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FOURIER METHOD FOR THE INVERSE COEFFICIENT OF THE PSEUDO-PARABOLIC EQUATION WITH NON-LOCAL BOUNDARY CONDITION

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Abstract. In this work, we have tried to find the inverse coefficient in the quasilinear pseudo-parabolic equation with over determination conditions. It shows the existence, stability of the solution by iteration method and examined numerical solution.

Keywords: Inverse Problem, Quasilinear Parabolic Equation, Crank-Nicolson difference scheme, Overdetermination Data.

AMS Subject Classification: 35K55, 35K70.

1. Introduction

Recently, there have been a lot of problems with inverse problems that have a lot of applications like chemical diffusion, applications in heat conduction, population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering. The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist [1, 3, 2, 4]. Nonlocal boundary conditions have played a lot of many important roles in heat transfer, termoelasticity, control theory, mathematical biology, etc.[2, 4]. Let's take the following problem with unknowns (q, u)

$$u_t - u_{xx} - \varepsilon u_{txx} - q(t)u = g(x,t,u),\ (x,t) \in\ \Omega, \tag{1}$$

$$u(x,0) = \theta(x), \quad x \in [0,1], \tag{2}$$

$$u(0,t) = 0,\ u_x(0,t) = u_x(1,t),\ 0 \le t \le T, \tag{3}$$

$$h(t) = \int_0^1 u(x,t)dx, 0 \le t \le T, \tag{4}$$

Here $\Omega := \{0 < x < 1,\ 0 < t < T\}$, $\theta(x) \epsilon\, [0,1]$ and $g(x,t,u) \epsilon\ \overline{\Omega} \times (-\infty, \infty)$.

Definition 1.1. $\{q(t), u(x,t)\} \in C[0,T] \times (C^{2,1}(\Omega) \cap C^{1,0}(\overline{\Omega}))$ *is called the classical solution.*

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2. Solution of the Inverse Problem

Consider the following system of functions on the interval $[0,1]: X_0(x)=x$, $X_{2k-1}(x)=x\cos 2\pi kx$, $X_{2k}(x)=\sin 2\pi kx$ $Y_0(x)=2$, $Y_{2k-1}(x)=4\cos 2\pi kx$, $Y_{2k}(x)=4(1-x)\sin 2\pi kx$ The systems of these functions arise for the solution of a nonlocal boundary value problem in heat conduction. It is easy to verify that the systems of functions $Xk(x)$ and $Yk(x), k=1,2,3,...$, are biorthonormal on $[0,1]$. They are also Riesz bases in $L_2\,[0,1]$ Let assume the following conditions are ensured. (C1) $h(t)\in C^1[0,T], q(t)\in C[0,T]$. (C2) $\theta(x)\in C^3[0,1]$, $\theta(x)|_{x=0}=0,\ \theta_x\,(x)|_{x=0}=\theta_x(x)|_{x=1}$, (C3) $g(x,t,u)$ is provided following conditions: (1)

$$\left|\frac{\partial^{(n)}g(x,t,u)}{\partial x^n}-\frac{\partial^{(n)}g(x,t,\tilde{u})}{\partial x^n}\right|\le b(x,t)\,|u-\tilde{u}|\ \ ,n=0,1,2,$$

where $b(x,t)\in L_2(\Omega),\ b(x,t)\ge 0$, (2) $g(x,t,u)\in C^2[0,1],\ t\in[0,T]$, (3) $g(x,t,u)|_{x=0}=0$, $g_x\,(x,t,u)|_{x=0}=g_x(x,t,u)|_{x=1}$, (4) $g_0(t)\ge 0,\ t\in[0,T]$.
By Fourier method,

$$\begin{aligned}
u(x,t) \;=\;& 2\left[\theta_0 e^{-\int\limits_0^t q(\tau)d\tau}+\int\limits_0^t g_0(\tau,u)\,e^{-\int\limits_\tau^t q(\tau)d\tau}d\tau\right] \qquad (5)\\
&+\sum_{k=1}^{\infty}\sin 2\pi kx\,(\theta_{2k-1}-4\pi kt\;\theta_{2k})\,e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}e^{-\int\limits_\tau^t q(\tau)d\tau}\\
&+\sum_{k=1}^{\infty}x\cos 2\pi kx(\theta_{2k-1}e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}e^{-\int\limits_\tau^t q(\tau)d\tau}+\int\limits_0^t g_{2k-1}(\tau,u)\,e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}e^{-\int\limits_\tau^t q(\tau)d\tau})d\tau\\
&+\sum_{k=1}^{\infty}\sin 2\pi kx\int\limits_0^t(g_{2k}(\tau,u)-4\pi k g_{2k-1}(\tau,u)(t-\tau))e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}e^{-\int\limits_\tau^t q(\tau)d\tau}d\tau.
\end{aligned}$$

