TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 216-227

BOUNDS ON HYPER-STATUS CONNECTIVITY INDEX OF GRAPHS

K. PATTABIRAMAN[1], A. SANTHAKUMAR[2], §

Abstract. In this paper, we obtain the bounds for the hyper-status connectivity indices of a connected graph and its complement in terms of other graph invariants. In addition, the hyper-status connectivity indices of some composite graphs such as Cartesian product, join and composition of two connected graphs are obtained. We apply some of our results to compute the hyper-status connectivity indices of some important classes of graphs.

Keywords: Wiener index, status connectivity index, composite graph.
AMS Subject Classification: 05C12, 05C76

1. Introduction

A *topological index* is a mathematical measure which correlates to the chemical structures of any simple finite graph. They are invariant under the graph isomorphism. They play an important role in the study of $QSAR/QSPR$. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, nanoscience, biological and other properties of chemical compounds. Wiener index is the first distance-based topological index that were defined by Wiener [17]. For more details, see [5, 7–9].

The status [6] of a vertex $v \in V(G)$ is defined as the sum of its distance from every other vertex in $V(G)$ and is denoted by $\sigma_G(v)$, that is, $\sigma_G(v) = \sum\limits_{u \in V(G)} d_G(u, v)$, where $d_G(u, v)$ is the distance between u and v in G. The status of a vertex is also called as transmission of a vertex [6].

The Wiener index $W(G)$ of a connected graph G is defined as the sum of the distances between all pairs of vertices of G, that is,$W(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u, v) = \frac{1}{2} \sum\limits_{u \in V(G)} \sigma_G(v)$.

The first Zagreb index is defined as $M_1(G) = \sum\limits_{u \in V(G)} (d_G(u))^2 = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$ and the second Zagreb index is defined as $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u)d_G(v)$ The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [4]. The first

[1] Department of Mathematics, Government Arts College(Autonomous), Kumbakonam 612 002, India.
Department of Mathematics, Annamalai University, Annamalainagar 608 002, India.
e-mail: pramank@gmail.com; ORCID: https://orcid.org/0000-0001-7236-0732.
[2] Department of Mathematics, CK College of Engineering and Technology, Cuddalore, India.
e-mail: Santha.santhasulo.kumar8@gmail.com; ORCID: https://orcid.org/0000-0002-3727-0536.
§ Manuscript received: October 20, 2019; accepted: April 2, 2020.

and second Zagreb coindices were first introduced by Ashrafi et al. [2]. They are defined as follows: $\overline{M}_1(G) = \sum\limits_{uv \notin E(G)} (d_G(u) + d_G(v))$ and the second Zagreb index is defined as $\overline{M}_2(G) = \sum\limits_{uv \notin E(G)} d_G(u)d_G(v)$. The Zagreb indices and their variants have been used to study molecular complexity, chirality, ZE-isomerism and heterosystems. Overall, Zagreb indices exhibited a potential applicability for deriving multilinear regression models. Details on the chemical applications of the two Zagreb indices can be found in the books by Todeschini and Consonni [13, 14].

Motivated by the invariants like Zagreb indices and coindices, Ramane et al. [11] proposed the first status connectivity index $S_1(G)$ and first status connectivity coindex $\overline{S}_1(G)$ of a connected graph G as

$$S_1(G) = \sum_{uv \in E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big) \text{ and } \overline{S}_1(G) = \sum_{uv \notin E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big).$$

Similarly, the second status connectivity index $S_2(G)$ and second status connectivity coindex $\overline{S}_2(G)$ of a connected graph G as

$$S_2(G) = \sum_{uv \in E(G)} \sigma_G(u)\sigma_G(v) \text{ and } \overline{S}_2(G) = \sum_{uv \notin E(G)} \sigma_G(u)\sigma_G(v).$$

Shirdel et al. [12] introduced a new Zagreb index of a graph G named hyper-Zagreb index and is defined as $HM(G) = \sum\limits_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big)^2$. Recently, Veylaki et al. [15] defined the hyper-Zagreb coindex as $\overline{HM}(G) = \sum\limits_{uv \notin E(G)} \Big(d_G(u) + d_G(v)\Big)^2$.

