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ON THE SOLUTIONS OF SOME NONLINEAR FREDHOLM INTEGRAL EQUATIONS IN TOPOLOGICAL HÖLDER SPACES

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Abstract. In this article, we show that the existence theorem for fredholm type quadratic integral equation in the space of functions satisfying Hölder the condition, based on the classical Schauder fixed point theorem, has new methods that perform with relative compactness in the Hölder spaces. In section 3, some axioms are introduced to solve the fredholm integral equation. In section 4, one example is presented to verify the effectiveness and applicability of our results.

Keywords: Fredholm integral equation, schauder fixed point theorem, Hölder condition, nonlinear equations.

AMS Subject Classification: 45B05.

1. Introduction

Integral equation has a wide range of applications in different branches of sciences and engineering. It arises naturally in a variety of models from biological science, applied mathematics, physics, and other disciplines, such as the theory of elasticity, biomechanics, electromagnetics, electrodynamics, fluid dynamics, heat and mass transfer, oscillating magnetic field, etc. [4, 9, 10, 11]. Quadratic integral equation is used to identify the problems that arise in physics, mathematics, engineering, biology and economics.

Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, they are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory.

Recently, J. Banaś and R. Nalepa et al. [2] have studied the following equation;

$$x(t) = p(t) + x(t) \int_a^b k(t, \tau) x(\tau) d\tau.$$

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Further, J. Caballero, B. Lopez and K. Sadarangani et al. [5] have studied the following equation;

$$x(t) = a(t) + (Tx)(t) \int_0^{\sigma(t)} u(t, s, x(s), x(\lambda s))ds.$$

Also, J. Cabelloro Mena, R. Nalepa and K. Sadarangani et al.[6] have studied the following equation;

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau) \left\{ \max_{\eta \in [0, r(\tau)]} |x(\eta)| \right\} d\tau.$$

Very recently, M. Temizer Ersoy and H. Furkan et al. [21] have studied the following equation;

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau.$$

Similar equations are examined by several authors,[7, 13, 18, 19, 20, 15, 1, 3, 8, 12, 14, 16, 17].

The object of this paper is to investigate the existence of solutions of the Fredholm integral equations of the form

$$x(t) = p(t) + (Tx)(t) \int_0^1 k(t,\tau) \frac{|x(\tau)|}{1 + |x(\tau)|} d\tau, \tag{1}$$

where p, k are given functions, T is a given operator satisfied conditions specified later and x is an unknown function.

At present, in this paper, we define relative compactness on the Hölder space, and then we study on the problem of existence of solutions of equation (1) using the technique of relative compactness in conjunction with classical Schauder fixed point theorem. Finally, one example illustrating the mentioned existence result are also included.

2. Preliminaries and Notations

In this section, we introduce definitions, lemmas and theorems which will be needed in our further considerations.

Let $[a, b]$ be a closed interval in $\mathbb{R}$, by $C[a, b]$, we denote the space of continuous functions illustrated on $[a, b]$ accoutred with the supremum norm, i.e.,

$$\|x\|_\infty = \sup \{|x(t)| : t \in [a, b]\},$$

for $x \in C[a, b]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[a, b]$ we will denote the spaces of the real functions x illustrated on $[a, b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that:

$$|x(t) - x(s)| \leq H_x^\alpha |t - s|^\alpha,$$

for all $t, s \in [a, b]$. It is easily proved that $H_\alpha[a, b]$ is a linear subspaces of $C[a, b]$. Furthermore, we put

$$H_x^\alpha = \sup \left\{ \frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b] \text{ and } t \neq s \right\}. \tag{2}$$

The space $H_\alpha[a, b]$ with $0 < \alpha \leq 1$ may be accoutred with the norm:

$$\|x\|_\alpha = |x(a)| + H_x^\alpha,$$

for $x \in H_\alpha[a, b]$. Here, H_x^α is illustrated by (2). In [2], the authors demonstrated that $(H_\alpha[a, b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Lemma 2.1. *For $x \in H_\alpha[a,b]$ with $0 < \alpha \leq 1$, the undermentioned inequality is satisfied:*

$$\|x\|_\infty \leq \max\left(1, (b-a)^\alpha\right) \|x\|_\alpha.$$

Lemma 2.2. *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[a,b] \subset H_\alpha[a,b] \subset C[a,b].$$