Under the condition (1)-(3), differentiating (4), we obtain

$$\int\limits_0^1 u_t(x,t)dx=h^{'}(t), 0\le t\le T. \qquad (6)$$

(5) and (6) yield

$$q(t)=\frac{\int\limits_0^t\int\limits_0^1\ g\left(\alpha,\beta,u(\alpha,\beta)\right)d\alpha d\beta-h^{'}(t)}{h(t)}. \qquad (7)$$

Definition 2.1. *Let* $\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$ *is satisfied that*

$$\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right)<\infty,$$

by $\mathbf{B}$.

$$\|u(t)\|=\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right),$$

be the norm where $\mathbf{B}$ *is Banach space.*

Theorem 2.1. *If the assumptions (C1)-(C3) be provided then the problem (1)-(4) has a unique solution.*

Proof. Using iteration to equation (5)

$$u_0^{(N+1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 g\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right) e^{-\int_\beta^t q^{(N)}(\tau)d\tau} d\alpha d\beta, \tag{8}$$

$$u_{2k}^{(N+1)}(t) = u_{2k}^{(0)}(t) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 g(\alpha,\beta,u^{(N)}(\alpha,\beta)) \sin 2\pi k\alpha \; e^{\frac{-(2\pi k)^2(t-\beta)}{1+\varepsilon(2\pi k)^2}} e^{-\int_\beta^t q^{(N)}(\tau)d\tau} d\alpha d\beta,$$

$$u_{2k-1}^{(N+1)}(t) = u_{2k-1}^{(0)}(t) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 g(\alpha,\beta,u^{(N)}(\alpha,\beta))\alpha \cos 2\pi k\alpha \; e^{\frac{-(2\pi k)^2(t-\beta)}{1+\varepsilon(2\pi k)^2}} e^{-\int_\beta^t q^{(N)}(\tau)d\tau} d\alpha d\beta$$

$$\frac{-4\pi k}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 (t-\beta) g(\alpha,\beta,u^{(N)}(\alpha,\beta)) \sin 2\pi k\alpha \; e^{\frac{-(2\pi k)^2(t-\beta)}{1+\varepsilon(2\pi k)^2}} e^{-\int_\beta^t q^{(N)}(\tau)d\tau} d\alpha d\beta,$$

$$u_0^{(0)}(t) = \theta_0, u_{2k-1}^{(0)}(t) = (\theta_{2k-1} - 4\pi k t \; \theta_{2k}) \; e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_0^t q(\tau)d\tau}, u_{2k}^{(0)}(t) = \theta_{2k} e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_0^t q(\tau)d\tau}.$$

$$q^{(N)}(t) = \frac{\int_0^t \int_0^1 g\left(\alpha,\beta,u^{(N)}(\alpha,\beta)\right) d\alpha d\beta - h'(t)}{h(t)}.$$

From the theorem, we find $u^{(0)}(t) \in \mathbf{B}_1$, $t \in [0,T]$. For $N = 0$

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \int_0^t \int_0^1 g\left(\alpha,\beta,u^{(0)}(\alpha,\beta)\right) e^{-\int_\beta^t q^{(0)}(\tau)d\tau} d\xi d\beta,$$

Adding and subtracting $\int_0^t \int_0^1 f(\alpha,\beta,0) e^{-\int_\beta^t q^{(0)}(\tau)d\tau} d\alpha d\beta$, we find

$$u_0^{(1)}(t) = \theta_0(t) + \int_0^t \int_0^1 [g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)] e^{-\int_\beta^t q^{(0)}(\tau)d\tau} d\alpha d\beta + \int_0^t \int_0^1 g(\alpha,\beta,0) e^{-\int_\beta^t q^{(0)}(\tau)d\tau} d\alpha d\beta.$$