Motivated by the invariants like hyper-Zagreb indices and coindices, we propose the hyper-status connectivity index $HS(G)$ and hyper-status connectivity coindex $\overline{HS}(G)$ of a connected graph G as

$$HS(G) = \sum_{uv \in E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big)^2 \text{ and } \overline{HS}(G) = \sum_{uv \notin E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big)^2.$$

The bounds for the status connectivity indices are determined in [11]. Also they are discussed the linear regression analysis of the distance-based indices with the boiling points of benzenoid hydrocarbons and the linear model based on the status index is better than the models corresponding to the other distance based indices. In [10], the exact formulae for the first status connectivity indices and its coindices of some composite graphs are obtained. In this connection, here we obtain the bounds for the hyper-status connectivity indices of a connected graph and its complement in terms of other graph invariants. Further, we study the behavior of the hyper-status connectivity index and apply our results to different chemically interesting molecular graphs.

2. Bounds for Hyper-status connectivity index

In this section, we obtain the hyper-status connectivity index of a graph G and its complements.

Theorem 2.1. *Let G be a connected graph on n vertices and m edges with $d(G) \leq 2$, then $HS(G) = HM(G) - 8(n-1)M_1(G) + 16(n-1)^2 m$.*

Proof: Since $d(G) \leq 2$, for each vertex of $u \in V(G)$, the $d_G(u)$ vertices at distance 1 from the vertex u and $n - 1 - d_G(u)$ vertices of G at distance 2 from u in G. Thus for each vertex u in G, we have $\sigma_G(u) = d_G(u) + 2(n-1-d_G(u)) = 2n-2-d_G(u)$. Hence,

$$
\begin{aligned}
HS(G) &= \sum_{uv \in E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big)^2 \\
&= \sum_{uv \in E(G)} \Big(2n-2-d_G(u) + 2n-2-d_G(v)\Big)^2 \\
&= \sum_{uv \in E(G)} \Big(4n-4-(d_G(u)+d_G(v))\Big)^2 \\
&= \sum_{uv \in E(G)} \Big((4n-4)^2 + (d_G(u)+d_G(v))^2 - 2(4n-4)(d_G(u)+d_G(v))\Big)
\end{aligned}
$$

From the definitions of hyper Zagreb index and first Zagreb index, we obtain

$$
HS(G) = HM(G) - 8(n-1)M_1(G) + 16(n-1)^2 m.
$$

Using Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let G be a r-regular graph on n vertices and m edges with $d(G) \leq 2$. Then $HS(G) = 4m\Big(4n(n-2) + r(r-4n+4) + 4\Big)$.*

Example 2.1. *For a complete graph K_n on n vertices, $HS(K_n) = 2n(n-1)^3$.*

Lemma 2.1. *Let P_n be a path on n vertices. Then $HS(P_n) = \frac{(n-1)}{15}\Big(12(n-1)^4 + 30(n-1)^3 + 20(n-1)^2 - (5n^4 - 10n^3 + 2)\Big)$.*

Proof: Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$, where u_k is adjacent to u_{k+1}, $k = 1, 2, \ldots, n-1$. Thus for $k = 1, 2, \ldots, n-1$, we have

$$
\begin{aligned}
\sigma_G(u_k) &= (k-1) + (k-2) + \ldots + 1 + 1 + 2 + \ldots + (n-k) \\
&= \frac{(k-1)i}{2} + \frac{(n-k)(n-k+1)}{2} \\
&= \frac{n(n+1)}{2} + k(k-n-1). \qquad (1)
\end{aligned}
$$

Hence by (1), we obtain:

$$
\begin{aligned}
HS(P_n) &= \sum_{k=1}^{n-1}(\sigma_{P_n}(v_k)+\sigma_{P_n}(v_{k+1}))^2 \\
&= \sum_{k=1}^{n-1}\Big(\frac{n^2+n}{2}+k(k-n-1)+\frac{n^2+n}{2}+(k+1)(k-n)\Big)^2 \\
&= \sum_{k=1}^{n-1}\Big(n^2+2k^2-2kn\Big)^2 \\
&= \sum_{k=1}^{n-1}\Big(n^4+4k^4-4kn^3-8k^3n+8k^2n^2\Big) \\
&= (n-1)n^4+4\sum_{k=1}^{n-1}k^4-8n\sum_{k=1}^{n-1}k^3+8n^2\sum_{k=1}^{n-1}k^2-4n^3\sum_{k=1}^{n-1}k \\
&= n^4(n-1)+\frac{4}{30}\Big(6(n-1)^5+15(n-1)^4+10(n-1)^3-(n-1)\Big) \\
&\quad -8n\frac{(n-1)^2n^2}{4}+8n^2\frac{(n-1)n(2n-1)}{6}-4n^3\frac{(n-1)n}{2} \\
&= \frac{4}{5}(n-1)^5+2(n-1)^4+\frac{4}{3}(n-1)^3-2n^3(n-1)^2+\frac{1}{15}(n-1)(25n^4-20n^3-2).
\end{aligned}
$$

