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ON THE CONCEPT OF FUZZY COMULTISETS AND ITS PROPERTIES

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Abstract. The notion of fuzzy multigroups is an application of fuzzy multisets or fuzzy bags to classical group theory. In this paper, we propose the concept of fuzzy comultiset and explicate some of its properties. The relationship between fuzzy comultisets of a fuzzy multigroup and the cosets of a group is established. Some results on the concept of fuzzy comultisets are deduced with proofs.

Keywords: Fuzzy comultiset, fuzzy multiset, fuzzy multigroup, fuzzy submultigroup.

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1. Introduction

The theory of fuzzy sets proposed in [1], has been extensively researched with various applications ranging from engineering and computer science to medical diagnosis and social behavior, etc. In [2], the notion of fuzzy groups was first introduced as an extension of group theory to fuzzy sets and some number of results were obtained. Several researches have been carried out on some group theoretic notions in fuzzy group setting such as seen in [3, 4, 5, 6, 7, 8, 9].

Motivated by the work in [1], the idea of fuzzy multisets or fuzzy bags was proposed in [10] as a generalization of fuzzy sets in multisets framework. But, the concept of multiset is an extension of set whereby elements are allowed to repeat in a collection [11, 12, 13]. Many researches have gone into the concept of fuzzy multisets, for some details see [14, 15, 16]. Recently, in [17], the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory. In [18, 19], the notion of abelian fuzzy multigroups were explored with a number of results. The concepts of fuzzy submultigroups, normal fuzzy submultigroups, characteristic fuzzy submultigroups, and homomorphic properties of fuzzy multigroups were studied with some relevant results, as can be found in [20, 21, 22, 23].

The present paper is a further study of fuzzy multigroups in group theoretic analogs. We propose the concept of fuzzy comultisets as an extension of fuzzy cosets [3, 7] and obtain some related results. A number of some properties of cosets in fuzzy multigroup

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context are explored, thereby enriching the theory of fuzzy multigroups. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy multisets and fuzzy multigroups. In Section 3, we propose the idea of fuzzy comultiset and discuss some of its properties with results. Finally, Section 4 concludes the paper and provides direction for future studies.

2. Preliminaries

In this section, we present some existing definitions and results to be used in the sequel.

Definition 2.1. [10] Assume X is a set of elements. Then a fuzzy bag/multiset A drwan from X can be characterized by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From [16], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By [24], it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ [25, 26].

We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.1. Let $X = \{a, b, c\}$ be a set. Then a fuzzy multiset of X is given as

$$A = \{\langle \frac{0.5, 0.4, 0.3}{a} \rangle, \langle \frac{0.6, 0.4, 0.4}{b} \rangle, \langle \frac{0.7, 0.4, 0.2}{c} \rangle\}.$$

Definition 2.2. [10] Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. [10] Let $A, B \in FMS(X)$. Then A and B are equal if and only if $CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.4. [15] Let $A \in FMS(X)$. Then the cardinality of A denoted by $|A|$ is the length of the membership sequence $CM_A(x) = \mu_A^1(x), \mu_A^2(x), ..., \mu_A^m(x)$. We define the length $L(x; A)$, that is, the length of $\mu_A^i(x), i = 1, ..., m$ as

$$L(x; A) = \vee\{i \mid \mu_A^i(x) \neq 0\},$$

where $\vee$ stands for maximum.

The cardinality between two fuzzy multisets, say A and B of X, is the lengths of the membership sequences

$$CM_A(x) = \mu_A^1(x), \mu_A^2(x), ..., \mu_A^m(x) \text{ and } CM_B(x) = \mu_B^1(x), \mu_B^2(x), ..., \mu_B^n(x)$$

defined as $L(x; A, B) = \vee\{L(x; A), L(x; B)\}$. Where no ambiguity arises, we write $L(x) = L(x; A, B)$ for simplicity.