Applying Cauchy inequality,

$$\left|u_0^{(1)}(t)\right| \leq |\theta_0| + \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 [g(\alpha,\beta,u^{(0)}(\alpha,\beta)) - g(\alpha,\beta,0)] d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}}$$

$$+ \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 |g(\alpha,\beta,0)| \, d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}},$$

and using Lipschitzs condition , we obtain

$$\begin{aligned}\left|u_0^{(1)}(t)\right| &\leq |\theta_0| + \sqrt{t}\left(\int_0^t\left\{\int_0^1 b(\alpha,\beta)\left|u^{(0)}(\alpha,\beta)\right|d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}} \\ &+\sqrt{t}\left(\int_0^t\left\{\int_0^1 |g(\alpha,\beta,0)|\,d\alpha\right\}^2 d\beta\right)^{\frac{1}{2}},\end{aligned}$$

and taking maximum, we find:

$$\begin{aligned}\max_{0\leq t\leq T}\left|u_0^{(1)}(t)\right| &\leq |\theta_0| + \sqrt{T}\,\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B \\ &+\sqrt{T}\,\|g(x,t,0)\|_{L_2(\Gamma)}\,.\end{aligned}$$

using the same estimations and Hölder, Bessel inequality and taking maximum,

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{2k-1}^{(1)}(t)\right| &\leq \sum_{k=1}^{\infty}|\theta_{2k-1}| + \frac{\sqrt{3}}{12}\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\|_B \\ &+\frac{\sqrt{3}}{12}\|g(x,t,0)\|_{L_2(\Gamma)}\,,\end{aligned}$$

and applying the same estimations we obtain,

$$\begin{aligned}\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{2k}^{(1)}(t)\right| &\leq \sum_{k=1}^{\infty}|\theta_{2k-1}| + 4\pi\,|T|\sum_{k=1}^{\infty}\left|\theta_{2k}'\right| \\ &+\frac{\sqrt{3}}{12}\|b(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_B \\ &+\frac{\sqrt{3}}{12}\|g(x,t,0)\|_{L_2(\Omega)} \\ &+\frac{\sqrt{2}\,|T|}{4\pi}\|b(x,t)\|_{L_2(\Gamma)}\left\|u^{(0)}(t)\right\|_B \\ &+\frac{\sqrt{2}\,|T|}{4\pi}\|g(x,t,0)\|_{L_2(\Omega)}\,,\end{aligned}$$

and then, we find,

$$\begin{aligned}&\left\|u^{(1)}(t)\right\|_{\mathbf{B}} \\ \leq\ & 2\,|\theta_0| + 4\sum_{k=1}^{\infty}|\theta_{2k-1}| + 4\pi\,|T|\sum_{k=1}^{\infty}\left|\theta_{2k}'\right| \\ &+\left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+3\sqrt{2}\,|T|}{3\pi}\right)\|b(x,t)\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_B \\ &+\left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+3\sqrt{2}\,|T|}{3\pi}\right)\|g(x,t,0)\|_{L_2(\Omega)}\,.\end{aligned}$$

$u^{(1)}(t) \in \mathbf{B}$. Same estimations for N,

$$\begin{aligned} &\left\| u^{(N+1)}(t) \right\|_{\mathbf{B}} \\ \leq \quad & 2\left|\theta_0\right| + 8\sum_{k=1}^{\infty}\left|\theta_{2k-1}\right| + 16\pi\left|T\right|\sum_{k=1}^{\infty}\left|\theta_{2k}^{'}\right| \\ & + \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_{B_1} \\ & + \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|g(x,t,0)\right\|_{L_2(D)}. \end{aligned}$$

According to $u^{(N)}(t) \in \mathbf{B}$ and theorem, $u^{(N+1)}(t) \in \mathbf{B}$,

$$\{u(t)\} = \{u_0(t), u_{2k}(t), u_{2k-1}(t),\ k = 1,2,...\} \in \mathbf{B}.$$

If we used with same estimations,we obtain

$$\begin{aligned} \left\| q^{(N+1)} \right\|_{C[0,T]} \quad & \leq \quad \left|\frac{h^{'}(t)}{h(t)}\right| + \frac{\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(N)}(t)\right\|_B}{|h(t)|} \\ & + \frac{\left\|g(x,t,0)\right\|_{L_2(\Omega)}}{|h(t)|}. \end{aligned}$$

