{\mathrm{C}}\widetilde{\mathrm{B}}}. \tag{5}$$

Namely the proper ERCD algebra SO(8), given by the 29 orts (4), is our [3]-[5] direct generalization of the standard 16-dimensional CD algebra. It is also the basis for our dual FB consideration of a spinor field, which enabled us to prove the additional bosonic properties of this field. For the physical applications we consider here the realizations of the proper ERCD algebra in the field space $\mathrm{S}^*(\mathrm{M}(1,3)) \otimes \mathrm{C}^{\otimes 4} \equiv \mathrm{S}^{4,4*}$ of the Schwartz generalized functions and in the quantum mechanical Hilbert space $\mathrm{H}^{3,4}$ (1). These realizations are found with the help of transformations $V^+\mathrm{SO}(8)V^-$, $v\mathrm{SO}(8)v$, where the operators of transformations have the form

$$V^{\pm} \equiv \frac{\pm i\overrightarrow{\gamma}\cdot\nabla + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\ v = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & \hat{C}\mathrm{I}_2 \end{vmatrix},\ \widehat{\omega} \equiv \sqrt{-\Delta + m^2},\ \nabla \equiv (\partial_\ell),\ \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \tag{6}$$

Furthermore, the realizations of the proper ERCD algebra for bosonic fields are presented.

We put into consideration the ERCD algebra (64 orts) and proper ERCD algebra (29 orts) into the FW representation of the spinor field [49] (advantages in comparison with the standard Dirac equation in definitions of coordinate, velocity and spin operators are well known from [49]). In this representation the equation for the spinor field (the FW equation) has the form

$$(\partial_0 + i\gamma^0\widehat{\omega})\phi(x) = 0,\ x \in \mathrm{M}(1,3),\ \phi \in \mathrm{H}^{3,4}; \tag{7}$$

and is linked with the Dirac equation

$$(\partial_0 + iH)\psi(x) = 0,\ H \equiv \overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta m, \tag{8}$$

by the FW transformation $V^{\pm}$:

$$\phi(x) = V^{-}\psi(x),\ \psi(x) = V^{+}\phi(x),\ V^{+}\gamma^{0}\widehat{\omega}V^{-} = \overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta m. \tag{9}$$

Below the ERCD algebra and proper ERCD algebra (4) are essentially used in our proofs of bosonic properties of the Dirac and FW equations. The proper ERCD algebra has 29 independent orts given in (4). In comparison with 16 independent orts of standard CD algebra we can operate now with additional elements. These additional generators of SO(8) algebra enabled us to prove the additional bosonic symmetries of the FW and Dirac equations [3]-[5] and to construct the additional bosonic solutions of these equations (section 7 below). Moreover, the anticommutation relations (3) were used in calculations.

4. Extended real Clifford-Dirac algebra

Proper ERCD algebra has been found in [3]-[5] as an important subalgebra of 64-dimensional ERCD algebra (proper ERCD algebra is the direct generalization of a standard CD algebra). The ERCD algebra itself has been found in [3]-[5] as the complete set of operators of standard CD algebra together with the operators of a Pauli-Gursey-Ibragimov algebra [53], [54]:

$$\{\gamma^{2}\hat{C},\, i\gamma^{2}\hat{C},\, \gamma^{2}\gamma^{4}\hat{C},\, i\gamma^{2}\gamma^{4}\hat{C},\, \gamma^{4},\, i\gamma^{4},\, i,\, \mathrm{I}\}. \tag{10}$$

Hence, such generalization of the real CD algebra is constructed with the help of imaginary unit $i=\sqrt{-1}$ together with the operator $\hat{C}$ of complex conjugation, i. e. these operators are here the nontrivial orts of the algebra.

It is known from [3]-[5] that 16 orts of the standard CD algebra can be written in the form

$$\{\mathrm{ind} \quad \mathrm{CD}\} \equiv \{\mathrm{I},\, \alpha^{\tilde{\mu}\tilde{\nu}} = 2s^{\tilde{\mu}\tilde{\nu}}\},\ \tilde{\mu},\tilde{\nu} = \overline{0,5}, \tag{11}$$

where

$$s^{\bar{\mu}\bar{\nu}} \equiv \frac{1}{4}\left[\gamma^{\bar{\mu}},\gamma^{\bar{\nu}}\right],\ s^{\bar{\mu}5} = -s^{5\bar{\mu}} \equiv \frac{1}{2}\gamma^{\bar{\mu}};\ \gamma^{4} \equiv \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3},\ \bar{\mu},\bar{\nu} = \overline{0,4}. \tag{12}$$

Matrices (12) satisfy the commutation relations of nontrivial generators of the SO(1,5) algebra in the form

$$\left[s^{\tilde{\mu}\tilde{\nu}}, s^{\tilde{\rho}\tilde{\sigma}}\right] = -g^{\tilde{\mu}\tilde{\rho}}s^{\tilde{\nu}\tilde{\sigma}} - g^{\tilde{\rho}\tilde{\nu}}s^{\tilde{\sigma}\tilde{\mu}} - g^{\tilde{\nu}\tilde{\sigma}}s^{\tilde{\mu}\tilde{\rho}} - g^{\tilde{\sigma}\tilde{\mu}}s^{\tilde{\rho}\tilde{\nu}};\ (g^{\tilde{\mu}}_{\tilde{\nu}}) = \mathrm{diag}(+1,-1,-1,-1,-1,-1). \tag{13}$$

