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HYDRODYNAMIC LIMIT OF THE BOLTZMANN-MONGE-AMPERE SYSTEM

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ABSTRACT. This paper investigates the hydrodynamic limit of the Boltzmann-Monge-Ampere system in the so-called quasineutral regime and more precisely the convergence of the Boltzmann-Monge-Ampere system to the Euler equation by using the relative entropy method.

Keywords: Boltzman equation, Monge-Ampère equation, Euler equations of the incompressible fluid.

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1. Introduction

The behavior of dilute charged particules when the magnetic forces are neglected is described by the Vlasov-Poisson-Boltzman (VPB) system defined by

$$\partial_t f^\varepsilon + \xi.\nabla_x f^\varepsilon + \nabla_x \varphi^\varepsilon.\nabla_\xi f^\varepsilon = Q(f^\varepsilon, f^\varepsilon) \quad (1)$$

$$\varepsilon^2 \triangle \varphi^\varepsilon = \rho^\varepsilon - 1 \quad (2)$$

where

$$\rho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon(t,x,\xi) d\xi \quad (3)$$

and $f^\varepsilon(t,x,\xi) \geq 0$ is the electronic density at time $t \geq 0$ point $x \in [0,1]^d = \mathbb{T}^d$, and with a velocity $\xi \in \mathbb{R}^d$. The periodic electric potential φ^ε is coupled with f^ε through the Poisson equation (2).The quantities $\varepsilon > 0$ and $Q(f^\varepsilon, f^\varepsilon)$ denote respectively the vacum electric permitivity and the Boltzman collision integral. This latter, is given by (see([4],[9]))

$$Q(f^\varepsilon, f^\varepsilon)(t,x,\xi) = \int\int_{S_+^{d-1}\times\mathbb{R}^d} \left((f^\epsilon)' (f_1^\varepsilon)' - f^\varepsilon f_1^\varepsilon\right) b\,(\xi - \xi_1, \sigma)\, d\sigma d\xi_1,$$

where the terms f_1^ε, $(f^\varepsilon)'$ and $(f_1^\varepsilon)'$ define, respectively the values $f^\varepsilon(t,x,\xi_1)$, $f^\varepsilon(t,x,\xi')$ and $f^\varepsilon(t,x,\xi_1')$ with ξ' and ξ_1' given in terms of ξ, $\xi_1 \in \mathbb{R}^d$,

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and $\sigma \in S_{+}^{d-1} = \left\{\sigma \in S^{d-1}/\sigma.\xi \geq \sigma.\xi_1\right\}$ by

$$\xi' = \frac{\xi+\xi_1}{2} + \frac{\xi-\xi_1}{2}\sigma,\ \ \xi_1' = \frac{\xi+\xi_1}{2} - \frac{\xi-\xi_1}{2}\sigma.$$

The VPB system, has been investigated many authors. In [6] DiPerna and Lions showed the existence of renormalized solution. Desvilletes and Dolbeault [8] are interested in the long-time behavior of the weak solutions of the VPB system for the initial boundary problem. In [10] Guo established the global existence of smooth solutions to the VPB system in periodic boundary condition case. For more references for this subject, Boltzmann equation or Vlasov–Poisson system, one can see ([1],[4],[5],[7],[8],[12], [13])

Consider the Boltzman-Monge-Ampere system (BMA)

$$\partial_t f^\varepsilon + \xi.\nabla_x f^\varepsilon + \nabla_x \varphi^\varepsilon.\nabla_\xi f^\varepsilon = Q(f^\varepsilon, f^\varepsilon), \tag{4}$$

$$\det\left(\mathbb{I}_d + \varepsilon^2 D^2\varphi^\varepsilon\right) = \rho^\varepsilon, \tag{5}$$

where $\mathbb{I}_d$ is the identity matrix.

By linearising the determinant about the identity matrix $\mathbb{I}_d$,

$$\det\left(\mathbb{I}_d + \varepsilon^2 D^2\varphi^\varepsilon\right) = 1 + \varepsilon^2\triangle\varphi^\varepsilon + O\left(\varepsilon^4\right).$$

Formally, as ε is small, the BVP and BMA systems asymptotically approach each other up to order $O\left(\varepsilon^4\right)$.