We show that the iterations $u^{(N+1)}(t), q^{(N+1)}$ converge $\mathbf{B}$ and $C[0,T]$, respectively for $N \to \infty$ Using Cauchy, Bessel, Hölder inequality, Lipschitzs condition and taking maximum of both side of the last inequality, we obtain:

$$\begin{aligned} \left\| u^{(1)}(t) - u^{(0)}(t) \right\|_{B_1} \quad \leq \quad & \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|b(x,t)\right\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_B \\ & + \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|g(x,t,0)\right\|_{L_2(\Omega)}. \end{aligned}$$

$$\begin{aligned} A \quad = \quad & \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|b(x,t)\right\|_{L_2(\Omega)}\left\|u^{(0)}(t)\right\|_B \\ & + \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|g(x,t,0)\right\|_{L_2(\Omega)}. \end{aligned}$$

$$\left\| q^{(1)}(t) - q^{(0)}(t) \right\|_{C[0,T]} \leq \frac{\sqrt{T}}{|h(t)|}\left\| u^{(1)}(t) - u^{(0)}(t) \right\|_B \left\|b(x,t)\right\|_{L_2(\Omega)},$$

$$\begin{aligned} \left\| u^{(2)}(t) - u^{(1)}(t) \right\|_{B_1} \quad \leq \quad & \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left\|b(x,t)\right\|_{L_2(\Omega)}\left\| u^{(1)}(t) - u^{(0)}(t) \right\|_B \\ & + \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\frac{M\sqrt{T}}{|h(t)|}\left\| u^{(1)}(t) - u^{(0)}(t) \right\|_B \left\|b(x,t)\right\|_{L_2(\Omega)}. \end{aligned}$$

$$\left\| u^{(2)}(t) - u^{(1)}(t) \right\|_{B_1} \leq \left\{ \left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left(1+\frac{M\sqrt{T}}{|h(t)|}\right) \right\} A \left\|b(x,t)\right\|_{L_2(\Omega)},$$

$$\left\| q^{(2)}(t) - q^{(0)}(t) \right\|_{C[0,T]} \leq \frac{\sqrt{T}}{|h(t)|} \left\| u^{(2)}(t) - u^{(2)}(t) \right\|_B \|b(x,t)\|_{L_2(\Omega)},$$

$$\left\| u^{(3)}(t) - u^{(2)}(t) \right\|_{B_1} \leq \left\{ \left(\frac{2\pi\sqrt{3} + 6\pi\sqrt{T} + \left(1 + 2\sqrt{2}\pi\right)|T|}{3\pi} \right) \left(1 + \frac{M\sqrt{T}}{|h(t)|} \right) \right\}^2 \frac{A}{\sqrt{2!}} \|b(x,t)\|_{L_2(\Omega)}.$$

For N :

$$\left\| q^{(N+1)}(t) - q^{(N)}(t) \right\|_{C[0,T]} \leq \frac{\sqrt{T}}{|h(t)|} \left\| u^{(N+1)}(t) - u^{(N)}(t) \right\|_B \|b(x,t)\|_{L_2(\Omega)},$$

$$\left\| u^{(N+1)}(t) - u^{(N)}(t) \right\|_B \leq \frac{A}{\sqrt{N!}} \left\{ \left(\frac{2\pi\sqrt{3} + 6\pi\sqrt{T} + 3\sqrt{2}|T|}{3\pi} \right) \left(1 + \frac{M\sqrt{T}}{|h(t)|} \right) \right\}^N \|b(x,t)\|_{L_2(\Omega)}.$$

For $N \to \infty$,$u^{(N+1)}(t), q^{(N+1)}$ are converged. Let show that there exists u and q such that

$$\lim_{N\to\infty} u^{(N+1)}(t) = u(t), \ \lim_{N\to\infty} q^{(N+1)}(t) = q(t).$$

Using same inequality and Gronwall's inequality, we obtain

$$\begin{aligned} \left\| u(t) - u^{(N+1)}(t) \right\|_{\mathbf{B}}^2 \leq \\ & 2\left[\frac{A}{\sqrt{N!}} \left\{ \left(\frac{2\pi\sqrt{3} + 6\pi\sqrt{T} + \left(1 + 2\sqrt{2}\pi\right)|T|}{3\pi} \right) \right\}^N \|b(x,t)\|_{L_2(\Omega)}^N \right]^2 \\ & \times \exp 2\left\{ \left(\frac{2\pi\sqrt{3} + 6\pi\sqrt{T} + \left(1 + 2\sqrt{2}\pi\right)|T|}{3\pi} \right) \left(1 + \frac{M\sqrt{T}}{|h(t)|} \right) \right\}^2 \|b(x,t))\|_{L_2(\Omega)}^2. \end{aligned} \tag{9}$$

$$\left\| q(t) - q^{(N+1)}(t) \right\|_{C[0,T]} \leq \frac{\sqrt{T}}{|h(t)|} \left\| u(t) - u^{(N+1)}(t) \right\|_B \|b(x,t)\|_{L_2(\Omega)},$$