Furthermore, for $x \in H_\beta[a,b]$ the undermentioned inequality holds

$$\|x\|_\alpha \leq \max\left(1, (b-a)^{\beta-\alpha}\right) \|x\|_\beta.$$

Lemma 2.3. *Let us accept that $0 < \alpha < \beta \leq 1$ and E is bounded subset in $H_\beta[a,b]$, then E is a relatively compact subset in $H_\alpha[a,b]$,[7].*

Lemma 2.4. *Accept that $0 < \alpha < \beta \leq 1$ and by B_r^β we denote the ball centered at θ and radius r in the space $H_\beta[a,b]$, i.e., $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$. B_r^β is a closed subset of $H_\alpha[a,b]$,[7].*

Corollary 2.1. *Accept that $0 < \alpha < \beta \leq 1$ and B_r^β is a relatively compact subset in $H_\alpha[a,b]$ and is a closed subset of $H_\alpha[a,b]$, then B_r^β is a compact subset in the space $H_\alpha[a,b]$, [21].*

Now, we offer the following theorem and this is the basic tool used in our work.

Theorem 2.1 (Schauder's fixed point theorem)**.** *Let E be a nonempty, compact subset of a Banach space $(X, \|\cdot\|)$, convex and let $T : E \to E$ be a continuity mapping. Then T has at least one fixed point in E,[20].*

Let us now provide information that we will need in the future sections.

Definition 2.1. *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is relevant to be subadditive if*

$$f(x+y) \leq f(x) + f(y),$$

for any $x, y \in \mathbb{R}_+$,[7].

Lemma 2.5. *Let us suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$ is subadditive and $y \leq x$ then*

$$f(x) - f(y) \leq f(x-y),$$

[7].

Remark 2.1. *From Lemma 2.5, we infer that if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is subadditive then*

$$|f(x) - f(y)| \leq f(|x-y|),$$

for any $x, y \in \mathbb{R}_+$,[7].

Lemma 2.6. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function with $f(0) = 0$. Then f is subadditive, [7].*

Remark 2.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by $f(x) = \sqrt[p]{x}$, where $p > 1$. Since this functions is concave and $f(0) = 0$, Lemma 2.6 says us that f is subadditive. By Remark 2.1, we have*

$$|f(x) - f(y)| = |\sqrt[p]{x} - \sqrt[p]{y}| \leq \sqrt[p]{|x-y|},$$

for any $x, y \in \mathbb{R}_+$,[7].

3. Main Result

Now, we are ready to give the main result of the article. In this chapter,we will define some axioms in order to solve the integral equation defined in Hölder spaces and we will prove the solvability of the equation (1) in Hölder spaces.

Theorem 3.1. *Under the following assumptions* $(i)-(iv)$, *(1) equation has at least one* $x = x(t)$ *solution belonging to space* $H_\alpha[0,1]$, *where* α *is arbitrarily fixed number satisfying* $0 < \alpha < \beta \leq 1$.

(i) *The function* $p = p(t)$ *pertains to* $H_\beta[a,b]$.
(ii) $k : [0,1] \times [0,1] \to \mathbb{R}$ *is a continuous function such that exists a constant* $k_\beta > 0$ *such that:*

$$|k(t,\tau) - k(s,\tau)| \leq k_\beta |t-s|^\beta,$$

for any $t, s, \tau \in [0,1]$.
(iii) *The operator* $T : H_\beta[0,1] \to H_\beta[0,1]$ *is defined by*

$$\|Tx\|_\beta \leq \|x\|_\beta,$$

for $x \in H_\beta[0,1]$ *and it is a continuous on the space* $H_\beta[0,1]$ *according to the norm* $\|\cdot\|_\alpha$.
(iv) *The following inequality is satisfied*

$$\|p\|_\beta (2K + k_\beta) \leq \frac{1}{4},$$

where the constant K *is defined by*

$$\sup\left\{\int_0^1 |k(t,\tau)| d\tau : t \in [0,1]\right\} \leq K.$$

Then the equation (1) has at least one solution belonging to space $H_\alpha[0,1]$.

