GLOBAL COLOR CLASS DOMINATION PARTITION OF A GRAPH

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ABSTRACT. Color class domination partition was suggested by E. Sampathkumar and it was studied in [1]. A proper color partition of a finite, simple graph $G$ is called a color class domination partition (or cd-partition) if every color class is dominated by a vertex. This concept is different from dominator color partition introduced in [2], [3] where every vertex dominates a color class. Suppose $G$ has no full degree vertex (that is, a vertex which is adjacent with every other vertex of the graph). Then a color class may be independent from a vertex outside the class. This leads to Global Color Class Domination Partition. A proper color partition of $G$ is called a Global Color Class Domination Partition if every color class is dominated by a vertex and each color class is independent of a vertex outside the class. The minimum cardinality of a Global Color Class Domination Partition is called the Global Color Class Domination Partition Number of $G$ and is denoted by $\chi_{gcd}(G)$. In this paper a study of this new parameter is initiated and its relationships with other parameters are investigated.

Keywords: Color class domination partition, Global color class domination partition, Dominator color class partition, Global color class domination number.

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1. Introduction

Let $G$ be a finite, simple and undirected graph. A proper color partition of $G$ is a partition of $V(G)$ into independent sets of $G$. Several types of proper color partitions have been studied earlier. One of them is dominator coloring [2], [3]. In this coloring, each vertex dominates a color class. The minimum cardinality of a dominator color class partition is denoted by $\chi_d(G)$. A slight variation of this coloring is called a color class domination partition. In this partition, each color class is dominated by a vertex. In graphs without any full degree vertex, Global counter part of this concept can be defined. In this paper this new concept is introduced and studied.

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681
2. Global color class domination partition

**Definition 2.1.** Let $G$ be a finite, simple and undirected graph. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a proper color partition of $G$. $\Pi$ is called a global color class domination partition if for every color class $V_i$, there exists a vertex $u_i$ which dominates $V_i$ and there exists a vertex $w_i \notin V_i$ which is independent of $V_i$, $1 \leq i \leq k$. The minimum cardinality of a Global color class domination partition is called the Global color class domination number of $G$ and is denoted by $\chi_{gcd}(G)$.

If $G$ does not have a full degree vertex, then $\Pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is a global color class domination partition of $G$.

3. $\chi_{gcd}(G)$ for Standard Graphs

(1) $\chi_{gcd}(K_n) = n$.
(2) $\chi_{gcd}(D_{r,s}) = 4$, $r, s \geq 1$.
(3) $\chi_{gcd}(K_{m,n}) = 4$, where $m, n \geq 2$.
(4) $\chi_{gcd}(P_n) = \begin{cases} 4 & \text{if } n = 4, 5 \\ \gamma_{cd}(P_n) & \text{if } n \geq 6 \end{cases}$
\begin{align*}
&\chi_{gcd}(P_2) \text{ and } \chi_{gcd}(P_3) \text{ do not exist.} \\
&\chi_{gcd}(C_n) = \begin{cases} 4 & \text{if } n = 4 \\ 5 & \text{if } n = 5 \\ \chi_{cd}(C_n) & \text{if } n \geq 6 \end{cases}
\end{align*}
\begin{align*}
&\chi_{gcd}(C_3) \text{ does not exist.} \\
&\chi_{gcd}(P) = 5 \text{ where } P \text{ is the Petersen graph.}
\end{align*}

Here $\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9, v_{10}\}\}$ is a minimum global color class domination partition of $P$.

4. Main Results

**Theorem 4.1.** $\max\{\chi_{cd}(G), \frac{\gamma_{cd}(G)}{2}\} \leq \chi_{gcd}(G)$

**Proof.** Let $\Pi$ be a minimum global color class domination partition of $G$. Then $\Pi$ is a color class domination partition of $G$. Therefore $\chi_{cd}(G) \leq \chi_{gcd}(G)$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a minimum global color partition of $G$. Then there exist $x_1, x_2, \ldots, x_k$ such that $x_i$ dominates $V_i$, $(1 \leq i \leq k)$ and $y_1, y_2, \ldots, y_k$ such that $y_i$ is independent of $V_i$, $(1 \leq i \leq k)$.
Let $S = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$. Then $S$ is a global dominating set of $G$. Therefore
\[\gamma_g(G) \leq |S| \leq 2k, \quad \frac{\gamma_g(G)}{2} \leq k = \gamma_{cd}(G).\] Therefore $\max\{\gamma_{cd}(G), \frac{\gamma_g(G)}{2}\} \leq \gamma_{cd}(G)$. \hfill \Box

**Remark 4.1.** Let $G = P_6$. $\gamma_g(G) = 2$. $\chi_{cd}(G) = \frac{n+2}{2} = 4$. Therefore $\max\{\gamma_{cd}(G), \frac{\chi_{cd}(G)}{2}\} = 4 = \chi_{cd}(G)$.