In formulae (12), (13) we rewrite the result of [55], [56], where the 16 orts of the standard CD algebra are presented in a form of the SO(3,3) algebra. We give this result in the form, which is useful for our purposes, i. e. as the SO(1,5) algebra (similarly to the algebra of the proper ortochronous Lorentz group $\mathrm{L}^{\uparrow}_{+}$ = SO(1,3) in the Minkowski space M(1,3)⊂M(1,5)).

Thus, in the terms of (11) the complete set of 64 orts of the ERCD algebra (the complete set of operators (11) and (10)) has the form

$$\{\mathrm{ERCD}\} = \Big\{(\mathrm{ind} \quad \mathrm{CD}),\ i\cdot(\mathrm{ind} \quad \mathrm{CD}),\ \hat{C}\cdot(\mathrm{ind} \quad \mathrm{CD}),\ i\hat{C}\cdot(\mathrm{ind} \quad \mathrm{CD})\Big\}. \tag{14}$$

The ERCD algebra (14) has only some partial features of the CD algebra. The direct generalization of the standard CD algebra is only its subalgebra – the proper ERCD algebra (section 3). Nevertheless, the ERCD algebra has another important subalgebra – the 32-dimensional algebra $\mathrm{A}_{32} = \mathrm{SO}(6) \oplus i\gamma^{0}\cdot\mathrm{SO}(6) \oplus i\gamma^{0}$. The last one is the maximal set of pure matrix operators, which left the FW equation (7) invariant (the details are given in [3]-[5]).

5. Representations of the proper extended real Clifford-Dirac algebra

In **the fundamental FW representation,** the 29 orts of the proper ERCD algebra SO(8) are given by the formulae (4), where the 7 generating operators have the form (2).

In a standard **Pauli-Dirac representation**, the so called local representation, the corresponding 29 orts are the consequences of the FW transformation $V^{\pm}$ (6), (9) and are given by the elements ($\alpha^{\widetilde{A}\widetilde{B}} = 2\widetilde{s}^{\widetilde{A}\widetilde{B}}$, I), where

$$\widetilde{s}^{\widetilde{A}\widetilde{B}} = \{\widetilde{s}^{AB} = \frac{1}{4}[\widetilde{\gamma}^{A}, \widetilde{\gamma}^{B}],\ \widetilde{s}^{A8} = -\widetilde{s}^{8A} = \frac{1}{2}\widetilde{\gamma}^{A}\},\ \widetilde{A}, \widetilde{B} = \overline{1,8},\ A, B = \overline{1,7}. \tag{15}$$

Here 7 generating operators $\widetilde{\gamma}^{A}$ together with operators $\widetilde{\gamma}^{0} = V^{+}\gamma^{0}V^{-}$ and $\widetilde{C} = V^{+}\hat{C}V^{-}$ are nonlocal and have the form

$$\overrightarrow{\widetilde{\gamma}} = \overrightarrow{\gamma}\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}} + \overrightarrow{p}\frac{-\overrightarrow{\gamma}\cdot\nabla + \widehat{\omega} + m}{\widehat{\omega}(\widehat{\omega} + m)},\ \widetilde{\gamma}^{4} = \gamma^{4}\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}},$$

$$\widetilde{\gamma}^{5} = \widetilde{\gamma}^{1}\widetilde{\gamma}^{3}\widetilde{C},\ \widetilde{\gamma}^{6} = i\widetilde{\gamma}^{1}\widetilde{\gamma}^{3}\widetilde{C},\ \widetilde{\gamma}^{7} = i\widetilde{\gamma}^{0},\ \widetilde{\gamma}^{0} = \gamma^{0}\frac{-\overrightarrow{\gamma}\cdot\nabla + m}{\widehat{\omega}}, \tag{16}$$

$$\widetilde{C} = (\mathrm{I} + 2\frac{i\gamma^{1}\partial_{1} + i\gamma^{2}\partial_{2}}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}})\hat{C}; \quad \widehat{\omega} \equiv \sqrt{-\triangle + m^{2}}.$$

In **bosonic representation**, where the proof of the bosonic properties of the FW and Dirac equation is most convenient, the corresponding 29 orts of the proper ERCD algebra (SO(8) algebra) are given by the elements ($\alpha^{\widetilde{A}\widetilde{B}} = 2\breve{s}^{\widetilde{A}\widetilde{B}}$, I), where

$$\breve{s}^{\widetilde{A}\widetilde{B}} = \{\breve{s}^{AB} = \frac{1}{4}[\breve{\gamma}^{A}, \breve{\gamma}^{B}],\ \breve{s}^{A8} = -\breve{s}^{8A} = \frac{1}{2}\breve{\gamma}^{A}\},\ \breve{A}, \breve{B} = \overline{1,8},\ A, B = \overline{1,7}. \tag{17}$$