Proof. At the beginning, let us consider the operator F defined by

$$(Fx)(t) = p(t) + (Tx)(t) \int_0^1 k(t,\tau) \frac{|x(\tau)|}{1+|x(\tau)|} d\tau,$$

for $x \in H_\beta[0,1]$ and $t \in [0,1]$. We will prove that the operator F maps $H_\beta[0,1]$ into $H_\beta[0,1]$ space. Considering the axioms (i) and (ii), for $x \in H_\beta[a,b]$ and $t, s \in [0,1], (t \neq s)$

we have:

$$
\begin{aligned}
&\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^{\beta}}\\
=&\left|p(t)+(Tx)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau-p(s)-(Tx)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\right|\frac{1}{|t-s|^{\beta}}\\
=&|p(t)-p(s)+(Tx)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau-(Tx)(s)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\\
&+(Tx)(s)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau-(Tx)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau|\frac{1}{|t-s|^{\beta}}\\
\le&\frac{|p(t)-p(s)|}{|t-s|^{\beta}}+\frac{|(Tx)(t)-(Tx)(s)|}{|t-s|^{\beta}}\int_0^1|k(t,\tau)|\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\\
&+|(Tx)(s)|\int_0^1\frac{|k(t,\tau)-k(s,\tau)|}{|t-s|^{\beta}}\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\\
\le&H_p^{\beta}+\|Tx\|_{\beta}\int_0^1|k(t,\tau)||x(\tau)|d\tau+\|Tx\|_{\infty}\int_0^1 k_{\beta}\frac{|t-s|^{\beta}}{|t-s|^{\beta}}|x(\tau)|d\tau\\
\le&H_p^{\beta}+\|x\|_{\infty}\|Tx\|_{\beta}K+\|x\|_{\infty}\|Tx\|_{\infty}k_{\beta}.
\end{aligned}
$$

Because of $\|x\|_{\infty}\le\|x\|_{\beta}$ and $\|Tx\|_{\beta}\le\|x\|_{\beta}$, the last inequality is

$$
\begin{aligned}
\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^{\beta}}&\le H_p^{\beta}+K\|x\|_{\beta}\|x\|_{\beta}+\|x\|_{\beta}\|x\|_{\beta}k_{\beta}\\
&\le H_p^{\beta}+(K+k_{\beta})\|x\|_{\beta}^2.
\end{aligned}
$$

From $\|Fx\|_{\beta}=|(Fx)(0)|+H_{F_x}^{\beta}$, we get:

$$
\begin{aligned}
\|Fx\|_{\beta}&=\left|p(0)+(Tx)(0)\int_0^1 k(0,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\right|+\sup\left\{\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^{\beta}}:t,s\in[a,b]\text{ and }t\neq s\right\}\\
&\le|p(0)|+|(Tx)(0)|\int_0^1|k(0,\tau)||x(\tau)|d\tau+H_p^{\beta}+(K+k_{\beta})\|x\|_{\beta}^2\\
&\le\|p\|_{\beta}+\|Tx\|_{\infty}\|x\|_{\infty}K+(K+k_{\beta})\|x\|_{\beta}^2\\
&\le\|p\|_{\beta}+\|Tx\|_{\beta}\|x\|_{\infty}K+(K+k_{\beta})\|x\|_{\beta}^2\\
&\le\|p\|_{\beta}+\|x\|_{\beta}\|x\|_{\beta}K+(K+k_{\beta})\|x\|_{\beta}^2\\
&\le\|p\|_{\beta}+(2K+k_{\beta})\|x\|_{\beta}^2.
\end{aligned}
$$