For example, let

$$A = \{\langle\frac{0.3, 0.2}{x}\rangle, \langle\frac{1, 0.5, 0.5}{y}\rangle\} \text{ and } B = \{\langle\frac{0.7, 0.1}{w}\rangle, \langle\frac{0.6}{x}\rangle, \langle\frac{0.8, 0.6}{y}\rangle\}.$$

Then $L(w; A) = 0$, $L(x; A) = 2$, $L(y; A) = 3$, $L(w; B) = 2$, $L(x; B) = 1$ and $L(y; B) = 2$. Also, $L(w) = 2$, $L(x) = 2$ and $L(y) = 3$. Then $|A| = 3$ and $|B| = 2$.

We can rewrite A and B as

$$A = \{\langle\frac{0, 0, 0}{w}\rangle, \langle\frac{0.3, 0.2, 0}{x}\rangle, \langle\frac{1, 0.5, 0.5}{y}\rangle\} \text{ and } B = \{\langle\frac{0.7, 0.1}{w}\rangle, \langle\frac{0.6, 0}{x}\rangle, \langle\frac{0.8, 0.6}{y}\rangle\}$$

by completing the membership sequences.

Definition 2.5. [15] Let $A, B \in FMS(X)$. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$ are defined by the rules that for any object $x \in X$,

(i) $CM_{A\cap B}(x) = CM_A(x) \wedge CM_B(x)$,
(ii) $CM_{A\cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum operations, respectively.

Before finding the intersection and union of A and B, the membership sequences of A and B should be equal. If not, it could be completed by affixing zero(s).

Definition 2.6. [22] A fuzzy multiset B of a set X is said to have sup-property if for any subset $W \subset X$ $\exists$ $w_0 \in W$ such that

$$CM_B(w_0) = \bigvee_{w\in W} [CM_B(w)].$$

Definition 2.7. [17, 21] Let X be a group. A fuzzy multiset A of X is said to be a fuzzy multigroupoid of X if

$$CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \; \forall x, y \in X.$$

A fuzzy multigroupoid of a group X is a fuzzy multigroup of X if

$$CM_A(x^{-1}) = CM_A(x) \forall x \in X.$$

It can be easily verified that if A is a fuzzy multigroup of X, then

$$CM_A(e) = \bigvee_{x\in X} CM_A(x) \; \forall x \in X,$$

that is, $CM_A(e)$ is the tip of A. The set of all fuzzy multigroups of X is denoted by $FMG(X)$.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a group of modulo 4 with respect to addition. Again, let

$$A = \{\langle\frac{1, 1, 1}{0}\rangle, \langle\frac{0.9, 0.7, 0.5}{1}\rangle, \langle\frac{0.8, 0.7, 0.4}{2}\rangle, \langle\frac{0.9, 0.7, 0.5}{3}\rangle\}$$

be a fuzzy multiset of X. Using Definition 2.7, it follows that A is a fuzzy multigroup of X.

But,

$$B = \{\langle\frac{1,1,1}{0}\rangle, \langle\frac{0.9,0.7,0.5}{1}\rangle, \langle\frac{0.9,0.7,0.5}{2}\rangle, \langle\frac{0.9,0.8,0.7}{3}\rangle\}$$

is a fuzzy multigroupoid of X. The order of A is the analogue of the notion of cardinality in Definition 2.4.

We notice that, a fuzzy multigroup is a fuzzy group that admits repetition of membership values. That is, a fuzzy multigroup collapses into a fuzzy group whenever repetition of membership values is ignored.

Remark 2.1. [17, 21] We notice the following from Definition 2.7:

(i) every fuzzy multigroup is a fuzzy multiset but the converse is not always true.

(ii) a fuzzy multiset A of a group X is a fuzzy multigroup if $\forall x, y \in X$,

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM(y)$$

holds.

Definition 2.8. [21] Let $\{A_i\}_{i \in I}, I = 1, ..., n$ be an arbitrary family of fuzzy multigroups of X. Then

$$CM_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i \in I} CM_{A_i}(x) \,\forall x \in X$$

and

$$CM_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i \in I} CM_{A_i}(x) \,\forall x \in X.$$

The family of fuzzy multigroups $\{A_i\}_{i \in I}$ of X is said to have inf or sup assuming chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

Definition 2.9. [18] Let $A \in FMG(X)$. Then A is said to be commutative if for all $x, y \in X$,

$$CM_A(xy) = CM_A(yx).$$

Definition 2.10. [17] Let A be a fuzzy multigroup of a group X. Then A^{-1} is defined by $CM_{A^{-1}}(x) = CM_A(x^{-1}) \,\forall x \in X$.

