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LACUNARY STATISTICAL CONVERGENT FUNCTIONS VIA IDEALS WITH RESPECT TO THE INTUITIONISTIC FUZZY NORMED SPACES

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ABSTRACT. In this paper, we introduce the notion $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence of a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ with respect to the intuitionistic fuzzy norm (μ,v), investigate their relationship and make some observations about these classes.

Keywords: Ideal, Filter, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence, intuitionistic fuzzy normed spaces.

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1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [6] (see also [27]) as follows. Let K be a subset of $\mathbb{N}$. Then the asymptotic density of K is denoted by $\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L we write $st\text{-}\lim x_k = L$. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [7] and Šalát [21].

P. Kostyrko et al. (a similar concept was invented in [10]) introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [11, 12, 4, 22, 23, 24, 26]. Also, some preliminary and related results can be found in [3, 5].

In another direction, a new type of convergence, called lacunary statistical convergence, was introduced in [8] as follows. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

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A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or S_θ-convergent to L) if, for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [8], some inclusion theorem are proven.

Recently, Mohiuddine and Aiyub [16] studied lacunary statistical convergence as generalization of the statistical convergence and introduced the concept of θ-statistical convergence in random 2-normed space. In [17], Mursaleen and Mohiuddine extended the idea of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. Also lacunary statistically convergent double sequences in probabilistic normed space was studied by Mohiuddine and Savaş in [15].

Following the introduction of fuzzy set theory by Zadeh [30], there has been extensive research to find applications and fuzzy analogues of the classical theories. The theory of intuitionistic fuzzy sets was introduced by Atanassov [1]; it has been extensively used in decision-making problems [2]. The concept of an intuitionistic fuzzy metric space was introduced by Park [19]. Furthermore, Saadati and Park [20] gave the notion of an intuitionistic fuzzy normed space. In [17], Mursaleen and Mohiuddine extended the idea of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in [9, 18, 29, 24, 25].

The notion of ideal statistical convergence and ideal lacunary statistical convergence of functions has not been studied previously in the setting of fuzzy normed linear spaces. Motivated by this fact, in this paper, the notions of ideal statistical convergence and ideal lacunary statistical convergence of function are introduced in an intuitionistic fuzzy normed linear space and some important results are established. We use ideals to introduce the concept of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence of function with respect to the intuitionistic fuzzy norm (μ, v), which naturally extend the notions of statistical convergence and lacunary statistical convergence.

Throughout by function $x(t)$ we shall mean a nonnegative real-valued Lebesque measurable function in the interval $(1, \infty)$. $\mathbb{N}$ will stand for the set of natural numbers. First we collect some basic definitions used in the paper.

Definition 1.1. *A triangular norm (t-norm) is a continuous mapping* $* : [0,1] \times [0,1] \to [0,1]$ *such that* $(S, *)$ *is an abelian monoid with unit one and* $c * d \leq a * b$ *if* $c \leq a$ *and* $d \leq b$ *for all* $a, b, c, d \in [0,1]$.

Definition 1.2. *(*[28]*)A binary operation* $\Diamond : [0,1]\times[0,1] \to [0,1]$ *is said to be a continuous* t*-conorm if it satisfies the following conditions:*

(i) $\Diamond$ *is associate and commutative,*

(ii) $\Diamond$ *is continuous,*

(iii) $a\Diamond 0 = a$ *for all* $a \in [0,1]$,

(iv) $a\Diamond b \leq c\Diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

For example, we can give $a * b = ab$, $a * b = \min\{a, b\}$, $a\Diamond b = \min\{a + b, 1\}$ and $a\Diamond b = \max\{a, b\}$ for all $a, b \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [20] has recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 1.3. *(*[20]*)The 5-tuple* $(X, \mu, v, *, \Diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\Diamond$ *is a continuous* *z-conorm, and μ, v are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$, and $s, t > 0$:*