This shows that the operator F transforms the space $H_{\beta}[0,1]$ into itself. Also, in view of assumption (iv), we infer that the operator F converts $B_{r_0}^{\beta}=\{x\in H_{\beta}[0,1]:\|x\|_{\beta}\le r_0\}$ into itself, where r_0 is an arbitrary number from the interval $[r_1,r_2]$, while

$$r_1=\frac{1-\sqrt{1-4(2K+k_{\beta})\|p\|_{\beta}}}{2(2K+k_{\beta})},$$

and

$$r_2=\frac{1+\sqrt{1-4(2K+k_{\beta})\|p\|_{\beta}}}{2(2K+k_{\beta})},$$

with

$$\|Fx\|_{\beta}\le\|p\|_{\beta}+(2K+k_{\beta})r_0^2\le r_0.$$

Here it is seen that for $r_1 \leq r_0 \leq r_2$, $F : B^{\beta}_{r_0} \to B^{\beta}_{r_0}$. Then, we conclude that the set $B^{\beta}_{r_0}$ is relatively compact in $H_\alpha[0,1]$. $B^{\beta}_{r_0}$ is also closed in the space $H_\alpha[0,1]$. Thus, gathering the above obtained facts, we infer that the set $B^{\beta}_{r_0}$ is a compact and convex subset of the space $H_\alpha[0,1]$.

At present, we will prove that the operator F is continuous on $B^{\beta}_{r_0}$, where $B^{\beta}_{r_0}$ considering the induced norm by $\|.\|_\alpha$, where $0 < \alpha < \beta \leq 1$. In addition, arbitrarily $x \in B^{\beta}_{r_0}$ and $\varepsilon > 0$. Assume that $y \in B^{\beta}_{r_0}$ and $\|x - y\|_\alpha < \delta$, where δ is a nonnegative number such that

$$0 < \delta < \frac{\varepsilon}{2(2K + k_\beta) r_0}.$$

In that case, for arbitrarily $t, s \in [0,1]$ we take

$$\frac{|(Fx)(t) - (Fy)(t) - [(Fx)(s) - (Fy)(s)]|}{|t-s|^\alpha}$$

$$\begin{aligned}
= \quad & |[p(t) + (Tx)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - p(t) - (Ty)(t)\int_0^1 k(t,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau] \\
& -[p(s) + (Tx)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - p(s) - (Ty)(s)\int_0^1 k(s,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau]|\frac{1}{|t-s|^\alpha} \\
= \quad & |[(Tx)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - (Ty)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau \\
& +(Ty)(t)\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - (Ty)(t)\int_0^1 k(t,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau] \\
& -[(Tx)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - (Ty)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau \\
& +(Ty)(s)\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau \\
& -(Ty)(s)\int_0^1 k(s,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau]|\frac{1}{|t-s|^\alpha} \\
= \quad & |[(Tx)(t) - (Ty)(t)]\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau + (Ty)(t)\int_0^1 k(t,\tau)\left[\frac{|x(\tau)|}{1+|x(\tau)|} - \frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau \\
& -[(Tx)(s) - (Ty)(s)]\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau + \\
& (Ty)(s)\int_0^1 k(s,\tau)\left[\frac{|x(\tau)|}{1+|x(\tau)|} - \frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau|\frac{1}{|t-s|^\alpha} \\
= \quad & |[(Tx)(t) - (Ty)(t)]\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - [(Tx)(s) - (Ty)(s)]\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau \\
& +[(Tx)(s) - (Ty)(s)]\int_0^1 k(t,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau - [(Tx)(s) - (Ty)(s)]\int_0^1 k(s,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau \\
& +(Ty)(t)\int_0^1 k(t,\tau)\left[\frac{|x(\tau)|}{1+|x(\tau)|} - \frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau - (Ty)(s)\int_0^1 k(t,\tau)[\frac{|x(\tau)|}{1+|x(\tau)|} - \frac{|y(\tau)|}{1+|y(\tau)|}]d\tau \\
& +(Ty)(s)\int_0^1 k(t,\tau)\left[\frac{|x(\tau)|}{1+|x(\tau)|} - \frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau
\end{aligned}$$

$$-(Ty)(s)\int_0^1 k(s,\tau)\left[\frac{|x(\tau)|}{1+|x(\tau)|}-\frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau|\frac{1}{|t-s|^\alpha}$$

$$\leq \frac{\left[|(Tx)(t)-(Ty)(t)-[(Tx)(s)-(Ty)(s)]|\int_0^1|k(t,\tau)|\frac{|x(\tau)|}{1+|x(\tau)|}d\tau\right]}{|t-s|^\alpha}$$

$$+\frac{[|[(Tx)(s)-(Ty)(s)]\int_0^1|k(t,\tau)-k(s,\tau)|\frac{|x(\tau)|}{1+|x(\tau)|}d\tau]}{|t-s|^\alpha}$$