Here 7 generating operators $\breve{\gamma}^{A}$ together with operators $\breve{\gamma}^{0}$, operator i and $\breve{C}$ have the form

$$\breve{\gamma}^{0} = \begin{vmatrix} \sigma^{3} & 0 \\ 0 & \sigma^{1} \end{vmatrix},\ \breve{\gamma}^{1} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & i & i \\ -1 & i & 0 & 0 \\ 1 & i & 0 & 0 \end{vmatrix},\ \breve{\gamma}^{2} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -i & i \\ 0 & 0 & -1 & -1 \\ -i & 1 & 0 & 0 \\ i & 1 & 0 & 0 \end{vmatrix},\ \breve{\gamma}^{3} = -\begin{vmatrix} \sigma^{2} & 0 \\ 0 & i\sigma^{2} \end{vmatrix}\hat{C},$$

$$\breve{\gamma}^{4} = \begin{vmatrix} i\sigma^{2} & 0 \\ 0 & -\sigma^{2} \end{vmatrix}\hat{C},\ \breve{\gamma}^{5} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & i & -i \\ 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \end{vmatrix},\ \breve{\gamma}^{6} = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 0 & -i & -i \\ 0 & 0 & 1 & -1 \\ -i & -1 & 0 & 0 \\ -i & 1 & 0 & 0 \end{vmatrix}, \tag{18}$$

$$\breve{\gamma}^{7} = \gamma^{7} = i\gamma^{0},\ \breve{i} = \begin{vmatrix} i\sigma^{3} & 0 \\ 0 & -i\sigma^{1} \end{vmatrix},\ \breve{C} = \begin{vmatrix} \sigma^{3} & 0 \\ 0 & \mathrm{I}_{2} \end{vmatrix}\hat{C}.$$

Transition from the fundamental representation of the proper ERCD algebra to the bosonic representation is fulfilled by the transformation $\breve{\gamma}^{A} = W\gamma^{A}W^{-1}$ with the help of the operator W:

$$W = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2}\hat{C} & 0 \\ 0 & -\hat{C} & 0 & 1 \\ 0 & -\hat{C} & 0 & -1 \end{array}\right|, \; W^{-1} = \left|\begin{array}{cccc} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -\hat{C} & -\hat{C} \\ 0 & i\sqrt{2}\hat{C} & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right|, \; WW^{-1} = W^{-1}W = \mathrm{I}_4. \tag{19}$$

6. Bosonic spin s=(1,0) symmetry of the Foldy-Wouthuysen and Dirac equations

An example of construction of the important bosonic symmetry of the FW and Dirac equation is under consideration. The fundamental assertion is that the subalgebra SO(6) of the proper ERCD algebra (4), which is determined by the operators

$$\{\mathrm{I},\, \alpha^{\bar{\mathrm{A}}\bar{\mathrm{B}}} = 2s^{\bar{\mathrm{A}}\bar{\mathrm{B}}}\},\ \bar{\mathrm{A}}, \bar{\mathrm{B}} = \overline{1,6}, \tag{20}$$

$$\{s^{\bar{\mathrm{A}}\bar{\mathrm{B}}}\} = \{s^{\bar{\mathrm{A}}\bar{\mathrm{B}}} \equiv \frac{1}{4}[\gamma^{\bar{\mathrm{A}}}, \gamma^{\bar{\mathrm{B}}}]\}, \tag{21}$$

is the algebra of invariance of the Dirac equation in the FW representation (7) (in (21) the six matrices $\{\gamma^{\bar{\mathrm{A}}}\} = \{\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6,\}$ are known from (2)). The algebra SO(6) contains two different realizations of the SU(2) algebra for the spin s=1/2 doublet. By taking the sum of the two independent sets of SU(2) generators from (21) one can obtain the SU(2) generators of spin s=(1,0) multiplet, which generate the transformation of invariance of the FW equation (7). These operators can be presented in a form

$$\overrightarrow{\breve{s}} \equiv (\breve{s}^j) = (\breve{s}_{mn}) = \frac{1}{2}(\breve{\gamma}^2\breve{\gamma}^3 - \breve{\gamma}^0\breve{\gamma}^2\breve{C},\, \breve{\gamma}^3\breve{\gamma}^1 + \breve{i}\breve{\gamma}^0\breve{\gamma}^2\breve{C},\, \breve{\gamma}^1\breve{\gamma}^2 - \breve{i}), \tag{22}$$

where corresponding orts of the ERCD algebra in bosonic representation are given in (18).