Lemma 2.2. *Let C_n be a cycle on n vertices. Then* $HS(C_n)=\begin{cases}\frac{n^5}{4} & \text{if } n \text{ is even}\\ \frac{n(n^2-1)^2}{4} & \text{if } n \text{ is odd.}\end{cases}$

Proof: If n is even, then for every vertex v of C_n, $\sigma_{C_n}(v)=2\Big(1+2+\ldots+\frac{n-1}{2}\Big)+\frac{n}{2}=\frac{n^2}{4}$. Thus $HS(C_n)=\sum\limits_{uv\in E(C_n)}\big(\sigma_{C_n}(u)+\sigma_{C_n}(v)\big)^2=\frac{n^5}{4}$.

If n is odd, then for every vertex v of C_n,$\sigma_{C_n}(v)=2\Big(1+2+\ldots+\frac{n-1}{2}\Big)=\frac{n^2-1}{4}$. Thus $HS(C_n)=\sum\limits_{uv\in E(C_n)}\big(\sigma_{C_n}(u)+\sigma_{C_n}(v)\big)^2=\frac{n(n^2-1)^2}{4}$.

Now we obtain the lower bounds for hyper-status connectivity index of the complement of a given graph G.

Theorem 2.2. *Let G be a graph on n vertices and m edges. If $\overline{G}$ be a complement of G, then $HS(\overline{G})\geq\overline{HM}(G)+4(n-1)\overline{M_1}(G)+2(n-1)^2(n-2m-1)$. Equality holds if and only if $d(\overline{G})\leq 2$.*

proof. Let $v\in V(\overline{G})$. There are $d_G(v)$ vertices which are at distance at least 2 from v and remaining $n-1-d_G(v)$ vertices at distance 1 from the vertes v in G. Thus $\sigma_{\overline{G}}(v)\geq(n-1+d_G(v))+2d_G(v)=n-1+d_G(v)$. Hence

$$\begin{aligned}
HS(\overline{G}) &= \sum_{uv\in E(\overline{G})} \Big(\sigma_{\overline{G}}(u)+\sigma_{\overline{G}}(v)\Big)^2\\
&\geq \sum_{uv\in E(\overline{G})} \Big(n-1+d_G(u)+n-1+d_G(v)\Big)^2\\
&= \sum_{uv\notin E(G)} \Big(2(n-1)+d_G(u)+d_G(v)\Big)^2\\
&= \sum_{uv\notin E(G)} \Big(4(n-1)^2+(d_G(u)+d_G(v))^2+4(n-1)(d_G(u)+d_G(v))\Big).
\end{aligned}$$

From the definitions of first Zagreb coindex and hyper Zagreb coindex, we obtain:

$$\begin{aligned}
HS(\overline{G}) &= 4(n-1)^2\Big(\frac{n(n-1)}{2}-m\Big)+\overline{HM}(G)+4(n-1)\overline{M_1}(G)\\
&= \overline{HM}(G)+4(n-1)\overline{M_1}(G)+2(n-1)^2(n-2m-1). \qquad (2)
\end{aligned}$$

Conversely, Let the value of $HS(\overline{G})$ be given in (2). Suppose $d(\overline{G})\geq 3$, then there exists at least one pair of vertices , say u and v such that $d_G(u,v)\geq 3$. Thus $\sigma_{\overline{G}}(u)\geq d_{\overline{G}}(u)+3+2(n-2-d_{\overline{G}}(u))=n+d_G(u)$. Similarly, $\sigma_{\overline{G}}(v)\geq n+d_G(v)$ and for all other vertices $x\neq u,v$ of $\overline{G},\sigma_{\overline{G}}(x)\geq n-1-d_G(x)$.