$$+\frac{|(Ty)(t)-(Ty)(s)|\int_0^1|k(t,\tau)|\left[\frac{|x(\tau)|}{1+|x(\tau)|}-\frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau}{|t-s|^\alpha}$$

$$+\frac{|(Ty)(s)|\int_0^1|k(t,\tau)-k(s,\tau)|\left[\frac{|x(\tau)|}{1+|x(\tau)|}-\frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau}{|t-s|^\alpha}$$

$$\leq \|Tx-Ty\|_\alpha\int_0^1|k(t,\tau)||x(\tau)|d\tau+\|Tx-Ty\|_\infty\int_0^1 k_\beta\frac{|t-s|^\beta}{|t-s|^\alpha}|x(\tau)|d\tau$$

$$+\|Ty\|_\alpha\int_0^1|k(t,\tau)||x(\tau)-y(\tau)|d\tau+\|Ty\|_\infty\int_0^1 k_\beta\frac{|t-s|^\beta}{|t-s|^\alpha}|x(\tau)-y(\tau)|d\tau$$

$$\begin{aligned}&\leq \|Tx-Ty\|_\alpha\|x\|_\infty K+\|Tx-Ty\|_\alpha\|x\|_\infty k_\beta+\|Ty\|_\alpha\|x-y\|_\infty K+\|Ty\|_\infty\|x-y\|_\infty k_\beta\\&\leq (K+k_\beta)\|Tx-Ty\|_\alpha\|x\|_\infty+(K+k_\beta)\|Ty\|_\alpha\|x-y\|_\infty.\end{aligned} \tag{3}$$

Here is $|(Fx-Fy)(0)|=|(Fx)(0)-(Fy)(0)|$. Further,

$$|(Fx)(0)-(Fy)(0)|$$

$$= \left|p(0)+(Tx)(0)\int_0^1 k(0,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau-p(0)-(Ty)(0)\int_0^1 k(0,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau\right|$$

$$\leq |(Tx)(0)\int_0^1 k(0,\tau)\frac{|x(\tau)|}{1+|x(\tau)|}d\tau-(Tx)(0)\int_0^1 k(0,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau|$$

$$+|(Tx)(0)\int_0^1 k(0,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau-(Ty)(0)\int_0^1 k(0,\tau)\frac{|y(\tau)|}{1+|y(\tau)|}d\tau|$$

$$\leq |(Tx)(0)|\int_0^1|k(0,\tau)|\left[\frac{|x(\tau)|}{1+|x(\tau)|}-\frac{|y(\tau)|}{1+|y(\tau)|}\right]d\tau$$

$$+|(Tx)(0)-(Ty)(0)|\int_0^1|k(0,\tau)|\frac{|y(\tau)|}{1+|y(\tau)|}d\tau$$

$$\leq \|Tx\|_\infty\int_0^1|k(0,\tau)||x(\tau)-y(\tau)|d\tau+\|Tx-Ty\|_\infty\int_0^1|k(0,\tau)||y(\tau)|d\tau$$

$$\leq \|Tx\|_\infty\|x-y\|_\infty K+\|Tx-Ty\|_\infty\|y\|_\infty K. \tag{4}$$

According to (3) and (4) and from

$$\|Fx-Fy\|_\alpha=|(Fx-Fy)(0)|+H^\alpha_{Fx-Fy},$$

we obtain following estimates;

$$\begin{aligned}
\|Fx - Fy\|_\alpha &= |(Fx - Fy)(0)| + H^\alpha_{Fx-Fy} \\
&\leq \|Tx\|_\infty \|x - y\|_\infty K + \|Tx - Ty\|_\infty \|y\|_\infty K \\
&\quad +(K + k_\beta)\|Tx - Ty\|_\alpha \|x\|_\infty + (K + k_\beta)\|Ty\|_\alpha \|x - y\|_\infty \\
&\leq \|x\|_\beta \|x - y\|_\alpha K + \|Tx - Ty\|_\infty \|y\|_\beta K \\
&\quad +(K + k_\beta)\|Tx - Ty\|_\alpha \|x\|_\infty + (K + k_\beta)\|y\|_\beta \|x - y\|_\alpha \\
&\leq Kr_0\|x - y\|_\alpha + Kr_0\|Tx - Ty\|_\alpha + (K + k_\beta)r_0\|Tx - Ty\|_\alpha + (K + k_\beta)r_0\|x - y\|_\alpha \\
&\leq (2K + k_\beta)r_0\|x - y\|_\alpha + (2K + k_\beta)r_0\|Tx - Ty\|_\alpha.
\end{aligned}$$