The spin operators (22) of SU(2) algebra, which commute with the operator $\partial_0 + i\gamma^0\widehat{\omega}$ of the FW equation (7), can also be presented in the explicit form

$$\breve{s}^1 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc} 0 & 0 & i\hat{C} & 0 \\ 0 & 0 & -\hat{C} & 0 \\ -i\hat{C} & \hat{C} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|, \; \breve{s}^2 = \frac{1}{\sqrt{2}}\left|\begin{array}{cccc} 0 & 0 & \hat{C} & 0 \\ 0 & 0 & -i\hat{C} & 0 \\ -\hat{C} & i\hat{C} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|, \; \breve{s}^3 = \left|\begin{array}{cccc} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|. \tag{23}$$

Calculation of the Casimir operator for the SU(2) generators (23) gives the result $\overrightarrow{\breve{s}}^{\,2} = -1(1+1)\left|\begin{array}{cc} \mathrm{I}_3 & 0 \\ 0 & 0 \end{array}\right|$.

On the basis of the spin operators (22), (23) the bosonic spin (1,0) representation of the Poincaré group $\mathcal{P}$ is constructed. It is easy to show (after our consideration in [3]-[5] and above) that generators

$$p_0 = -i\gamma_0\widehat{\omega},\ p_n = \partial_n,\ j_{ln} = x_l\partial_n - x_n\partial_l + \breve{s}_{ln},\ j_{0k} = x_0\partial_k + i\gamma_0\{x_k\widehat{\omega} + \frac{\partial_k}{2\widehat{\omega}} + \frac{(\overrightarrow{\breve{s}} \times \overrightarrow{\partial})_k}{\widehat{\omega} + m}\}, \tag{24}$$

of group $\mathcal{P}$ commute with the operator of the FW equation (7) and satisfy the commutation relations of the Lie algebra of the group $\mathcal{P}$ in manifestly covariant form. The operators (24) generate in the space $\mathrm{H}^{3,4}$ another than the fermionic $\mathcal{P}^{\mathrm{F}}$-generators D-64 - D-67 of

[25] unitary $\mathcal{P}$ representation, i. e. the bosonic $\mathcal{P}^{\mathrm{B}}$ representation of the group $\mathcal{P}$, with respect to which the FW equation (7) is invariant. For the generators (24) the Casimir operators have the form:

$$p^{\mu}p_{\mu} = m^2,\ W^{\mathrm{B}} = w^{\mu}w_{\mu} = m^2\overrightarrow{\breve{s}}^2 = -1(1+1)m^2\begin{vmatrix} \mathrm{I}_3 & 0 \\ 0 & 0 \end{vmatrix}. \tag{25}$$

Hence, according to the Bargman-Wigner classification, we consider here the spin s=(1,0) representation of the group $\mathcal{P}$.

The corresponding bosonic spin s=(1,0) symmetries of the Dirac equation (8) can be found from the generators (24) with the help of the FW operator (6) in bosonic representation, i. e. $WV^{\pm}W^{-1}$.

More complete and detailed consideration of the bosonic symmetries of the FW and Dirac equation was given in [3]-[5].

7. Bosonic spin s=(1,0) multiplet solution of the Foldy-Wouthuysen and Dirac equations

Here as the next step in the FB duality investigation we consider the bosonic solution of the Dirac (FW) equation. A bosonic solution of the FW equation (7) is found completely similarly to the procedure of construction of standard fermionic solution. Thus, the bosonic solution is determined by some stationary diagonal complete set of operators of bosonic physical quantities for the spin s=(1,0)-multiplet in the FW representation, e. g., by the set "momentum-spin projection $\breve{s}^3$":

$$(\overrightarrow{p} = -\nabla,\ \breve{s}^3), \tag{26}$$

where the spin operators $\overrightarrow{\breve{s}}$ and $\breve{s}^3$ for the spin s=(1,0)-multiplet are given in (22), (23). The fundamental solutions of equation (7), which are the common eigen solutions of the bosonic complete set (26), have the form

$$\varphi^{-}_{\vec{k}\mathrm{r}}(t,\overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-ikx}\mathrm{d}_{\mathrm{r}},\ \varphi^{+}_{\vec{k}\acute{\mathrm{r}}}(t,\overrightarrow{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{ikx}\mathrm{d}_{\acute{\mathrm{r}}},\ kx = \omega t - \overrightarrow{k}\overrightarrow{x}, \tag{27}$$

where $\mathrm{d}_{\alpha} = (\delta^{\beta}_{\alpha})$ are the Cartesian orts in the space $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$, numbers $\mathrm{r} = (1,2)$, $\acute{\mathrm{r}} = (3,4)$ mark the eigen values $(+1,-1,0,\underline{0})$ of the operator $\breve{s}^3$ from (22), (23).