**Theorem 4.2.** \(\min\{\Delta(G), \frac{n}{n-1-\delta(G)}\} \leq \chi_{cd}(G)\)

**Proof.** Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a minimum global color partition of $G$. Since each $V_i$ is dominated by a vertex say $x_i$.

\[\deg(x_i) \geq |V_i|, \quad (1 \leq i \leq k).\]

Therefore $|V_i| \leq \Delta(G)$, $(1 \leq i \leq k)$. That is, $\max\{|V_i|\} \leq \Delta(G)$. Since each $V_i$ is independent of some $y_i$, $(1 \leq i \leq k)$, each $V_i$ is dominated by $y_i$ in $\overline{G}$, $(1 \leq i \leq k)$, therefore $|V_i| \leq \deg_{\overline{G}}(y_i) \leq \Delta(\overline{G})$.

\[\delta(G) \leq n - \Delta(\overline{G}) - 1. \quad \Delta(\overline{G}) \leq n - \delta(G) - 1.\]

Therefore $|V_i| \leq \min\{\Delta(G), |n - \delta(G) - 1\}$, $(1 \leq i \leq k)$. $n = |V_1| + |V_2| + \ldots + |V_k| \leq \min\{|V_1|\} + \min\{|V_2|\} + \ldots + \min\{|V_k|\}$. $n = k \min\{\Delta(G), n - \delta(G) - 1\}$. $\min\{\Delta(G), \frac{n}{n-1-\delta(G)}\} \leq k = \chi_{cd}(G)$.

**Remark 4.2.** The above bound is sharp. For: Let $G = P_6$. $\chi_{cd}(G) = 4$, $\Delta(G) = 2$, $\delta(G) = 1$. Therefore $\min\{\Delta(P_6), n-1-\delta(P_6)\} = \chi_{cd}(G)$. \hfill \Box

**Observation 4.1.** Let $G = C_{20}$. $\chi_{cd}(C_{20}) = \chi_{cd}(C_{20}) = 2$. $\gamma_{cd}(C_{20}) = 7$. Therefore $\chi(C_{20}) + \gamma_{cd}(C_{20}) = 2 + 7 = 9 < \chi_{cd}(G)$ where $G = C_{20}$.

Let $G = C_6$. $\chi_{cd}(C_6) = 3$. $\chi(C_6) = 2$ and $\gamma_{cd}(C_6) = 2$. Therefore $\chi(G) + \gamma_{cd}(G) = 2 + 2 = 4 \geq \chi_{cd}(G) = 6$. $\chi(C_6) = 6$.

Let $G = P_4$: $\chi_{cd}(P_4) = 4$. $\chi((P_4) = 2$ and $\gamma_{cd}(P_4) = 2$. Therefore $\chi(G) + \gamma_{cd}(G) = 2 + 2 = 4 = \chi_{cd}(G)$ where $G = P_4$. There is therefore no relationship between $\chi_{cd}(G)$ and $\chi(G) + \gamma_{cd}(G)$.

**Observation 4.2.** Let $G$ be the disjoint union of connected graphs $G_1, G_2, \ldots, G_k$. Then $\chi_{cd}(G) = \chi_{cd}(G_1) + \chi_{cd}(G_2) + \ldots + \chi_{cd}(G_k)$.

**Theorem 4.3.** Let $G$ have isolates. Then $\chi_{cd}(G) = \chi_{cd}(G)$.

**Proof.** Let $u_1, u_2, \ldots, u_k$ be the isolates of $G$. Let $\Pi$ be a minimum color class domination partition of $G$. Since $u_i$, $(1 \leq i \leq k)$, are isolates, $\{u_1\}, \{u_2\}, \ldots, \{u_k\}$ all belong to $\Pi$. Therefore $\Pi$ is also a global color class domination partition of $G$. Therefore $\chi_{cd}(G) \leq |\Pi| = \chi_{cd}(G)$. But $\chi_{cd}(G) \leq \chi_{cd}(G)$.

**Theorem 4.4.** Let $G$ be a bipartite graph without isolates and the cardinalities of the bipartite sets of $G$ are $\geq 2$. Then $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = \chi_{cd}(G)$ if $N(u_i) \neq Y$ for any $u_i$ in $X$ and $N(v_i) = X$ for some $v_i$ in $Y$.

If $N(u_i) = Y$ for any $u_i$ in $X$ and $N(v_i) = X$ for some $v_i$ in $Y$, then $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = 2$ and $\chi_{cd}(G) = 4$.

If $N(u_i) = Y$ for any $u_i$ in $X$ and $N(v_i) = X$ for some $v_i$ in $Y$, then $\gamma(G) = \gamma_g(G) = \chi_{cd}(G) = k + 1$ and $\chi_{cd}(G) = k + 2$.