Let us take on arbitrary $\varepsilon > 0$. Since, the operator T is continuous on $H_\beta[0, 1]$ with respect to the norm $\|.\|_\alpha$, it is also continuous at the point $y \in B^\beta_{r_0}$. Hence,

$$\|Tx - Ty\|_\alpha < \frac{\varepsilon}{2(2K + k_\beta)r_0}$$

for all $x \in B^\beta_{r_0}$, where $\|x - y\|_\alpha < \delta$ and

$$0 < \delta < \frac{\varepsilon}{2(2K + k_\beta)r_0}.$$

Then, we have

$$\begin{aligned}
\|Fx - Fy\|_\alpha &\leq (2K + k_\beta)r_0\|x - y\|_\alpha + (2K + k_\beta)r_0\|Tx - Ty\|_\alpha \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This shows that the operator F is continuous on $B^\beta_{r_0}$, with respect to the norm $\|.\|_\alpha$, where $0 < \alpha < \beta \leq 1$. Thus, taking into account the fact that the set $B^\beta_{r_0}$ is compact in the space $H_\alpha[0, 1]$ and applying the classical Schauder fixed point principle, we complete the proof. □

At present, we exemplify the above result by one example.

4. Example

In this part, we conclude the article by presenting one example which illustrates the generality and efficiency of our results.

Example 4.1. *Let us consider the following quadratic integral equation:*

$$x(t) = \sqrt[3]{qt + \hat{q}} + (x(t)) \int_0^1 \sqrt{mt^2 + \tau}\frac{|x(\tau)|}{1 + |x(\tau)|}d\tau, \; t \in I = [0, 1]. \tag{5}$$

Equation (5) is a special case of equation (1) with $p(t) = \sqrt[3]{qt + \hat{q}}$, $(Tx)(t) = x(t)$ *and* $k(t, \tau) = \sqrt{mt^2 + \tau}$. *For all* $t, s \in [0, 1]$,

$$\begin{aligned}
|p(t) - p(s)| &= |\sqrt[3]{qt + \hat{q}} - \sqrt[3]{qs + \hat{q}}| \\
&\leq |\sqrt[3]{qt + \hat{q} - qs - \hat{q}}| \\
&\leq \sqrt[3]{q}\sqrt[3]{|t - s|} \\
&\leq \sqrt[3]{q}.|t - s|^{\frac{1}{3}}.
\end{aligned}$$

This says that $p \in H_{\frac{1}{3}}[0,1]$ *and* $H_p^{\frac{1}{3}} = \sqrt[3]{q}$. *Further,*

$$\begin{aligned} \|p\|_{\frac{1}{3}} &= |p(0)| + H_p^{\frac{1}{3}} \\ &= \sqrt[3]{q} + \sqrt[3]{\hat{q}}. \end{aligned}$$

Then, for $t, s \in [0,1]$,

$$\begin{aligned} |k(t,\tau) - k(s,\tau)| &= \left|\sqrt{mt^2+\tau} - \sqrt{ms^2+\tau}\right| \\ &\leq \left|\sqrt{m(t^2-s^2)}\right| \\ &\leq \sqrt{m}\sqrt{|(t^2-s^2)|} \\ &= \sqrt{m}|t-s|^{\frac{1}{2}}|t+s|^{\frac{1}{2}} \\ &\leq \sqrt{m}|t-s|^{\frac{1}{2}} \\ &= \sqrt{m}|t-s|^{\frac{1}{3}}|t-s|^{\frac{1}{6}} \\ &\leq \sqrt{m}|t-s|^{\frac{1}{3}}, \end{aligned}$$

and it is seen that $k_\beta = k_{\frac{1}{3}} = \sqrt{m}$. *This shows that functions* $p(t)$ *and* $k(t,\tau)$ *involved in (5) satisfy assumptions (i) and (ii) of Theorem 3.1.*