The bosonic solutions of equation (7) are the generalized states belonging to the space $\mathrm{S}^{3,4*}$; they form a complete orthonormalized system of bosonic states. Therefore, any bosonic physical state of the FW field ϕ from the dense in $\mathrm{H}^{3,4}$ manifold $\mathrm{S}^{3,4}$ (the general bosonic solution of the equation (7)) is uniquely presented in the form

$$\phi_{(1,0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k[\xi^{\mathrm{r}}(\overrightarrow{k})\mathrm{d}_{\mathrm{r}}e^{-ikx} + \xi^{*\acute{\mathrm{r}}}(\overrightarrow{k})\mathrm{d}_{\acute{\mathrm{r}}}e^{ikx}], \tag{28}$$

where $\xi(\overrightarrow{k})$ are the coefficients of expansion of the bosonic solution of the FW equation (7) with respect to the Cartesian basis (27). The relationships of amplitudes $\xi(\overrightarrow{k})$ with quantum-mechanical bosonic amplitudes $b(\overrightarrow{k})$ of probability distribution according to the eigen values of the stationary diagonal complete set of operators of quantum-mechanical bosonic s=(1,0)-multiplet are given by

$$\xi^1 = b^1,\ \xi^2 = -\frac{1}{\sqrt{2}}(b^3 + b^4),\ \xi^3 = -ib^2,\ \xi^4 = \frac{1}{\sqrt{2}}(b^3 - b^4);\quad b^{1,2,3,4}(\overrightarrow{k}) \equiv b^{+,-,0,\underline{0}}(\overrightarrow{k}), \tag{29}$$

where the 4 amplitudes $b^{1,2,3,4}(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes with the eigen values (+1,-1,0,$\underline{0}$) of the quantum-mechanical (1,0) multiplet $\breve{s}^3$ operator projections respectively (last eigen value $\underline{0}$ is related to the proper zero spin). And if $\phi_{(1,0)}(x) \in \mathrm{S}^{3,4}$, then the bosonic amplitudes $\xi(\overrightarrow{k})$ belong to the Schwartz complexvalued test function space too.

Moreover, the set $\{\phi_{(1,0)}(x)\}$ of solutions (28) is invariant just with respect to the unitary bosonic representation of the group $\mathcal{P}$, which is determined by the generators (24) and Casimir operators (25). Therefore, the Bargman-Wigner analysis of the Poincaré symmetry of the set $\{\phi_{(1,0)}(x)\}$ of solutions (28) completes the demonstration that it is the set of Bose-states $\phi_{(1,0)}$ of the field ϕ, i. e. the s=(1,0)-multiplet states. Hence, the existence of bosonic solutions of the FW equation is proved.

In the terms of quantum-mechanical momentum-spin amplitudes $b^\alpha(\vec{k})$ from (29), the bosonic spin (1,0)-multiplet solution $\psi = V^+\phi$ of the Dirac equation (8) is given by

$$\psi_{(1,0)}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\{e^{-ikx}[b^1 v_1^-(\overrightarrow{k}) - \frac{1}{\sqrt{2}}(b^3+b^4)v_2^-(\overrightarrow{k})] + e^{ikx}[ib^{*2}v_1^+(\overrightarrow{k}) + \frac{1}{\sqrt{2}}(b^{*3}-b^{*4})v_2^+(\overrightarrow{k})]\}, \tag{30}$$

where the 4-component spinors are the same as in the Dirac theory of fermionic doublet

$$v_{\mathrm{r}}^-(\overrightarrow{k}) = N\begin{vmatrix} (\widehat{\omega}+m)\mathrm{d_r} \\ (\overrightarrow{\sigma}\cdot\overrightarrow{k})\mathrm{d_r} \end{vmatrix},\ v_{\mathrm{r}}^+(\overrightarrow{k}) = N\begin{vmatrix} (\overrightarrow{\sigma}\cdot\overrightarrow{k})\mathrm{d_r} \\ (\widehat{\omega}+m)\mathrm{d_r} \end{vmatrix};\ N \equiv \frac{1}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\ \mathrm{d}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix},\ \mathrm{d}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}. \tag{31}$$

The well known (standard) Fermi solution of the Dirac equation for the spin s=1/2 doublet has the form

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k[e^{-ikx}a_{\mathrm{r}}^-(\overrightarrow{k})v_{\mathrm{r}}^-(\overrightarrow{k}) + e^{ikx}a_{\mathrm{r}}^+(\overrightarrow{k})v_{\mathrm{r}}^+(\overrightarrow{k})], \tag{32}$$

where the physical meaning of the amplitudes $a_{\mathrm{r}}^-(\overrightarrow{k})$, $a_{\mathrm{r}}^+(\overrightarrow{k})$ is explained in [57].

All the above given assertions about the FB duality of the spinor field are valid both in FW and PD representation, i. e. for both equations FW (7) and Dirac (8). The transition between FW and PD representations is fulfilled by the FW transformation (6).

8. Lagrangian for the Foldy-Wouthuysen equation

Before the Noether analysis of conservation laws we must consider the Lagrange approach (L-approach) for the spinor field $\phi(x)$ in the FW representation. The L-approach in this representation has been formulated first in [58], [59]. Representation of the operator $\widehat{\omega} \equiv \sqrt{-\triangle + m^2}$ in a form of the series over the Laplace operator $\triangle$ powers has been used. Author applied a nonstandard formulation of the least action principle in the terms of infinite order derivatives from the field functions. Mathematical correctness was not considered.