(a) $\mu(x, z) + v(x, t) \leq 1,$
(b) $\mu(x, z) > 0,$
(c) $\mu(x, z) = 1$ *if and only if* $x = 0,$
(d) $\mu(\alpha x, z) = \mu\left(x, \frac{z}{|\alpha|}\right)$ *for each* $\alpha \neq 0,$
(e) $\mu(x, z) * \mu(y, s) \leq \mu(x + y, z + s),$
(f) $\mu(x, .) : (0, \infty) \to [0, 1]$ *is continuous,*
(g) $\lim_{z\to\infty} \mu(x, t) = 1$ *and* $\lim_{z\to 0} \mu(x, z) = 0,$
(h) $v(x, z) < 1,$
(i) $v(x, z) = 0$ *if and only if* $x = 0,$
(j) $v(\alpha x, z) = \mu\left(x, \frac{z}{|\alpha|}\right)$ *for each* $\alpha \neq 0,$
(k) $v(x, z) \lozenge v(y, s) \geq v(x + y, z + s),$
(l) $v(x, .) : (0, \infty) \to [0, 1]$ *is continuous,*
(m) $\lim_{z\to\infty} v(x, z) = 0$ *and* $\lim_{z\to 0} v(x, z) = 1.$

In this case (μ, v) is called an intuitionistic fuzzy norm. As a standard example, we can give the following:

Let $(X, \|.\|)$ be a normed space, and let $a * b = ab$ and $a \lozenge b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $z > 0$, consider

$$\mu(x, t) = \frac{z}{z + \|x\|} \text{ and } v_0(x, z) = \frac{\|x\|}{z + \|x\|}.$$

Then observe that $(X, \mu, v, *, \lozenge)$ is an intuitionistic fuzzy normed space.

We also recall that the concept of convergence in an intuitionistic fuzzy normed space is studied in [20].

Definition 1.4. *([20])Let $(X, \mu, v, *, \lozenge)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, v) if, for every $\varepsilon > 0$ and $z > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, z) > 1 - \varepsilon$ and $v(x_k - L, z) < \varepsilon$ for all $k \geq k_0$. It is denoted by (μ, v)-$\lim x = L$ or $x_k \overset{(\mu,v)}{\to} L$ as $k \to \infty$.*

2. $\mathcal{I}$-Statistical and $\mathcal{I}$-Lacunary statistical convergence on IFNS

In this section we deal with the ideal statistical convergence and ideal lacunary statistical convergence on the intuitionistic fuzzy norm spaces. Before proceeding further, we should recall some notation on the ideal.

Definition 2.1. *A non-empty family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$.

Definition 2.2. *A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin \mathcal{F}$,
(b) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,
(c) $A \in \mathcal{F}$, $A \subset B$ imply $B \in \mathcal{F}$.

If $\mathcal{I}$ is a proper nontrivial ideal of $\mathbb{N}$ (i.e. $\mathbb{N} \notin \mathcal{I}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \ \exists\, A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal $\mathcal{I}$. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.3. *([10]). (i) A sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$.*

(ii) A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$ if there exists $M \in \mathcal{F}(\mathcal{I})$ such that $(x_k)_{k \in M}$ converges to L.

We now ready to obtain our main definitions and results.

Definition 2.4. *Let $(X, \mu, v, *, \Diamond)$ be an IFNS. Then, a function $x(t)$ is said to be $\mathcal{I}$-statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, v) if, for each $\varepsilon > 0$, $z > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ t \leq n : \mu\left(x(t) - L, z\right) \leq 1 - \varepsilon \ \text{or} \ v\left(x(t) - L, z\right) \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write $x(t) \overset{(\mu,v)}{\to} L\left(S^{(\mu,v)}(\mathcal{I})\right)$.

Definition 2.5. *Let $(X, \mu, v, *, \Diamond)$ be an IFNS and θ be a lacunary sequence. A function $x(t)$ is said to be $\mathcal{I}$-lacunary statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, v) if, for any $\varepsilon > 0$, $z > 0$ and $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ t \in \mathcal{I}_r : \mu\left(x(t) - L, z\right) \leq 1 - \varepsilon \ \text{or} \ v\left(x(t) - L, z\right) \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x(t) \overset{(\mu,v)}{\to} L\left(S_\theta^{(\mu,v)}(\mathcal{I})\right)$. The class of all $\mathcal{I}$-lacunary statistically convergent functions will be denoted by $S_\theta^{(\mu,v)}(\mathcal{I})$.