Now, we will show that the operator $T : H_{\frac{1}{3}}[0,1] \to H_{\frac{1}{3}}[0,1]$ *continuous according to be norm with* $\|.\|_\alpha$ *defined in* $H_{\frac{1}{3}}[0,1]$ *space. To do this, fix arbitrarily* $y \in H_\beta[0,1]$ *and* $\varepsilon > 0$. *Assume that* $x \in H_\beta[0,1]$ *is an arbitrary function and* $\|x-y\|_\alpha < \delta$, *where* δ *is a positive number such that*

$$0 < \delta \leq \frac{\varepsilon}{2}.$$

Then, for arbitrary $t, s \in [0,1]$ *we obtain*

$$(Tx-Ty)(t) - (Tx-Ty)(s) = x(t) - y(t) - (x(s) - y(s)). \tag{6}$$

By (6), we get

$$|(Tx-Ty)(t) - (Tx-Ty)(s)| \leq |x(t) - y(t) - (x(s) - y(s))|. \tag{7}$$

By (7), we have:

$$\sup\left\{\frac{|(Tx-Ty)(t) - (Tx-Ty)(s)|}{|t-s|^\alpha} : t, s \in [0,1], \ \textit{and } t \neq s\right\}$$

$$\begin{aligned} &\leq \sup\left\{\frac{|x(t) - y(t) - (x(s) - y(s))|}{|t-s|^\alpha} : t, s \in [0,1], \ \textit{and } t \neq s\right\} \\ &\leq \|x-y\|_\alpha. \end{aligned} \tag{8}$$

From (8), we obtain the following inequality:

$$\begin{aligned}\|Tx-Ty\|_\alpha &= |(Tx-Ty)(0)|+\sup\left\{\frac{|(Tx-Ty)(t)-(Tx-Ty)(s)|}{|t-s|^\alpha}:t,s\in[0,1]\ and\ t\neq s\right\}\\ &\leq |x(0)-y(0)|+\|x-y\|_\alpha\\ &\leq \|x-y\|_\infty+\|x-y\|_\alpha\\ &\leq \|x-y\|_\alpha+\|x-y\|_\alpha\\ &\leq 2\|x-y\|_\alpha\\ &< \varepsilon,\end{aligned}$$

which yields that the operator T is continuous on $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$.

Further, we get

$$\begin{aligned}\sup\left\{\int_0^1|k(t,\tau)|d\tau:t\in[0,1]\right\} &= \sup\left\{\int_0^1\left|\sqrt{mt^2+\tau}\right|d\tau:t\in[0,1]\right\}\\ &= \sup\left\{\frac{2}{3}\left(\sqrt{(mt^2+1)^3}-\sqrt{(mt^2)^3}\right):t\in[0,1]\right\}\\ &\leq \sup\left\{\frac{2}{3}\sqrt{(mt^2+1)^3}:t\in[0,1]\right\}\\ &\leq \sqrt{(m+1)^3}\\ &= K.\end{aligned}$$

Hence; we deduce that the inequality from assumption (iv) of Theorem 3.1 is satisfied provided

$$\|p\|_{\frac{1}{3}}+(2K+k_\beta)r^2\leq r,$$

which is equivalent to

$$\sqrt[3]{q}+\sqrt[3]{\hat{q}}+\left(2\sqrt{(m+1)^3}+\sqrt{m}\right){r_0}^2\leq r_0.$$

Thus, by choosing $q=\hat{q}=\frac{1}{10^{15}}$ and $m=\frac{1}{2^{18}}$, it is easy to see that a number $r_0=3,10^{-5}$ satisfies the inequality in condition (iv). Consequently, all the conditions of Theorem 3.1 are satisfied. This implies that the integral equation (5) has at least one solution which belongs to the space $H_\alpha[0,1]$ with $0<\alpha<\frac{1}{3}$.

5. Conclusion

Our main aim in this paper is to study the existence of solutions of equation (1) using the technique of relative compactness in conjunction with Schauder's fixed point theorem. Further, the example is presented to verify the effectiveness and applicability of our results.

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