Therefore, we present here briefly a well-defined L-approach for the spinor field in the FW representation, which is based on the standard formulation of the least action principle. The quantum-mechanical rigged Hilbert space (both in the coordinate and momentum realizations of this space) is used, but the start is well-defined from momentum realization. In such realization rigged Hilbert space is given by

$$\widetilde{\mathrm{S}}^{3,4} \subset \widetilde{\mathrm{H}}^{3,4} \subset \widetilde{\mathrm{S}}^{3,4*}; \quad \widetilde{\mathrm{H}}^{3,4} = \mathrm{L}_2(\mathrm{R}^3_k) \otimes \mathrm{C}^{\otimes 4} = \{\phi = (\phi^\mu) : \mathrm{R}^3_k \to \mathrm{C}^{\otimes 4}; \int d^3k |f(t, \overrightarrow{k})|^2 < \infty\}\}, \tag{33}$$

Here R^3_k is the momentum operator $\overrightarrow{p}$ spectrum, which is canonically conjugated to the coordinate $\overrightarrow{x}$, $([x^j, p^l] = i\delta^{jl})$. Corresponding $\overrightarrow{x}$-realization is connected to (33) by 3-dimensional Fourier transformation. The alternative use of both realizations is based on the principle of heredity with classical and non-relativistic quantum mechanics of single mass point and with the mechanics of continuous media. The Lagrange function and the action (in alternative $\overrightarrow{x}$ or $\overrightarrow{k}$-realizations) are constructed in complete analogy with their consideration in the classical mechanics of a system with finite number of freedom degrees $q = (q_1, q_2, ...)$. The difference is only in the fact that here the continuous variable $\overrightarrow{k} \in \mathrm{R}^3_k$ is the carrier of freedom degrees.

In the $\overrightarrow{k}$-realization, where this analogy is maximally clear, the Lagrange function has the form

$$L = L(\widetilde{\phi}, \widetilde{\phi}^\dagger, \widetilde{\phi}_{,0}, \widetilde{\phi}^\dagger{}_{,0}) = \frac{i}{2}[\widetilde{\phi}^\dagger(\widetilde{\phi}_{,0} + i\gamma^0\widetilde{\omega}\widetilde{\phi}) - (\widetilde{\phi}^\dagger{}_{,0} - i\widetilde{\omega}\widetilde{\phi}^\dagger\gamma^0)\widetilde{\phi}], \tag{34}$$

and in the $\overrightarrow{x}$-realization this function can be found from (34) by the Fourier transformation. The Euler - Lagrange equations coincide with the FW equation in both realizations. For example, in the $\overrightarrow{k}$-realization the Euler - Lagrange equations

$$\frac{\delta W}{\delta \widetilde{\phi}^\dagger} \equiv \frac{\partial L}{\partial \widetilde{\phi}^\dagger} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \widetilde{\phi}^\dagger{}_{,0}} = 0, \quad \frac{\delta W}{\delta \widetilde{\phi}} \equiv \frac{\partial L}{\partial \widetilde{\phi}} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \widetilde{\phi}_{,0}} = 0, \tag{35}$$

coincide with the FW equation for the vectors $\widetilde{\phi} \in \widetilde{\mathrm{H}}^{3,4}$:

$$(i\partial_0 - \gamma^0\widetilde{\omega})\widetilde{\phi}(t, \overrightarrow{k}) = 0; \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^2 + m^2}, \ \overrightarrow{k} \in \mathrm{R}^3_k, \tag{36}$$

and with conjugated equation for $\widetilde{\phi}^\dagger$.

The well defined L-approach for the FW field becomes essentially actual problem after the construction in [60] of the quantum electrodynamics in the FW representation.

9. The Fermi-Bose conservation laws for the spinor field

Note briefly the FB conservation laws (CL) for the spinor field. It is preferable to calculate them in the FW (not local PD) representation too. In FW representation the Fermi spin s^{12} from (21) (together with the "boost spin") is the independent symmetry operator for the FW equation. The orbital angular momentum and pure Lorentz angular momentum (the carriers of external statistical degrees of freedom) are in this representations the independent symmetry operators too (one can find the corresponding independent spin and angular momentum symmetries in the PD representation for the Dirac equation too, but the corresponding operators are essentially nonlocal). Hence, one obtains 10 Poincaré and 12 additional (3 spin, 3 pure Lorentz spin, 3 angular momentum, 3 pure angular momentum) CL.

Therefore, in the FW representation one can find very easily the 22 fermionic and 22 bosonic CL. Separation into bosonic and fermionic set is caused by the existence of the FB symmetries and solutions. Indeed, if substitution of bosonic $\mathcal{P}$ generators q (24) and bosonic solutions (28) into the Noether formula $Q = \int d^3x\phi^\dagger(x)q\phi(x)$ is made, then automatically the bosonic CL for s=(1,0)-multiplet are obtained. The standard substitution of corresponding well known fermionic generators and solutions gives fermionic CL.