**Proof.** Let $G$ be a bipartite graph without isolates and let $X$, $Y$ be the bipartite sets of $G$. Let $|X| \geq 2$, $|Y| \geq 2$. Since $G$ is bipartite without isolates, $G = K_x \cup K_y$. Any subset of $V(G)$ containing a vertex from $X$ and a vertex from $Y$ is a dominating set of $\overline{G}$. Any dominating set of $G$ contains at least one vertex from $X$ and at least one vertex from $Y$. Therefore any dominating set of $G$ is also a dominating set of $\overline{G}$. Therefore $\gamma(G) = \gamma_g(G)$.

Let $\{u_1, u_2, \ldots, u_r\}$ be a $\gamma$-set of $G$. Let $u_1, u_2, \ldots, u_k \in X$ and $u_{k+1}, u_{k+2}, \ldots, u_r \in Y$. 
Consider $V_i = N(u_i) - \bigcup_{j=1}^{i-1} N(u_j)$. If $u_i \in X$, then $V_i \subseteq Y$. If $u_i \in Y$, then $V_i \subseteq X$. Let $u_{i_1}$ and $u_{i_2} \in X$. Without loss of generality $i_1 < i_2$. Then $V_{i_2} \cap V_{i_1} = \emptyset$. Therefore $V_1, V_2, \ldots, V_r$ are mutually disjoint. If $u_i \in X$, $V_i \subseteq Y$, then $V_i$ is independent. Therefore $\Pi = \{V_1, V_2, \ldots, V_r\}$ is a partition of $G$ into independent sets. $V_i$ is dominated by $u_i$, $1 \leq i \leq k$. If $N(u_i) = Y$, then $V_2, V_3, \ldots, V_k$ are empty. If $N(u_{k+1}) = X$, then $V_{k+2}, V_{k+3}, \ldots, V_r$ are empty. Therefore $\{u_1, u_{k+1}\}$ is a minimum dominating as well as a global dominating set of $G$, that is, $\gamma(G) = \gamma_g(G) = 2$. Let $\Pi = \{V_1 - \{u_1\}, V_2 - \{u_r\}, \{u_k\}, \{u_r\}\}$ is a minimum global color class domination partition of $G$. Therefore $\chi_{gcd}(G) = 4$. $\Pi_1 = \{V_1, V_{k+1}\}$ is a minimum global color class domination partition of $G$. Therefore $\chi_{gcd}(G) = 2$. Suppose $N(u_1) \subseteq X$. But $N(u_{k+1}) = X$. Therefore $V_1 \not\subseteq Y$. Suppose $V_2 = N(u_2) - N(u_1) = \emptyset$. Then $N(u_2) \subseteq N(u_1)$. Therefore $D = \{u_1, u_3, \ldots, u_r\}$ is a dominating set of $G$. There $\gamma(G) < r$, a contradiction. Therefore $V_2 \neq \emptyset$. A similar argument shows that $V_3, V_4, \ldots, V_r$ are empty. Since $V_{k+1} = X, V_{k+2}, \ldots, V_r = \emptyset$, therefore $\Pi = \{V_1, \ldots, V_k, V_{k+1} - \{u_k\}, \{u_r\}\}$ is a minimum global color class domination partition. Therefore $\chi_{cd}(G) = k + 2$. Since $V_{k+1} = X$, $D = \{u_1, u_2, \ldots, u_k, u_{k+1}\}$ is a minimum global color class domination partition. Therefore $\chi_{gcd}(G) = k + 2$. Since $V_{k+1} = X$, $D = \{u_1, u_2, \ldots, u_k, u_{k+1}\}$ is a minimum dominating set of $G$. $|D| = k + 1 < r$. Therefore $\gamma(G) = k + 1$. $\gamma_g(G) = k + 1$. $\chi_{gcd}(G) = k + 2$. Suppose $N(u_1) \not\subseteq Y$, $N(u_{k+1}) \not\subseteq X$. Then $V_2, V_3, V_{k+2}, \ldots, V_r$ are non-empty. $\Pi = \{V_2, \ldots, V_k, V_{k+2}, \ldots, V_r\}$ is a minimum global color class domination partition of $G$. It is also a minimum color class domination partition of $G$. Therefore $\gamma(G) = \gamma_g(G) = \chi_{gcd}(G) = \chi_{gcd}(G) = r$.

**Proposition 4.1.** $\chi_{gcd}(G) = 2$ iff $G = \overline{K_2}$.

**Proof.** Suppose $\chi_{gcd}(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a $\chi_{gcd}$-partition of $G$. $V_1$ is dominated by a vertex of $V_2$ or $V_1$ is a singleton. Since there exists a vertex in $V_1$ which is not adjacent with any vertex of $V_2$, $V_1$ is a singleton. Similarly $V_2$ is a singleton. Let $V_1 = \{u\}, V_2 = \{v\}$. If $u$ and $v$ are adjacent, then $G = K_2$ and hence $G$ has a full degree vertex, a contradiction. Therefore $u$ and $v$ are not adjacent. Therefore $G = \overline{K_2}$.