In [11] L. Hsiao and al. studied the convergence of the VPB system to the Incompressible Euler Equations. It is clear that the case $Q(f^\varepsilon, f^\varepsilon) = 0$ corresponds to the the Vlasov-Monge-Ampère(VMA). This problem, was been considered by Y. Bernier and Grégoire [2]. They showed that weak solution of VMA converge to a solution of the incompressible Euler equations when the parameter ε goes to 0.

This article studies the hydrodynamical limit of the BMA.

First, Note that

$$\int_{\mathbb{R}^d} Q(f^\varepsilon, f^\varepsilon)d\xi = \int_{\mathbb{R}^d} \xi_i Q(f^\varepsilon, f^\varepsilon)d\xi = \int_{\mathbb{R}^d} |\xi|^2 Q(f^\varepsilon, f^\varepsilon)d\xi = 0,\ i = 1, 2, ..., d.$$

The conservation of total energy reads

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |\xi|^2 f^\varepsilon(t,x,\xi)\,dxd\xi + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(t,x)|^2\,dx = E_0 \tag{6}$$

where

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |\xi|^2 f_0^\varepsilon(x,\xi) + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} |\nabla\varphi^\varepsilon(0,x)|^2\,dx$$

Denote

$$J^\varepsilon(t,x) = \int_{\mathbb{R}^d} \xi f^\varepsilon(t,x,\xi)\,d\xi. \tag{7}$$

The conservation laws of mass and momentum are

$$\partial_t\rho^\varepsilon + \nabla.J^\varepsilon = 0 \tag{8}$$

and

$$\partial_t J^\varepsilon + \nabla_x.\int_{\mathbb{R}^d}(\xi\otimes\xi)f^\varepsilon d\xi + \nabla\varphi^\varepsilon + \frac{\varepsilon}{2}\nabla\left(|\nabla\varphi^\varepsilon|^2\right) - \varepsilon\nabla.\left(\nabla\varphi^\varepsilon\otimes\nabla\varphi^\varepsilon\right) = 0 \tag{9}$$

with

$$\varepsilon^2\triangle\varphi^\varepsilon = \rho^\varepsilon - 1.$$

The periodic boundary problem of Euler equations to the incompressible fluid is defined by

$$\nabla .u = 0,\ t > 0,\ x \in \mathbb{T}^d \tag{10}$$

$$\partial_t u + (u.\nabla)\, u + \nabla p = 0\, t > 0,\ x \in \mathbb{T}^d \tag{11}$$

$$u\,(0,x) = u_0\,(x) \in \mathcal{H}^s, \tag{12}$$

where the function space $\mathcal{H}^s$ is given by $\mathcal{H}^s = \left\{ u \in H^s\left(\mathbb{T}^d\right),\ \nabla .u = 0 \right\}$. At the end of this introduction define the modulated energy functional

$$H^\varepsilon\,(t) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |\xi - u\,(x)|^2 f^\varepsilon\,(t,x,\xi)\, dxd\xi + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} \left|\nabla\varphi^2\,(t,x)\right|^2 dx.$$

Elementary calculations lead to

$$\begin{aligned}\tfrac{d}{dt}H_\varepsilon\,(t) = \quad & -\textstyle\int d\,(u)\,(t,x) : (\xi - u\,(t,x)) \otimes (\xi - u\,(t,x))\, f\,(t,x,\xi)\, dxd\xi \\ & +\varepsilon \int d\,(u)\,(t,x) : \nabla\varphi\,(t,x) \otimes \nabla\varphi\,(t,x)\, dx \\ & + \int A\,(u)\,(t,x)\,.\,(\rho\,(t,x)\, u\,(t,x) - J\,(t,x))\, dx\end{aligned}$$

where $d\,(u)$ is the symmetrized gradient of u defined by

$$d_{ij}\,(u) = \frac{1}{2}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right)$$

and $A\,(u)$ is the acceleration operator given by

$$A\,(u) = \partial_t u + (u.\nabla)\, u.$$

It follows that

$$\frac{d}{dt}H_\varepsilon\,(t) \le 2\,\|d\,(u\,(t))\|\, H_\varepsilon\,(t) + \int A\,(u)\,(\rho^\varepsilon u - J^\varepsilon)\, dx,$$

where $\|d\,(u\,(t))\|$ is the supremum in x of the spectral radius of $d\,(u)\,(t,x)$. Integrating in t,

$$\begin{aligned} H_\varepsilon\,(t) \le H_\varepsilon\,(0) \exp\left(\textstyle\int_0^t 2\,\|d\,(u\,(s))\|\, ds\right) \\ + \textstyle\int_0^t \exp\left(\int_s^t 2\,\|d\,(u\,(\theta))\|\, d\theta\right)\left(\int A\,(u)\,(s,x)\,.\,(\rho^\varepsilon u - J^\varepsilon)\,(s,x)\right) dsdx.\end{aligned}$$