By Definition 2.7, we get $CM_{A^{-1}}(x) = CM_A(x^{-1}) = CM_A(x)$. That is, $A^{-1} = A$. Thus $A \in FMG(X) \Leftrightarrow A^{-1} \in FMG(X)$.

Definition 2.11. [22] Let $A, B \in FMG(X)$. Then the product $A \circ B$ of A and B is defined to be a fuzzy multiset of X as follows:

$$CM_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz}\{CM_A(y) \wedge CM_B(z)\}, & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.12. [21] Let $B \in FMG(X)$. A fuzzy submultiset A of B is called a fuzzy submultigroup of B denoted by $A \sqsubseteq B$ if A is a fuzzy multigroup. A fuzzy submultigroup A of B is a proper fuzzy submultigroup denoted by $A \sqsubset B$, if $A \sqsubseteq B$ and $A \neq B$.

Theorem 2.1. [22] *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid C_A(g) = C_A(e)\}, \\ K &= \{x \in X \mid C_{Ax}(y) = C_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again, $H = K$.

Definition 2.13. [17, 21] Let $A \in FMG(X)$. Then the set A_* defined as

$$A_* = \{x \in X \mid CM_A(x) > 0\}$$

is the level set or support of A. It follows that A_* is a subgroup of X.

3. Fuzzy comultiset and some of its properties

In this section, we define fuzzy comultiset with examples, and explicate some of its properties.

Definition 3.1. Let X be a group. For any fuzzy submultigroup A of a fuzzy multigroup B of X, the fuzzy submultiset yA of B for $y \in X$ defined by

$$CM_{yA}(x) = CM_A(y^{-1}x) \; \forall x \in X$$

is called the left fuzzy comultiset of A. Similarly, the submultiset Ay of B for $y \in X$ defined by

$$CM_{Ay}(x) = CM_A(xy^{-1}) \; \forall x \in X$$

is called the right fuzzy comultiset of A.

Example 3.1. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on the set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1),\ \rho_1 = (123),\ \rho_2 = (132),\ \rho_3 = (23),\ \rho_4 = (13),\ \rho_5 = (12).$$

Suppose

$$B = \{\langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.9,0.8}{\rho_1}\rangle, \langle\frac{0.9,0.8}{\rho_2}\rangle, \langle\frac{0.8,0.7}{\rho_3}\rangle, \langle\frac{0.8,0.7}{\rho_4}\rangle, \langle\frac{0.8,0.7}{\rho_5}\rangle\}$$

is a fuzzy multigroup of X. Then

$$A = \{\langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle\}$$

is a fuzzy submultigroup of B.

The left fuzzy comultisets of A in B for $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \in X$ are:

$$\rho_0 A = \{\langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle\},$$

$$\rho_1 A = \{\langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle\},$$

$$\rho_2 A = \{\langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle\},$$

$$\rho_3 A = \{\langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle\},$$

$$\rho_4 A = \{\langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.7,0.6}{\rho_5}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{1,1}{\rho_0}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle\},$$

$$\rho_5 A = \{\langle\frac{0.7,0.6}{\rho_5}\rangle, \langle\frac{0.7,0.6}{\rho_4}\rangle, \langle\frac{0.7,0.6}{\rho_3}\rangle, \langle\frac{0.8,0.7}{\rho_2}\rangle, \langle\frac{0.8,0.7}{\rho_1}\rangle, \langle\frac{1,1}{\rho_0}\rangle\}.$$

The right fuzzy comultisets can be computed similarly.

Remark 3.1. Since a fuzzy multiset is unordered, it follows that $xA = Ax \; \forall x \in X$. It is easily seemed that $xA = yA = zA = A \; \forall x, y, z \in X$ for the same reason. These are not so in the classical case.

The following result proves that the right and left fuzzy comultisets of a fuzzy submultigroup in a fuzzy multigroup are equal.