From the above discussion, we partition the edge set of $\overline{G}$ into three sets, namely E_1,E_2 and E_3 such that

$$\begin{aligned}
E_1 &= \left\{ux|\sigma_{\overline{G}}(u)\geq n+d_G(u) \text{ and } \sigma_{\overline{G}}(x)\geq n-1+d_G(x)\right\},\\
E_2 &= \left\{vx|\sigma_{\overline{G}}(v)\geq n+d_G(v) \text{ and } \sigma_{\overline{G}}(x)\geq n-1+d_G(x)\right\} \text{ and}\\
E_3 &= \left\{xy|\sigma_{\overline{G}}(x)\geq n-1+d_G(x) \text{ and } \sigma_{\overline{G}}(y)\geq n-1+d_G(y)\right\}.
\end{aligned}$$

One can easily obtain that $|E_1|=d_{\overline{G}}(u)$,$|E_2|=d_{\overline{G}}(v)$ and $|E_3|=\frac{n(n-1)}{2}-m-d_{\overline{G}}(u)-d_{\overline{G}}(v)$. Hence

$$\begin{aligned}
HS(\overline{G}) &= \sum_{uv\in E(\overline{G})} \Big(\sigma_{\overline{G}}(u)+\sigma_{\overline{G}}(v)\Big)^2\\
&= \sum_{uv\in E_1} \Big(2n-1+d_G(u)+d_G(v)\Big)^2+\sum_{uv\in E_2} \Big(2n-1+d_G(u)+d_G(v)\Big)^2\\
&+ \sum_{uv\in E_3} \Big(2n-2+d_G(u)+d_G(v)\Big)^2\\
&= \sum_{uv\in E_1} \Big((2n-1)^2+(d_G(u)+d_G(v))^2+2(2n-1)(d_G(u)+d_G(v))\Big)\\
&+ \sum_{uv\in E_2} \Big((2n-1)^2+(d_G(u)+d_G(v))^2+2(2n-1)(d_G(u)+d_G(v))\Big)\\
&+ \sum_{uv\in E_3} \Big((2n-2)^2+(d_G(u)+d_G(v))^2+2(2n-2)(d_G(u)+d_G(v))\Big)\\
&= (2n-1)^2 d_{\overline{G}}(u)+(2n-1)^2 d_{\overline{G}}(v)+(2n-2)^2\Big(\frac{n(n-1)}{2}-m-d_{\overline{G}}(u)-d_{\overline{G}}(v)\Big)\\
&+ 2(2n-1)\sum_{uv\in E(\overline{G})}(d_G(u)+d_G(v))-2\sum_{uv\in E_3}(d_G(u)+d_G(v)).
\end{aligned}$$

From the definitions of first Zagreb coindex and hyper Zagreb coindex, we obtain: $HS(\overline{G}) = \overline{HM}(G)+2(2n-1)\overline{M_1}(G)+(4n-3)(d_{\overline{G}}(u)+d_{\overline{G}}(v))+2(n-1)^2(n^2-n-2m)-2\sum_{uv\in E_3}(d_G(u)+d_G(v))$, which is a contradiction. Therefore $d(\overline{G}) \leq 2$.

Finally, we obtain the lower and upper bounds for hyper-status connectivity index of a given graph G.

Theorem 2.3. *Let G be a connected graph on n vertices and m edges with $d(G) = d$. Then, $HM(G) - 8(n-1)M_1(G) + 16(n-1)^2m \leq HS(G) \leq (d-1)^2HM(G) - 4d(d-1)(n-1)M_1(G) + 4(n-1)^2d^2m$ with equality on both sides if and only if $d(G) \leq 2$.*

Proof: First we prove the lower bound. For any vertex $v \in V(G)$, there are $d_G(v)$ vertices which are at distance 1 from v and the remaining $n-1-d_G(v)$ vertices are at distance at least 2 from v in G. Thus $\sigma_G(v) \geq d_G(v) + 2(n-1-d_G(v)) = 2n-2-d_G(v)$. Hence

$$HS(G) = \sum_{uv\in E(G)} (\sigma_G(u)+\sigma_G(v))^2$$
$$\geq \sum_{uv\in E(G)} \Big(4n-4-(d_G(u)+d_G(v))\Big)^2$$
$$= \sum_{uv\in E(G)} \Big((4n-4)^2+(d_G(u)+d_G(v))^2-2(4n-4)(d_G(u)+d_G(v))\Big)^2$$
$$= HM(G) - 8(n-1)M_1(G) + 16(n-1)^2m.$$