In the case $u = 0$, the total energy bound is recovered

$$\frac{1}{2}\int |\xi|^2 f\,(t,x,\xi)\, dxd\xi + \frac{\varepsilon}{2}\int |\nabla\varphi\,(t,x)|^2\, dx \le E_0.$$

2. Main results

Theorem 2.1. *Let* $0 < T < T^*$ *and* u_0 *in* $\mathcal{H}^s\left(s > 1 + \frac{d}{2}\right)$, $\mathbb{Z}^d$ periodic in x. Assume that $f_0^\varepsilon\,(x,\xi) \ge 0$ to be smooth, $\mathbb{Z}^d$ periodic in x, and f_0^ε decays fast as $\xi \to \infty$. In addition, assume that

$$\int_{\mathbb{R}^d} f_0^\varepsilon\,(x,\xi)\, d\xi = 1 + o\left(\varepsilon^{\frac{1}{2}}\right),\ \text{as } \varepsilon \to 0,$$

in the strong sense of the space $H^{-1}\left(\mathbb{T}^d\right)$ and

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |\xi - u_0\,(x)|^2 f_0^\varepsilon\,(x,\xi)\, dxd\xi + \frac{\varepsilon}{2}\int_{\mathbb{T}^d} \left|\nabla\phi^2\,(0,x)\right|^2 dx \to 0 \text{ as } \varepsilon \to 0.$$

Let f^ε be any nonnegative smooth solution of (4)-(5). Then, up to the extraction of a subsequence, the current J^ε converges weakly to the unique solution $u(x,t)$ of the Euler equations (10)-(11)-(12). Moreover, the divergence free part of f converges to u in $L^\infty\left([0,T], L^2\left(\mathbb{T}^d\right)\right)$.

3. Proof of the theorem 2.1

The proof will be given only for $d=2$. For $d>2$, one can operate differently. We recall (see [3]) a result of existence of smooth solutions to (5) for Hölder continuous, positive right-hand sides. This result gives us the a priori bound needed to prouve the Lemma 3.1.

Theorem 3.1. *Let ρ be a probability density over $\mathbb{T}^d$ such that $m \leq \rho \leq M$ for some pair $(m,M)>0$. Let $u:\mathbb{T}^d \to \mathbb{R}$ be solution of $\det\left(\mathbb{I}_d + D^2u\right)=\rho$, with $u+|.|^2/2$ convex. Then there exists a non- decreasing function $\mathcal{H}_{m,M}$ such that, $\|u\|_{C^{2,\alpha}\left(\mathbb{T}^d\right)} \leq \mathcal{H}_{m,M}\left(\|\rho\|_{C^\alpha\left(\mathbb{T}^d\right)}\right)$.*

In the sequel the following two Lemmas are needed

Lemma 3.1. *Under the hypothesis of the Theorem 2.1, one has up to the extraction of a sequence, ρ^ε converges to 1 in $C^0\left([0,T], \mathcal{D}'\left(\mathbb{T}^d\right)\right)$, the current J^ε converges to J in $L^\infty\left([0,T], \mathcal{D}'\left(\mathbb{T}^d\right)\right)$, $J \in L^\infty\left([0,T], L^2\left(\mathbb{T}^d\right)\right)$, and the divergence free parts of J^ε converges to J in $C^0\left([0,T], \mathcal{D}'\left(\mathbb{T}^d\right)\right)$.*

Proof. For $d=2$, one decompose

$$\det\left(I+\varepsilon^2 D^2\varphi^\varepsilon\right) = 1+\varepsilon^2 \triangle\varphi^\varepsilon + \varepsilon^4 \det D^2\varphi^\varepsilon. \tag{13}$$