In the following, we investigate the relationship between $\mathcal{I}$-statistical and $\mathcal{I}$-lacunary statistical convergence of function with respect to the intuitionistic fuzzy norm (μ, v).

Definition 2.6. *Let θ be a lacunary sequence. Then $x(t)$ is said to be $N_\theta^{(\mu,v)}(\mathcal{I})$-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, v) if, for any $\varepsilon > 0$, $z > 0$ and $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \int\limits_{t \in I_r} \{ \mu\left(x(t) - L, z\right) \leq 1 - \varepsilon \ \text{or} \ v\left(x(t) - L, z\right) \geq \varepsilon \} \, dt \right| \geq \delta \right\} \in \mathcal{I}.$$

This convergence is denoted by $x(t) \to L\left(N_\theta^{(\mu,v)}(\mathcal{I})\right)$, and the class of such sequences will be denoted simply by $N_\theta^{(\mu,v)}(\mathcal{I})$.

Theorem 2.1. *Let $\theta = \{k_r\}_{r \in \mathbb{N}}$ be a lacunary sequence. Then*

(i) $x(t) \to L\left(N_\theta^{(\mu,v)}(\mathcal{I})\right) \Rightarrow x(t) \to L\left(S_\theta^{(\mu,v)}(\mathcal{I})\right)$, and

(ii) $x(t)$ is bounded and $x(t) \to L\left(S_\theta^{(\mu,v)}(\mathcal{I})\right) \Rightarrow x(t) \to L\left(N_\theta^{(\mu,v)}(\mathcal{I})\right)$,

Proof. (i) (a) If $\varepsilon > 0$ and $x(t) \to L\left(N_{\theta}^{(\mu,v)}(\mathcal{I})\right)$, we can write

$$\begin{aligned}
&\int\limits_{t\in I_r} (\mu(x(t)-L,z) \text{ or } v(x(t)-L,z))\,dt \\
&\geq \int\limits_{\substack{t\in I_r,\mu(x(t)-L,z)\leq 1-\varepsilon \\ \text{or } v(x(t)-L,z)\geq\varepsilon}} (\mu(x(t)-L,z) \text{ or } v(x(t)-L,z))\,dt \\
&\geq \varepsilon\left|\{t\in I_r : \mu(x(t)-L,z)\leq 1-\varepsilon \text{ or } v(x(t)-L,z)\geq\varepsilon\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{\varepsilon h_r}\int\limits_{t\in I_r} (\mu(x(t)-L,z) \text{ or } v(x(t)-L,z))\,dt \\
&\geq \frac{1}{h_r}\left|\{t\in I_r : \mu(x(t)-L,z)\leq 1-\varepsilon \text{ or } v(x(t)-L,z)\geq\varepsilon\}\right|.
\end{aligned}$$

Then, for any $\delta > 0$ and $z > 0$

$$\begin{aligned}
&\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\{t\in I_r : \mu(x(t)-L,z)\leq 1-\varepsilon \text{ or } v(x(t)-L,z)\geq\varepsilon\}\right|\geq\delta\right\} \\
&\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\int\limits_{t\in I_r}\mu(x(t)-L,z)\,dt \leq (1-\varepsilon)\,\delta \text{ or } \frac{1}{h_r}\int\limits_{t\in I_r} v(x(t)-L,z)\,dt\geq\varepsilon.\delta\right\}\in\mathcal{I}.
\end{aligned}$$

This proves the result.