Proposition 3.1. *If A is a fuzzy submultigroup of $B \in FMG(X)$, then the right and left fuzzy comultisets of A in B are identical.*

Proof. Assume A is a fuzzy submultigroup of B. Then $\forall x \in X$ we have

$$\begin{aligned} CM_{yA}(x) = CM_A(y^{-1}x) &\geq CM_A(y) \wedge CM_A(x) \\ &= CM_A(x) \wedge CM_A(y) \\ &= CM_A(x) \wedge CM_A(y^{-1}). \end{aligned}$$

Suppose by hypothesis, $CM_A(x) \wedge CM_A(y) = CM_A(xy)$. Then we have

$$CM_{yA}(x) \geq CM_{Ay}(x).$$

Again,

$$\begin{aligned} CM_{Ay}(x) = CM_A(xy^{-1}) &\geq CM_A(x) \wedge CM_A(y) \\ &= CM_A(y) \wedge CM_A(x) \\ &= CM_A(y^{-1}) \wedge CM_A(x). \end{aligned}$$

By the same hypothesis, we get

$$CM_{Ay}(x) \geq CM_{yA}(x).$$

Hence, $CM_{yA}(x) = CM_{Ay}(x) \Rightarrow yA = Ay$. □

Remark 3.2. Let A be a fuzzy submultigroup of $B \in FMG(X)$. We notice there is a one-to-one correspondence between the set of right fuzzy comultisets and the set of left fuzzy comultisets of A in B.

The following propositions are straightforward, hence they are stated without proves.

Proposition 3.2. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then the number of fuzzy comultisets of A in B equals the cardinality of A_*.*

Proposition 3.3. *If A is a fuzzy submultigroup of $B \in FMG(X)$, then*

$$xA \cap yA \cap zA = A = xA \cup yA \cup zA$$

$\forall x, y, z \in X$.

Proposition 3.4. *If A is a fuzzy submultigroup of $B \in FMG(X)$, then both the right and left fuzzy comultisets of A in B are fuzzy submultigroups of B.*

Theorem 3.1. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $gA = hA$ for $g, h \in X$ if and only if*

$$CM_A(g^{-1}h) = CM_A(h^{-1}g) = CM_A(e).$$

Proof. Let $gA = hA$. Then $CM_{gA}(g) = CM_{hA}(g)$ and $CM_{gA}(h) = CM_{hA}(h)$ $\forall g, h \in X$. Hence

$$CM_A(g^{-1}h) = CM_A(h^{-1}g) = CM_A(e).$$

Conversely, let $CM_A(g^{-1}h) = CM_A(h^{-1}g)$ $\forall g, h \in X$. For every $x \in X$, we have

$$\begin{aligned} CM_{gA}(x) = CM_A(g^{-1}x) &= CM_A(g^{-1}hh^{-1}x) \\ &\geq CM_A(g^{-1}h) \wedge CM_A(h^{-1}x) \\ &= CM_A(h^{-1}x) \\ &= CM_{hA}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} CM_{hA}(x) = CM_A(h^{-1}x) &= CM_A(h^{-1}gg^{-1}x) \\ &\geq CM_A(h^{-1}g) \wedge CM_A(g^{-1}x) \\ &= CM_A(g^{-1}x) \\ &= CM_{gA}(x). \end{aligned}$$

Hence $CM_{gA}(x) = CM_{hA}(x) \Rightarrow gA = hA$. □

Corollary 3.1. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $Ag = Ah$for $g, h \in X$ if and only if*

$$CM_A(gh^{-1}) = CM_A(hg^{-1}) = CM_A(e).$$

Proof. Straightforward from Theorem 3.1. □

Proposition 3.5. *Let $A, B \in FMG(X)$ such that $A \subseteq B$. If $gA = hA$, then $CM_A(g) = CM_A(h)\, \forall g, h \in X$.*

Proof. Let X be a group and $g \in X$. Suppose $gA = hA$, then we have

$$\begin{aligned} CM_{gA}(g) = CM_{hA}(g) &\Rightarrow CM_A(g^{-1}g) = CM_A(h^{-1}g) \\ &\Rightarrow CM_A(e) = CM_A(h^{-1}g) \end{aligned}$$

$\forall g, h \in X$. Since $CM_A(e) = CM_A(h^{-1}g) \Rightarrow CM_A(h) = CM_A(g)$, the result follows.