For $\eta \in C_0^\infty\left(\mathbb{T}^d\right)$,

$$\begin{aligned}\int\left(\rho^\varepsilon(t,x)-1\right)\eta(x)\,dx &= \int\left(\det\left(I+\varepsilon^2D^2\varphi^\varepsilon\right)-1\right)\eta(x)\,dx\\ &= \int\left(\varepsilon^2\triangle\varphi^\varepsilon+\varepsilon^4\det D^2\varphi^\varepsilon\right)\eta(x)\,dx.\end{aligned}$$

But

$$\det D^2\varphi^\varepsilon = \frac{1}{2}\mathrm{tr}\left(\left(\mathrm{cof}D^2\varphi^\varepsilon\right)D^2\varphi^\varepsilon\right) = \frac{1}{2}\mathrm{div}\left(\left(\mathrm{cof}D^2\varphi^\varepsilon\right)\nabla\varphi^\varepsilon\right),$$

it follows by integrating by parts that

$$\begin{aligned}\int\left(\rho^\varepsilon(t,x)-1\right)\eta(x)\,dx &= \varepsilon^2\int\nabla\varphi^\varepsilon\nabla\eta(x)\,dx + \frac{\varepsilon^4}{2}\int\mathrm{div}\left(\left(\mathrm{cof}D^2\varphi^\varepsilon\right)\nabla\varphi^\varepsilon\right)\eta(x)\,dx\\ &= \varepsilon^2\int\nabla\varphi^\varepsilon\nabla\eta(x)\,dx - \frac{\varepsilon^4}{2}\int\left(\mathrm{cof}D^2\varphi^\varepsilon\right)\nabla\varphi^\varepsilon.\nabla\eta(x)\,dx.\end{aligned}$$

Thus, by the Hölder inequality one has

$$\begin{aligned}\int\left(\rho^\varepsilon(t,x)-1\right)\eta(x)\,dx \leq \varepsilon^{\frac{3}{2}}\left(\varepsilon\int|\nabla\varphi^\varepsilon|^2\right)^{\frac{1}{2}}\left(\int|\nabla\eta|^2\right)^{\frac{1}{2}}+\\ +\frac{\varepsilon^4}{2}\left\|\mathrm{cof}D^2\varphi^\varepsilon\right\|_{L^2}\|\nabla\varphi^\varepsilon\|_{L^2}\|\nabla\eta\|_{L^2}.\end{aligned}$$

From Theorem 3.1, one can deduce that

$$\left\|\mathrm{cof}D^2\varphi^\varepsilon\right\|_{L^2} \leq C\varepsilon^{-2},$$

So, by the conservation of the energy

$$\begin{aligned}\left|\int\left(\rho^{\varepsilon}(t,x)-1\right)\eta(x)\,dx\right| &\leq C_0\varepsilon^{\frac{3}{2}}\|\nabla\eta\|_{L^2}+C.\varepsilon^{2}\|\nabla\varphi^{\varepsilon}\|_{L^2}\|\nabla\eta\|_{L^2}.\\ &\leq C_0\varepsilon^{\frac{3}{2}}\|\nabla\eta\|_{L^2}+C.\varepsilon^{\frac{3}{2}}\|\nabla\eta\|_{L^2}\\ &\leq C\varepsilon^{\frac{3}{2}}\|\nabla\eta\|_{L^2}.\end{aligned}$$

Which shows that $\rho^{\varepsilon}\to 1$ in $C^0\left([0,T],\mathcal{D}'\left(\mathbb{T}^d\right)\right)$.

Moreover, using the total energy equality (6)

$$\int|J^{\varepsilon}(t,x)|\,dx\leq\left(\int\int|\xi|^2 f^{\varepsilon}(t,x,\xi)\,dxd\xi\right)^{\frac{1}{2}}\left(\int\int f^{\varepsilon}(t,x,\xi)\,dxd\xi\right)^{\frac{1}{2}}\leq C. \tag{14}$$