we obtain $u^{(N+1)} \to u, q^{(N+1)} \to q$, $N \to \infty$. For the uniqueness, let (u,q), (v,r) are two solution of (1)-(4). After applying Cauchy, Bessel, Lipschitz, Hölder inequality to $|u(t) - v(t)|$ and $|r(t) - q(t)|$, we obtain

$$\|u(t) - v(t)\|_{\mathbf{B}} \leq \left[\left(\frac{2\pi\sqrt{3} + 6\pi\sqrt{T} + \left(1 + 2\sqrt{2}\pi\right)|T|}{3\pi} \right) \left(1 + \frac{M\sqrt{T}}{|h(t)|} \right) \right] \left(\int_0^t \int_0^1 b^2(\alpha,\beta) |u(\beta) - v(\beta)|^2 \, d\alpha d\beta \right) \tag{10}$$

$u(t) = v(t)$ and then $r(t) = q(t)$. □

The proof is over.

3. Stability of problem

Theorem 3.1. *Assumption (C1)-(C3) the solution (q,u) of the problem (1)-(4) depends continuously upon the data θ, h.*

Proof. Let $\Phi = \{\theta,\ h,\ g\}$ and $\overline{\Phi} = \left\{\overline{\theta},\ \overline{h},\ g\right\}$ be two sets of the data, which satisfy the assumptions $(1)-(3)$. Let us denote $\|\Phi\| = (\|h\|_{C^1[0,T]} + \|\theta\|_{C^3[0,1]} + \|g\|_{C^{3,0}(\overline{\Gamma})})$. Let (q,u) and $(\overline{q},\overline{u})$ be solutions of problems (1)-(4).

$$
\begin{aligned}
u-\overline{u} \;=\;& 2(\theta_0-\overline{\theta_0})e^{-\int_\beta^t \overline{q(\tau)}d\tau} + 2\theta_0(e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau}) \\
&+4\sum_{k=1}^\infty (\theta_{2k-1}-\overline{\theta_{2k-1}})x\cos 2\pi kx e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\tau^t \overline{q(\tau)}d\tau} \\
&+4\sum_{k=1}^\infty \theta_{2k-1}x\cos 2\pi kx\ e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} (e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau}) \\
&+4\sum_{k=1}^\infty (\theta_{2k}-\overline{\theta_{2k}})\sin 2\pi kx e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\tau^t \overline{q(\tau)}d\tau} \\
&+4\sum_{k=1}^\infty \theta_{2k}\sin 2\pi kx\ e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} (e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau}) \\
&-16\pi\sum_{k=1}^\infty (\theta_{2k-1}-\overline{\theta_{2k-1}})kt\sin 2\pi kx e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\tau^t \overline{q(\tau)}d\tau} \\
&-16\pi\sum_{k=1}^\infty \theta_{2k-1}kt\sin 2\pi kx\ e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} (e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau}) \\
&+2\int_0^t\int_0^1 [g(\alpha,\beta,u(\alpha,\beta)) - g(\alpha,\beta,\overline{u}(\alpha,\beta))]\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\tau^t \overline{q(\tau)}d\tau} \\
&+2\int_0^t\int_0^1 g(\alpha,\beta,u(\alpha,\beta)) e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} (e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau}) \\
&+4\sum_{k=1}^\infty \int_0^t\int_0^1 [g(\alpha,\beta,u(\alpha,\beta)) - g(\alpha,\beta,\overline{u}(\alpha,\beta))]\,(1-x)\sin 2\pi kx e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\beta^t \overline{q(\tau)}d\tau} \\
&-16\pi\sum_{k=1}^\infty [g(\alpha,\beta,u(\alpha,\beta)) - g(\alpha,\beta,\overline{u}(\alpha,\beta))]\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} e^{-\int_\tau^t \overline{q(\tau)}d\tau} (t-\tau)\cos 2\pi kx \\
&-16\pi\sum_{k=1}^\infty [g(\alpha,\beta,u(\alpha,\beta)) - g(\alpha,\beta,\overline{u}(\alpha,\beta))]\, e^{\frac{-(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} (e^{-\int_\beta^t q(\tau)d\tau} - e^{-\int_\beta^t \overline{q(\tau)}d\tau})(t-\tau)\cos 2\pi kx.
\end{aligned}
$$