Alternatively, suppose $z \in X$, we get

$$\begin{aligned} CM_{gA}(z) = CM_{hA}(z) &\Rightarrow CM_A(g^{-1}z) = CM_A(h^{-1}z) \\ &\Rightarrow CM_{Az^{-1}}(g) = CM_{Az^{-1}}(h) \\ &\Rightarrow CM_A(g) = CM_A(h), \end{aligned}$$

since $Az^{-1} = A$. □

Theorem 3.2. *Let $B \in FMG(X)$. Any fuzzy submultigroup A of B and for any $z \in X$, the fuzzy submultiset zAz^{-1}, where $CM_{zAz^{-1}}(x) = CM_A(z^{-1}xz)$ for each $x \in X$ is a fuzzy submultigroup of B.*

Proof. Let $x, y \in X$ and $A \sqsubseteq B$. We prove that $zAz^{-1} \sqsubseteq B\ \forall z \in X$. Now

$$\begin{aligned} CM_{zAz^{-1}}(xy^{-1}) &= CM_A(z^{-1}xy^{-1}z) \\ &= CM_A(z^{-1}xzz^{-1}y^{-1}z) \\ &\geq CM_A(z^{-1}xz) \wedge CM_A(z^{-1}y^{-1}z) \\ &= CM_{zAz^{-1}}(x) \wedge CM_{zAz^{-1}}(y^{-1}) \\ &= CM_{zAz^{-1}}(x) \wedge CM_{zAz^{-1}}(y)\ \forall z \in X. \end{aligned}$$

Thus $CM_{zAz^{-1}}(xy^{-1}) \geq CM_{zAz^{-1}}(x) \wedge CM_{zAz^{-1}}(y)$. Hence zAz^{-1} is a fuzzy submultigroup of B. □

Corollary 3.2. *Let $\{A_i\}_{i\in I} \in FMG(X)$, then*

(i) $\bigcap_{i\in I} zA_iz^{-1} \in FMG(X)\, \forall z \in X$,
(ii) $\bigcup_{i\in I} zA_iz^{-1} \in FMG(X)\, \forall z \in X$ *provided $\{A_i\}_{i\in I}$ have sup/inf assuming chain.*

Proof. The results follow from Theorem 3.2. □

The following results are the application of product of fuzzy multigroups to the idea of fuzzy comultisets.

Proposition 3.6. *Let A be a fuzzy submultigroup of $B \in FMG(X)$ and for all $g, h \in X$, then the following statements hold;*

(i) $Ag \circ Ag = Ag$.
(ii) $Ag \circ Ah = Ah \circ Ag$.
(iii) $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$.
(iv) $(Ag \circ Ah)^{-1} = Ag \circ Ah$.

Proof. Let $g, h \in X$.
(i) From Definition 2.11, we have

$$\begin{aligned} CM_{Ag \circ Ag}(x) &= \bigvee_{x=yz} (CM_{Ag}(y) \wedge CM_{Ag}(z)) \, \forall y, z \in X \\ &= \bigvee_{y \in X} (CM_{Ag}(xy^{-1}) \wedge CM_{Ag}(y)) \, \forall x \in X. \end{aligned}$$

Hence $Ag \circ Ag = Ag$.

(ii) We get

$$\begin{aligned} CM_{Ag \circ Ah}(x) &= \bigvee_{x=yz} (CM_{Ag}(y) \wedge CM_{Ah}(z)) \, \forall y, z \in X \\ &= \bigvee_{x=yz} (CM_{Ah}(z) \wedge CM_{Ag}(y)) \, \forall y, z \in X \\ &= CM_{Ah \circ Ag}(x). \end{aligned}$$

Hence $Ag \circ Ah = Ah \circ Ag$.