We illustrate briefly the difference in fermionic and bosonic CL on the example of corresponding spin conservation. For the fermionic spin

$$\overrightarrow{s} = (s_{23}, s_{31}, s_{12}) \equiv (s^\ell) = \frac{1}{2}\left| \begin{array}{cc} \overrightarrow{\sigma} & 0 \\ 0 & \overrightarrow{\sigma} \end{array} \right| \rightarrow s_z \equiv s^3 = \frac{1}{2}\left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| \quad (37)$$

and bosonic spin (22), (23) the CL are given by

$$S^{\mathrm{F}}_{mn} = \int d^3x\phi^\dagger(x)s_{mn}\phi(x) = \int d^3kA^\dagger(\vec{k})s_{mn}A(\vec{k})), \quad (38)$$

$$S^{\mathrm{B}}_{mn} = \int d^3x\phi^\dagger(x)\breve{s}_{mn}\phi(x) = \int d^3kB^\dagger(\vec{k})\breve{s}_{mn}B(\vec{k})), \quad (39)$$

where

$$A(\vec{k}) = \mathrm{column}(a^-_+,\, a^-_-,\, a^{*+}_-,\, a^{*+}_+),\; B(\vec{k}) = \mathrm{column}(b^1,\, b^2,\, b^{*3},\, b^{*4}). \quad (40)$$

We present these CL in terms of quantum-mechanical Fermi and Bose amplitudes. Such explicit quantum-statistical form have all integral conserved quantities.

10. Relationship between the fermionic and bosonic solutions and some interpretation

Adequate statistical quantum-mechanical sense of the coefficients $a^-_{\mathrm{r}}(\overrightarrow{k})$, $a^+_{\mathrm{r}}(\overrightarrow{k})$ in the expansion (32) over the basis solutions (31) of the Dirac equation is found identically only with the help of transition $\phi(x) = V^-\psi(x)$ (9), (6) to the FW representation [49]. Indeed, the statistical sense of the FW field $\phi(x)$ is evidently related to the statistical sense of the particle-antiparticle doublet in relativistic canonical quantum mechanics [25], [61] of this doublet. It is shown in [25] that

$$\phi = \left| \begin{array}{c} \phi^- \\ 0 \end{array} \right| + \left| \begin{array}{c} 0 \\ \phi^{*+} \end{array} \right|, \quad (41)$$

where $\phi^\mp(x)$ are the relativistic quantum-mechanical wave functions of the particle-antiparticle doublet.

Solution of the FW equation (7) expanded over the eigenvectors of quantum-mechanical fermionic stationary diagonal complete set of operators (momentum $\overrightarrow{p}$, projection s^3 of the spin $\overrightarrow{s}^{\mathrm{quant.-mech.}}$ and sign of the charge $g = -\gamma^0$) has the form

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k\{e^{-ikx}[a^-_+(\overrightarrow{k})\mathrm{d}_1 + a^-_-(\overrightarrow{k})\mathrm{d}_2] + e^{ikx}[a^{*+}_-(\overrightarrow{k})\mathrm{d}_3 + a^{*+}_+(\overrightarrow{k})\mathrm{d}_4]\}, \quad (42)$$

where the coefficients of expansion $a^-_+(\overrightarrow{k})$, $a^-_-(\overrightarrow{k})$, $a^+_-(\overrightarrow{k})$, $a^+_+(\overrightarrow{k})$ have the meaning of statistical quantum-mechanical amplitudes of probability distribution over the eigen values of the above mentioned fermionic stationary complete set of operators. The 4-columns $\mathrm{d}_\alpha = (\delta^\beta_\alpha)$ are the Cartesian orts in the space $\mathrm{C}^{\otimes 4} \subset \mathrm{H}^{3,4}$. In order to obtain the most adequate and obvious statistical quantum-mechanical interpretation of the amplitudes and solutions the spin projection operator in the complete set (momentum $\overrightarrow{p}$, projection s^3 of the spin $\overrightarrow{s}^{\text{quant.-mech.}}$ and sign of the charge $g=-\gamma^0$) is taken in the quantum-mechanical form [61]

$$\overrightarrow{s}^{\text{quant.-mech.}}=\frac{1}{2}\begin{vmatrix}\overrightarrow{\sigma} & 0\\ 0 & -C\overrightarrow{\sigma}C\end{vmatrix}\rightarrow s_z^{\text{quant.-mech.}}\equiv s^3=\frac{1}{2}\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{vmatrix}. \tag{43}$$

and not in the canonical field theory form (37). The statistical sense of the amplitudes is conserved in the solution ($\psi(x)=V^+\phi(x)$ (9), (6))

$$\psi(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\{e^{-ikx}[a_+^-(\overrightarrow{k})v_1^-(\overrightarrow{k})+a_-^-(\overrightarrow{k})v_2^-(\overrightarrow{k})]+e^{ikx}[a_-^{*+}(\overrightarrow{k})v_1^+(\overrightarrow{k})+a_+^{*+}(\overrightarrow{k})v_2^+(\overrightarrow{k})]\}, \tag{44}$$

of the Dirac equation (8) in its standard local representation. The amplitudes $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$, $a_-^+(\overrightarrow{k})$, $a_+^+(\overrightarrow{k})$ in the fermionic solutions (42) and (44) of the FW and Dirac equations are one and the same. Thus, $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the particle with charge -e and eigen values of spin projection +1/2 and -1/2; $a_-^+(\overrightarrow{k})$, $a_+^+(\overrightarrow{k})$ are the quantum-mechanical momentum-spin amplitudes of the antiparticle with charge +e and eigen values of spin projection -1/2 and +1/2, respectively.