The converse is obvious.

**Theorem 4.5.** $2 \leq \chi_{gcd}(G) \leq n$.

**Theorem 4.6.** Let $G$ be disconnected. Then $\chi_{gcd}(G) = n$ iff $G = K_{r_1} \cup K_{r_2} \ldots \cup K_{r_k}$.

**Proof.** Let $\chi_{gcd}(G) = n$. By hypothesis, $G$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the components of $G$. Suppose $G_i$ has two independent points $u, v$ such that they are adjacent with a common vertex. Then $\{u, v\}$ is an element of a $\chi_{gcd}$-partition. Therefore $\chi_{gcd}(G) \leq n$, a contradiction. Hence either $G_i$ is complete or any two independent vertices of $G_i$ has no common adjacent vertex. In the latter case, there exists a path of length at least three between $u$ and $v$. Let $u = u_1, u_2, \ldots, u_n = v$ be a shortest path between $u$ and $v$ of length at least three. Then $u$ and $u_3$ are independent and have a common vertex, a contradiction. Therefore $G_i$ is complete. Therefore $G = K_{r_1} \cup K_{r_2} \ldots \cup K_{r_k}$.

The converse is obvious.

**Corollary 4.1.** If each $K_{r_i}$ is a singleton, then $G = \overline{K_n}$.

**Remark 4.3.** Let $G$ be a connected graph without full degree vertex. Suppose $|V(G)| = 3$. Then there exists no graph without full degree vertex. Let $|V(G)| = 4$. Then $P_4$ and $C_4$ are the only connected graphs without full degree vertex such that $\chi_{gcd}(G) = 4$. Let $|V(G)| = 5$. Let $G_i, 1 \leq i \leq 4$ be the graphs given below:
Then these are the four graphs without full degree vertex on five vertices such that $\chi_{gcd}(G) = 5$.

**Definition 4.1.** Let $G$ be a connected graph. Define $N_i(G)$ as follows: A vertex set of $N_i(G)$ is same as $V(G)$. Two vertices in $N_i(G)$ are adjacent if they are independent and they have a common adjacent vertex.

**Example 4.1.** Let $G = C_4$ and $N_i(G)$ be the graphs given below:

\[
\begin{align*}
G & \quad N_i(G) \\
v_1 & \quad v_2 \quad v_1 & \quad v_2 \\
v_4 & \quad v_3 \quad v_4 & \quad v_3
\end{align*}
\]

**Theorem 4.7.** Let $G$ be a connected graph without a full degree vertex. Then $\chi_{gcd}(G) = n$ iff for any edge $uv$ in $N_i(G)$, $\{u, v\}$ is a maximal independent set in $G$.

**Proof.** Suppose for any edge $xy$ in $N_i(G)$, $\{x, y\}$ is a maximal independent set in $G$. Since $G$ is connected and $G$ has no full degree vertex, there exist two independent vertices which have a common adjacent vertex. (For : if $u$ and $v$ are independent and $d(u, v) = 2$, then $u$ and $v$ have a common vertex. Suppose $d(u, v) \geq 3$. Let $u = u_1, u_2, \ldots, u_k = v$ be a shortest path between $u$ and $v$. Clearly $k \geq 4$. Then $u, u_3$ are independent and have a common vertex $u_2$). Hence $N_i(G)$ has at least one edge. Let $uv$ be an edge of $N_i(G)$. Then $\{u, v\}$ is a maximal independent set of $G$. Therefore there exists no vertex $w$ in $G$ such that $w$ is non-adjacent with $u$ and $v$. Therefore $\chi_{gcd}(G) = n$. Conversely, let $G$ be connected without full degree vertex and $\chi_{gcd}(G) = n$. Let $xy$ be an edge in $N_i(G)$. Then $x$ and $y$ have a common adjacent vertex in $G$. Since $\chi_{gcd}(G) = n$, $x$ and $y$ do not have a common non-adjacent vertex. Hence $\{x, y\}$ is a maximal independent set in $G$. \qed

**Example 4.2.** Let $G = C_4$ and $N_i(G)$ be the graphs given below:

\[
\begin{align*}
G & \quad N_i(G) \\
v_1 & \quad v_2 \quad v_1 & \quad v_2 \\
v_4 & \quad v_3 \quad v_4 & \quad v_3
\end{align*}
\]

Also $\{v_1, v_3\}$ is a maximal independent set in $G$ as well as $\{v_2, v_4\}$. Therefore $\chi_{gcd}(G) = 4$. 

REFERENCES


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