(iii) We show that, the left and right hand sides are equal.

$$CM_{(Ag \circ Ah)^{-1}}(x) = CM_{Ag \circ Ah}(x^{-1}) = CM_{Ag \circ Ah}(x).$$

Again, from the right hand side we get

$$\begin{aligned} CM_{(Ah)^{-1} \circ (Ag)^{-1}}(x) &= \bigvee_{y \in X} (CM_{(Ah)^{-1}}(y^{-1}) \wedge CM_{(Ag)^{-1}}(yx)) \, \forall x \in X \\ &= \bigvee_{y \in X} (CM_{Ah}(y^{-1}) \wedge CM_{Ag}(yx)) \, \forall x \in X \\ &= CM_{Ah \circ Ag}(x) = CM_{Ag \circ Ah}(x). \end{aligned}$$

Hence $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$.

(iv) Straightforward from (iii). □

Remark 3.3. Proposition 3.6 also holds for left fuzzy comultisets.

Proposition 3.7. *Let X be a group. If A is a fuzzy submultigroup of a commutative fuzzy multigroup B of X, then*

(i) $Ay \circ Az = Ayz \, \forall y, z \in X$,
(ii) $yA \circ zA = yzA \, \forall y, z \in X$.

Proof. (i) Let $A \in FMG(X)$ and $x \in X$, then we have

$$\begin{aligned} CM_{Ay \circ Az}(x) &= \bigvee_{x=zy} (CM_{Ay}(z) \wedge CM_{Az}(y)) \; \forall y, z \in X \\ &= \bigvee_{x=zy} (CM_A(zy^{-1}) \wedge CM_A(yz^{-1})) \; \forall y, z \in X \\ &= \bigvee_{x=zy} CM_{A \cap A}((zy^{-1})(yz^{-1})) \; \forall y, z \in X \\ &= CM_A(xz^{-1}y^{-1}), x = zy, \forall y, z \in X \\ &= CM_{Ayz}(x), \forall y, z \in X. \end{aligned}$$

Hence $Ay \circ Az = Ayz$.

(ii) Similar to (i). □

Corollary 3.3. *Let A be a fuzzy submultigroup of a commutative fuzzy multigroup B of a group X and $y, z \in X$. Then the following statements are equivalent.*

(i) $(Ay \circ Az)^{-1} = Ay \circ Az$,
(ii) $Ay \circ Az = Ayz$.

Proof. By implication, $A \in FMG(X)$ and it is commutative.
(i)⇒(ii). $CM_{(Ay \circ Az)^{-1}}(x) = CM_{Ay \circ Az}(x) = CM_{Ayz}(x) \; \forall x \in X$ by Proposition 3.7.
(ii)⇒(i). $CM_{Ayz}(x) = CM_{Ay \circ Az}(x) = CM_{Ay \circ Az}(x^{-1}) = CM_{(Ay \circ Az)^{-1}}(x) \; \forall x \in X$. □

Remark 3.4. If $(Ay \circ Ay)^{-1} = Ay \circ Az$ and $Ay \circ Az = Ayz$, then
$(Ay \circ Az)^{-1} = Ayz$.

Theorem 3.3. *Let A be a fuzzy submultigroup of $B \in FMG(X)$ such that B is commutative, then $Ag \circ Ah = Agh$ if and only if $gA \circ hA = ghA \; \forall \; g, h \in X$. Consequently, $Agh = ghA$.*

Proof. Let $A \in FMG(X)$ and $g, h \in X$. Suppose $Ag \circ Ah = Agh$. By Definition 2.11, we get

$$\begin{aligned} CM_{Agh}(x) = CM_{Ag \circ Ah}(x) &= \bigvee_{y \in X} (CM_{Ag}(y) \wedge CM_{Ah}(y^{-1}x)) \\ &= \bigvee_{y \in X} (CM_A(yg^{-1}) \wedge CM_A(y^{-1}xh^{-1})) \\ &= \bigvee_{y \in X} (CM_A(g^{-1}y) \wedge CM_A(h^{-1}y^{-1}x)) \\ &= \bigvee_{y \in X} (CM_{gA}(y) \wedge CM_{hA}(y^{-1}x)) \\ &= CM_{gA \circ hA}(x) = CM_{ghA}(x) \end{aligned}$$

$\Rightarrow gA \circ hA = ghA$.