By using same estimations, we obtain:

$$
\begin{aligned}
\|u-\overline{u}\|_B \;\leq\;& M_3\left\|\theta-\overline{\theta}\right\| \qquad (11)\\
&+\left(\frac{2\pi\sqrt{3}+6\pi\sqrt{T}+\left(1+2\sqrt{2}\pi\right)|T|}{3\pi}\right)\left(\int_0^t\int_0^1 b^2(\alpha,\beta)\,|u(\beta)-\overline{u}(\beta)|^2\, d\alpha d\beta\right)^{\frac{1}{2}}.
\end{aligned}
$$

$$\|q-\overline{q}\|_B \leq M_1 \left\|h(t)-\overline{h(t)}\right\|_{C^1[0,T]} + M_2 \left\|u(t)-\overline{u(t)}\right\|_B \|b(x,t)\|_{L_2(\Omega)},$$

applying Gronwall's inequality, we obtain:

$$\begin{aligned}\|u-\overline{u}\|_B &\leq 2M_3 \left\|\Phi-\overline{\Phi}\right\|^2 \\ &\quad \times \exp 2\left(\int_0^t \int_0^1 b^2(\alpha,\beta) d\alpha d\beta\right).\end{aligned}$$

For $\Phi \to \overline{\Phi}$ then $u \to \overline{u}$. Hence $q \to \overline{q}$. □

4. Numerical Procedure for the nonlinear problem (1)-(4)

An iteration algorithm for the linearization of the problem (1)-(4):

$$\frac{\partial u^{(n)}}{\partial t} - \frac{\partial^2 u^{(n)}}{\partial x^2} - \varepsilon \frac{\partial^3 u^{(n)}}{\partial t x^2} - q(t)u^{(n)} = g(x,t,u^{(n-1)}) \qquad (x,t) \in D. \tag{12}$$

$$u^{(n)}(x,0) = \theta(x), \qquad x \in [0,1]. \tag{13}$$

$$u^{(n)}(0,t) = 0, \qquad t \in [0,T]. \tag{14}$$

$$u_x^{(n)}(0,t) = u_x^{(n)}(1,t), \ t \in [0,T]. \tag{15}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $g(x,t,u^{(n-1)}) = \widetilde{g}(x,t)$ then a new linear problem :

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \varepsilon \frac{\partial^3 v}{\partial t \partial x^2} - q(t)v = \widetilde{g}(x,t) \qquad (x,t) \in D. \tag{16}$$

$$v(0,t) = 0, \qquad t \in [0,T]. \tag{17}$$

$$v_x(0,t) = v_x(1,t), \qquad t \in [0,T]. \tag{18}$$

$$v(x,0) = \theta(x), \qquad x \in [0,1]. \tag{19}$$

we use the method of the linearization and the finite difference method to solve (16)-(19).

We subdivide the intervals $[0,1]$ and $[0,T]$ into subintervals N_x and N_t of equal lengths $h = \frac{1}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. We use the Crank-Nicolson scheme which is absolutely stable and has a second-order accuracy in h and a first-order accuracy in τ. The Crank-Nicolson scheme for (16)-(19) is as follows:

$$\begin{aligned}\frac{1}{\tau}\left(v_i^{j+1} - v_i^{j+1}\right) &= \frac{1}{h^2}\left(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}\right) \\ &+\varepsilon \frac{1}{2h^2\tau}\left[(v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1}) - (v_{i-1}^{j+1} - 2v_i^{j+1} + v_{i+1}^{j+1})\right] \\ &+q^{j+1}v^{j+1} + \widetilde{g}_i^{j+1},\end{aligned} \tag{20}$$

$$v_i^0 = \theta_i, \tag{21}$$

$$v_0^j = 0, \tag{22}$$

$$v_{N_x+1}^j = v_1^j + v_{N_x}^j, \tag{23}$$

where $1 \leq i \leq N_x$ and $0 \leq j \leq N_t$ are the indices for the spatial and time steps respectively, $v_i^j = v(x_i,t_j)$, $\theta_i = \theta(x_i)$, $q^j = q(t_j)$, $\widetilde{g}_i^j = \widetilde{g}(x_i,t_j)$, $x_i = ih$, $t_j = j\tau$.

$$q^j = \frac{\left(\left(h^{j+1} - h^j\right)/\tau\right) - \widetilde{g}_i^{j+1}}{h_i^j}, \tag{24}$$

where $h_i^j = h(t_j)$, $j = 0, 1, ..., N_t$.

The system can be solved by the Gauss elimination method, and v_i^j is determined.

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