Thus J^{ε} is bounded in $L^{\infty}\left([0,T],L^1\left(\mathbb{T}^d\right)\right)$. Up to extracting a subsequence, assume that J^{ε} has a limit J in the sens of (Radon) measures on $[0,T]\times\dfrac{\mathbb{R}^d}{\mathbb{Z}^d}=\mathbb{T}^d$. Finally, define as in [11], for each non-negative function $z(t)\in C^0([0,T])$, the convex functional of a (Radon) measure

$$\begin{aligned}K\left(\rho^{\varepsilon},J^{\varepsilon}\right)&=\int_0^T\frac{|J^{\varepsilon}(t,x)|^2}{2\rho^{\varepsilon}(t,x)}z(t)\,dxdt\\ &=\sup_b\int_0^T\left\{-\frac{1}{2}|b(t,x)|^2\rho^{\varepsilon}(t,x)+b(t,x)\,J^{\varepsilon}(t,x)\right\}z(t)\,dt.\end{aligned}$$

where b belongs to the space of all continuous functions from $[0,T]\times\mathbb{T}^d$ to $\mathbb{R}^d$. From (14) and since the functional K is lower semi-continuous with respect to the convergence of measure, it follows that

$$\int_0^T z(t)\left(\int|J(t,x)|^2\,dx\right)dt\leq C\int_0^T z(t)\,dt,$$

which means that $J\in L^{\infty}\left([0,T],L^2\left(\mathbb{T}^d\right)\right)$.

From (8) and (7), one writes

$$\partial_t\rho^{\varepsilon}=\partial_t\det\left(\mathbb{I}+\varepsilon^2D^2\varphi^{\varepsilon}\right)=-\nabla J^{\varepsilon},$$

thus

$$\nabla J^{\varepsilon}=-\varepsilon\partial_t\triangle\phi^{\varepsilon}-\varepsilon^2\partial_t\det D^2\phi^{\varepsilon}.$$

Next, note that, for $\eta\in C_0^{\infty}\left(\mathbb{T}^d\right)$,

$$\int\nabla J^{\varepsilon}\eta(x)\,dx=-\varepsilon\int\partial_t\left(\triangle\phi^{\varepsilon}\eta\right)dx-\varepsilon^2\int\partial_t\det D^2\phi^{\varepsilon}\eta dx,$$

thus J is divergence free in x in the sense of distribution. By (7), it follows that $\partial_t J$ is bounded in $L^{\infty}\left([0,T],D'\left(\mathbb{T}^d\right)\right)$. So, we obtain that up to the exraction of a subsequance, $J\in C^0\left([0,T],L^2\left(\mathbb{T}^d\right)-w\right)$.

In the same way , the divergence -free part of J^{ε} converges to J in $C^0\left([0,T],D'\left(\mathbb{T}^d\right)\right)$.

□

Since J^{ε} converges to J, it remains to show that $J=u$ in $L^{\infty}\left([0,T],L^2\left(\mathbb{T}^d\right)\right)$. For this, it suffies to use the next Lemma (see [11])

Lemma 3.2. *Let u be the unique solution of the Euler equations (10)-(11) with initial datum and u_0 and the hypotheses of theorem 1 hold. Then, for any $t \in (0,T]$, $H^{\varepsilon}(t) \to 0$ as $\varepsilon \to 0$.*

To end the proof of the Theorem 2.1 , let

$$h^{\varepsilon}(t) = \int \frac{|J^{\varepsilon}(t,x) - \rho^{\varepsilon}(t,x)\, u(t,x)|^2}{2\rho^{\varepsilon}(t,x))} dx. \tag{15}$$

With b belongs to the space of all continuous functions from $\mathbb{T}^d$ to $\mathbb{R}^d$. By the Cauchy-Shwarz inequality,

$$h^{\varepsilon}(t) \leq \frac{1}{2} \int |\xi - u(t,x)|^2 f^{\varepsilon}(t,x,\xi)\, dx d\xi \leq H^{\varepsilon}(t).$$

Since $\rho^{\varepsilon} \to 1$, $J^{\varepsilon} \to J$ and from the convexity of the functional defined by (15), one concludes that

$$\int |J(t,x) - u(t,x)|^2 dx \leq 2 \lim_{\varepsilon \to 0} h^{\varepsilon}(t) \leq 2 \lim_{\varepsilon \to 0} H^{\varepsilon}(t) = 0.$$

This finishes the proof of Theorem 2.1.

4. Conclusions

The convergence of the BMA system to the incompressible Euler equations has been shown by using the entropy method. The proof is based on the decomposition 13, which is valid only in 2d. For $d > 2$ one operates differently.

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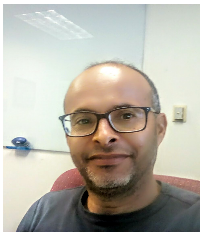

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