Statistical quantum mechanical sense of bosonic amplitudes $b^\alpha(\overrightarrow{k})$ of bosonic solution (30) of the Dirac equation (8) is found similarly and is explained in the section 5 in the course of this solution construction.

Relationship between the fermionic $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$, $a_-^+(\overrightarrow{k})$, $a_+^+(\overrightarrow{k})$ and bosonic $b^{1,2,3,4}(\overrightarrow{k})\equiv b^{+,-,0,\underline{0}}(\overrightarrow{k})$ amplitudes in one and the same (arbitrarily fixed) physical state of FB dual field ψ is given by the unitary operator U in the form:

$$\begin{vmatrix}a_+^-\\a_-^-\\a_-^+\\a_+^+\end{vmatrix}=\frac{1}{\sqrt{2}}\begin{vmatrix}\sqrt{2}&0&0&0\\0&0&-1&-1\\0&-i\sqrt{2}&0&0\\0&0&1&-1\end{vmatrix}\begin{vmatrix}b^+\\b^-\\b^0\\b^{\underline{0}}\end{vmatrix},\quad \begin{vmatrix}b^+\\b^-\\b^0\\b^{\underline{0}}\end{vmatrix}=\frac{1}{\sqrt{2}}\begin{vmatrix}\sqrt{2}&0&0&0\\0&0&i\sqrt{2}&0\\0&-1&0&1\\0&-1&0&-1\end{vmatrix}\begin{vmatrix}a_+^-\\a_-^-\\a_-^+\\a_+^+\end{vmatrix}. \tag{45}$$

Relationships (45) follow directly from the comparison of the solutions (30) and (44).

Note that the set of fermionic solutions $\{\psi^{\text{F}}\}$ (44) of the Dirac equation is invariant with respect to the well known induced fermionic $\mathcal{P}^{\text{F}}$ representation of the Poincaré group $\mathcal{P}$ [62], see also formula (19) in the paper [4]). The set of bosonic solutions $\{\psi^{\text{B}}\}$ (30) of the Dirac equation is invariant with respect to the induced bosonic $\mathcal{P}^{\text{B}}$ representation of the Poincaré group $\mathcal{P}$ (formula (21) in the paper [5]). However, the relationships (45) between the fermionic $a_+^-(\overrightarrow{k})$, $a_-^-(\overrightarrow{k})$, $a_-^+(\overrightarrow{k})$, $a_+^+(\overrightarrow{k})$ and bosonic $b^{1,2,3,4}(\overrightarrow{k})\equiv b^{+,-,0,\underline{0}}(\overrightarrow{k})$ amplitudes do not change in any inertial frame of references.

11. Conclusions

The property of the Fermi-Bose duality of the Dirac equation is proved on the three levels: bosonic symmetries, bosonic solutions and corresponding conservation laws. The role of the proper ERCD algebra SO(8) in the proof of this assertions is demonstrated.

The 64 dimensional ERCD and 29 dimensional proper ERCD algebras put into consideration in [2]-[5] are the useful generalizations of the standard 16 dimensional CD algebra.

Their application enabled us to prove the existence of additional bosonic symmetries, solutions and conservation laws for the spinor field, for the Foldy-Wouthuysen and the Dirac equations. The investigation of the spinor field in the Foldy-Wouthuysen representation has the independent meaning and purpose. This representation is of interest itself in connection with the recent result [60] of V. Neznamov, who developed the formalism of quantum electrodynamics in the Foldy-Wouthuysen representation, see also the results in [63]. The property of the Fermi - Bose duality of the Dirac equation (both in the Foldy-Wouthuysen and the Pauli-Dirac representations), which proof was started in [2]-[5], where the bosonic symmetries of this equation were found, is demonstrated here on the next level – on the level of existence of the spin (1,0) bosonic solutions of the equation under consideration and corresponding bosonic conservation laws. Similarly, the fermionic spin s=1/2 properties for the Maxwell equations both with nonzero and zero mass can be proved (see e. g. the procedure given in [31]).

In any case we do not change the main well known postulates and theory of the Fermi - Bose statistics. Our results have another, new principal, meaning. In our approach the Fermi - Bose duality of the spinor field found in [26] is proved by different method. We present the examples of the existence of the bosonic symmetries (section 6) and solutions (section 7) of the Dirac equation with nonzero mass together with obtaining the bosonic conservation laws (section 9) for the spinor field. It opens the new possibilities of the Dirac equation application for the description of bosonic states. Thus, the property of the Fermi - Bose duality of the Dirac equation proven in our publications [2]-[5] and here does not break the Fermi statistics for fermions (with the Pauli principle) and Bose statistics for bosons (with Bose condensation). We also never mixed the Fermi and Bose statistics between each other. Our assertion is following. One can apply with equal success both Fermi and Bose statistics for one and the same Dirac equation and one and the same spinor field, i. e. the Dirac equation can describe both fermionic and bosonic states.

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