(ii) Suppose that $x(t) \to L\left(S_{\theta}^{(\mu,v)}(\mathcal{I})\right)$ and $x(t)$ be a bounded function. Then there exists an $M > 0$ such that $\mu(x(t)-L,z) \geq 1-M$ or $v(x(t)-L,z) \leq M$ $\forall t$. Given $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{h_r}\int\limits_{t\in I_r} (\mu(x(t)-L,z) \text{ or } v(x(t)-L,z)\,dt) \\
&= \frac{1}{h_r}\int\limits_{\substack{t\in I_r,\mu(x(t)-L,z)\leq 1-\frac{\varepsilon}{2} \\ \text{or } v(x(t)-L,z)\geq\frac{\varepsilon}{2}}} (\mu(x(t)-L,t) \text{ or } v(x(t)-L,z))\,dt \\
&+ \frac{1}{h_r}\int\limits_{\substack{t\in I_r,\mu(x(t)-L,z)> 1-\frac{\varepsilon}{2} \\ \text{or } v(x(t)-L,z)<\frac{\varepsilon}{2}}} (\mu(x(t)-L,t) \text{ or } v(x(t)-L,z))\,dt \\
&\leq \frac{M}{h_r}\left|\left\{t\in I_r : \mu(x(t)-L,t) \text{ or } v(x(t)-L,z)\geq\frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&\left\{r\in\mathbb{N} : \frac{1}{h_r}\int\limits_{t\in I_r} (\mu(x(t)-L,z)\leq 1-\varepsilon \text{ or } \frac{1}{h_r}\int\limits_{t\in I_r} v(x(t)-L,z)\geq\varepsilon)dt\right\} \\
&\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{t\in I_r : \mu(x(t)-L,z)\leq 1-\frac{\varepsilon}{2} \text{ or } v(x(t)-L,z)\geq\frac{\varepsilon}{2}\right\}\right|\geq\frac{\varepsilon}{2M}\right\}\in\mathcal{I}.
\end{aligned}$$

This proves the result. □

Theorem 2.2. *For any lacunary sequence θ, $\mathcal{I}$-statistical convergence with respect to the intuitionistic fuzzy norm (μ, v) implies $\mathcal{I}$-lacunary statistical convergence with respect to the intuitionistic fuzzy norm (μ, v) if and only if* $\liminf_r q_r > 1$. *If* $\liminf_r q_r = 1$, *then there exists a bounded function $x = (x(t))$ which is $\mathcal{I}$-statistically convergent but not $\mathcal{I}$-lacunary statistically convergent.*

Proof. Suppose first that $\liminf_r q_r > 1$. Then there exists $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{\alpha + 1}.$$

Since $x(t) \to L\left(S^{(\mu,v)}(\mathcal{I})\right)$, for every $\varepsilon > 0$, $z > 0$, and for sufficiently large r, we have

$$\begin{aligned} &\frac{1}{k_r}\left|\{k \leq k_r : \mu(x(t) - L, z) \leq 1 - \varepsilon \text{ or } v(x(t) - L, z) \geq \varepsilon\}\right| \\ &\geq \frac{1}{k_r}\left|\{k \in I_r : \mu(x(t) - L, z) \leq 1 - \varepsilon \text{ or } v(x(t) - L, z) \geq \varepsilon\}\right| \\ &\geq \frac{\alpha}{\alpha + 1}\frac{1}{h_r}\left|\{k \in I_r : \mu(x(t) - L, z) \leq 1 - \varepsilon \text{ or } v(x(t) - L, z) \geq \varepsilon\}\right|. \end{aligned}$$

Then, for any $\delta > 0$, we get

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{t \in I_r : \mu(x(t) - L, z) \leq 1 - \varepsilon \text{ or } v(x(t) - L, z) \geq \varepsilon\}\right| \geq \delta\right\} \\ &\subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}\left|\{t \leq k_r : \mu(x(t) - L, z) \leq 1 - \varepsilon \text{ or } v(x(t) - L, z) \geq \varepsilon\}\right| \geq \frac{\delta\alpha}{(1 + \alpha)}\right\} \in \mathcal{I}. \end{aligned}$$

This proves the sufficiency. Conversely, suppose that $\liminf_r q_r = 1$. We can select a subsequence $\{k_{r_j}\}$ of the lacunary sequence θ such that

$$\frac{k_{r_j}}{k_{r_j} - 1} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j - 1}}{k_{r_{j-1}}} > j, \text{ where } r_j \geq r_{j-1} + 2.$$

Define a sequence $x(t)$ by

$$x = x(t) = \begin{cases} x(t) = 1, & \text{if } t \in I_{r_j} \\ x(t) = 0, & \text{otherwise.} \end{cases}$$

Then, for any real L,

$$\frac{1}{h_{r_j}}\int\limits_{t \in I_{r_j}} \left(\mu(x(t) - L, z) \text{ or } v(x(t) - L, z)\right) dt = \left(\mu(1 - L, z) \text{ or } v(1 - L, z)\right) \text{ for } j = 1, 2, ...$$

and

$$\frac{1}{h_r}\int\limits_{t \in I_r} \left(\mu(x(t) - L, z) \text{ or } v(x(t) - L, z)\right) dt = \left(\mu(L, z) \text{ or } v(L, z)\right) \text{ for } r \neq r_j$$

Then it is quite clear that $x(t)$ does not belong to $N_\theta^{(\mu,v)}(\mathcal{I})$. Since x is bounded, Theorem 2 (ii) implies that $x(t) \to L\left(S_\theta^{(\mu,v)}(\mathcal{I})\right)$.