Conversely, let $gA \circ hA = ghA$. Then we have

$$\begin{aligned} CM_{ghA}(x) = CM_{gA \circ hA}(x) &= \bigvee_{y \in X} (CM_{gA}(y) \wedge CM_{hA}(y^{-1}x)) \\ &= \bigvee_{y \in X} (CM_A(g^{-1}y) \wedge CM_A(h^{-1}y^{-1}x)) \\ &= \bigvee_{y \in X} (CM_A(yg^{-1}) \wedge CM_A(y^{-1}xh^{-1})) \\ &= \bigvee_{y \in X} (CM_{Ag}(y) \wedge CM_{Ah}(y^{-1}x)) \\ &= CM_{Ag \circ Ah}(x) = CM_{Agh}(x) \end{aligned}$$

$\Rightarrow Ag \circ Ah = Agh$. Hence the result follow. □

Theorem 3.4. *Let X be a finite group and A be a fuzzy submultigroup of $B \in FMG(X)$. Define*

$$H = \{g \in X \mid CM_A(g) = CM_A(e)\}.$$

Then $Hx = Hy \Leftrightarrow Ax = Ay\ \forall x, y \in X$. Similarly, $xH = yH \Leftrightarrow xA = yA$.

Proof. This result gives a relationship between fuzzy comultisets of a fuzzy submultigroup of a fuzzy multigroup and the cosets of a subgroup of a given group.

By Theorem 2.1, we know that H is a subgroup of X and

$$H = \{x \in X \mid CM_{Ax}(z) = CM_{Ae}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. Thus

$$CM_{Axy^{-1}}(z) = CM_{Ae}(z)\ \forall z \in X \text{ and so } CM_A(zyx^{-1}) = CM_A(z).$$

Put $z = zy^{-1}$, we get

$$\begin{aligned} CM_A(zy^{-1}yx^{-1}) = CM_A(zy^{-1}) &\Rightarrow CM_A(zx^{-1}) = CM_A(zy^{-1}) \\ &\Rightarrow CM_{Ax}(z) = CM_{Ay}(z) \end{aligned}$$

and so, $Ax = Ay$.

Conversely, suppose that $Ax = Ay$, that is $CM_{Ax}(z) = CM_{Ay}(z)\ \forall z \in X$. This implies that

$$CM_A(zx^{-1}) = CM_A(zy^{-1}).$$

Put $z = y$, we get

$$CM_A(yx^{-1}) = CM_A(e).$$

So $yx^{-1} \in H$. Thus $Hx = Hy$.

The proof of $xH = yH \Leftrightarrow xA = yA$ is similar. □

Corollary 3.4. *Let X be a group. If A is a fuzzy submultigroup of a fuzzy multigroup B of X and $x, y \in X$. Then $xA = yA$ and $Ax = Ay$ if and only if $CM_A(y^{-1}x) = CM_A(yx^{-1}) = CM_A(e)$.*

Proof. Let $x, y \in X$, and recall that $H = \{x \in X \mid CM_A(x) = CM_A(e)\}$. Suppose $xA = yA$ and $Ax = Ay$. Then, $y^{-1}x, yx^{-1} \in H$ as in Theorem 3.4. So, $CM_A(y^{-1}x) = CM_A(e) = CM_A(yx^{-1})$.

Conversely, assume $CM_A(y^{-1}x) = CM_A(e) = CM_A(yx^{-1})$. This implies that,

$$CM_A(y^{-1}x) = CM_A(x^{-1}x) \text{ and } CM_A(yx^{-1}) = CM_A(yy^{-1})$$

and hence,

$$CM_{yA}(x) = CM_{xA}(x) \text{ and } CM_{Ax}(y) = CM_{Ay}(y) \forall x, y \in X.$$

Thus $xA = yA$ and $Ax = Ay$. □

4. Conclusions

We have proposed the notion of coset in fuzzy multigroup setting, which we called fuzzy comultisets. Some relevant properties of fuzzy comultisets were explored, and we have deduced some results with their proofs. Nevertheless, more properties of fuzzy comultisets could still be exploited.

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