Next, let $k_{r_j-1} \leq n \leq k_{r_{j+1}-1}$. Then, by following theorem 2.1 in [8], we can write

$$\begin{aligned}
&\frac{\varepsilon}{n}\left|\{t \leq n : \mu\left(x(t)-L, z\right) \leq 1-\varepsilon \text{ or } v\left(x(t)-L, z\right) \geq \varepsilon\}\right| \\
&\leq \frac{1}{n}\int_{i=1}^{n}\left(\mu\left(x(t), z\right) \text{ or } v\left(x(t)-L, z\right)\right) \leq \frac{k_{r_j-1}+h_{r_j}}{k_{r_j-1}} \\
&\leq \frac{1}{j}+\frac{1}{j}=\frac{2}{j}.
\end{aligned}$$

Hence $x(t)$ is $\mathcal{I}$-statistically convergent with respect to the intuitionistic fuzzy norm (μ, v) for any admissible ideal $\mathcal{I}$. □

It is quite natural question when does $\mathcal{I}$ lacunary statistical convergence with respect to the intuitionistic fuzzy norm (μ, v) imply $\mathcal{I}$-statistical convergence with respect to the intuitionistic fuzzy norm (μ, v)?

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(\mathcal{I})$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$.

Theorem 2.3. *For a lacunary sequence θ satisfying the above condition, $\mathcal{I}$-lacunary statistical convergence implies $\mathcal{I}$-statistical convergence if* $\limsup_r q_r < \infty$.

Proof. If $\limsup\limits_{r} q_r < \infty$ then without any loss of generality we can assume that there exists a $0 < B < \infty$ such that $q_r < B$ for all $r \geq 1$. Suppose that $x(t) \to L(S_\theta(\mathcal{I}))$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : \mu\left(x(t)-L, z\right) \leq 1-\varepsilon \text{ or } v\left(x(t)-L, t\right) \geq \varepsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n}|\{t \leq n : \mu\left(x(t)-L, z\right) \leq 1-\varepsilon \text{ or } v\left(x(t)-L, z\right) \geq \varepsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal I. Further observe that

$$A_j = \frac{1}{h_j}|\{t \in I_j : \mu\left(x(t)-L, t\right) \leq 1-\varepsilon \text{ or } v\left(x(t)-L, z\right) \geq \varepsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
& \frac{1}{n}|\{t \leq n : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
\leq \ & \frac{1}{k_{r-1}}|\{t \leq k_r : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
= \ & \frac{1}{k_{r-1}}|\{t \in I_1 : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
+ \ & \cdots + \frac{1}{k_{r-1}}|\{t \in I_r : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
= \ & \frac{k_1}{k_{r-1}}\frac{1}{h_1}|\{t \in I_1 : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
+ \ & \frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}|\{t \in I_2 : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,z) \geq \varepsilon\}| \\
+ \ & \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}|\{t \in I_r : \mu(x(t)-L,z) \leq 1-\varepsilon \text{ or } v(x(t)-L,t) \geq \varepsilon\}| \\
= \ & \frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r \leq \sup_{j\in C} A_j . \frac{k_r}{k_{r-1}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem.

□

3. Conclusion

We should note that the notion of lacunary statistical convergence of a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ has not been studied in the setting of fuzzy normed linear spaces until now. It is natural to consider the notion of lacunary statistical convergence in an intuitionistic fuzzy normed linear space and also establish some important results. Our study of lacunary statistical convergence of a nonnegative real-valued Lebesque measurable function in the interval $(1,\infty)$ in intuitionistic fuzzy normed spaces also provides a tool to deal with convergence problems of sequences of fuzzy real numbers.

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