Next we prove the upper bound. For any vertex $v \in V(G)$, there are $d_G(v)$ vertices which are at distance 1 from v and the remaining $n-1-d_G(v)$ vertices are at distance at most 2. Thus $\sigma_G(v) \leq d_G(v) + d(n-1-d_G(v)) = d(n-1)-(d-1)d_G(v)$. Hence

$$HS(G) = \sum_{uv\in E(G)} (\sigma_G(u)+\sigma_G(v))^2$$
$$\leq \sum_{uv\in E(G)} \Big(2d(n-1)-(d-1)(d_G(u)+d_G(v))\Big)^2$$
$$= \sum_{uv\in E(G)} \Big(4d^2(n-1)^2+(d-1)^2(d_G(u)+d_G(v))^2-4d(n-1)(d-1)(d_G(u)+d_G(v))\Big)^2$$
$$= (d-1)^2HM(G) - 4d(d-1)(n-1)M_1(G) + 4(n-1)^2d^2m.$$

Equality holds if diameter of G is 1 or 2. If $d \geq 3$, then there exists at least one pair of vertices u and v such that $d_G(u,v)$. Thus $\sigma_G(v) \geq 2n-1-d_G(v)$. Therefore $HS(G) \geq HM(G) - 8(n-1)M_1(G) + 16(n-1)^2m$, which is a contradiction to $HS(G) = HM(G) - 8(n-1)M_1(G) + 16(n-1)^2m$. Hence $d(G) \leq 2$.

3. Composite graphs

In this section, we obtain the hyper status connectivity indices of Cartesian product, join and composition of two given graphs.

3.1. **Cartesian product.** The *Cartesian product,* $G\,\square\,H$, of the graphs G and H has the vertex set $V(G\,\square\,H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G\,\square\,H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G\,\square\,H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G\,\square\,H$.

Theorem 3.1. *Let G and G' be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then* $HS(G\Box G') = n_2^3 HS(G) + n_1^3 HS(G') + 4n_1^2 m_1 \sum_{v_s \in V(G')} \sigma_{G'}^2(v_s) + 4n_2^2 m_2 \sum_{u_i \in V(G)} \sigma_G^2(u_i) + 8n_1 n_2 \Big(S_1(G)W(G') + S_1(G')W(G)\Big)$.

Proof: From the structure of $G\Box G'$, the distance between two vertices (u_i, v_r) and (u_k, v_s) of $G\Box G'$ is $d_G(u_i, u_k) + d_{G'}(v_r, v_s)$. Moreover, the degree of a vertex (u_i, v_r) in $V(G\Box G')$ is $d_G(u_i) + d_{G'}(v_r)$. By the definition of $\sigma((u_i, v_r))$ for the graph $G\Box G'$ and a vertex $(u_i, v_r) \in V(G\Box G')$, we have

$$\begin{aligned} \sigma_{G\Box G'}((u_i, v_r)) &= \sum_{(u_k,v_s)\in V(G\Box G')} d_{G\Box G'}((u_i, v_r), (u_k, v_s)) \\ &= \sum_{u_k\in V(G)} \sum_{v_s\in V(G')} \Big(d_G(u_i, u_k) + d_{G'}(v_r, v_s)\Big) \\ &= n_2\sigma_G(u_i) + n_1\sigma'_G(v_r). \end{aligned} \tag{3}$$

Hence by the definitions of HS and $G\Box G'$, we have

$$\begin{aligned} HS(G\Box G') &= \sum_{(u_i,v_r)(u_k,v_s)\in E(G\Box G')} \Big(\sigma_{G\Box G'}((u_i, v_r)) + \sigma_{G\Box G'}((u_k, v_s))\Big)^2 \\ &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box G')} \Big(\sigma_{G\Box G'}((u_i, v_s)) + \sigma_{G\Box G'}((u_k, v_s))\Big)^2 \\ &+ \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box G')} \Big(\sigma_{G\Box G'}((u_i, v_r)) + \sigma_{G\Box G'}((u_i, v_s))\Big)^2 \\ &= A_1 + A_2, \end{aligned} \tag{4}$$

where

$$\begin{aligned} A_1 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box G')} \Big(\sigma_{G\Box G'}((u_i, v_s)) + \sigma_{G\Box G'}((u_k, v_s))\Big)^2 \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_s\in V(G')} \Big(n_2\sigma_G(u_i) + n_1\sigma_{G'}(v_s) + n_2\sigma_G(u_k) + n_1\sigma_{G'}(v_s)\Big)^2, \text{ by (3)} \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_s\in V(G')} \Big(n_2\sigma_G(u_i) + 2n_1\sigma_{G'}(v_s) + n_2\sigma_G(u_k)\Big)^2 \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_s\in V(G')} \Big(n_2(\sigma_G(u_i) + \sigma_G(u_k))^2 + 4n_1^2\sigma_{G'}^2(v_s) + 4n_1n_2(\sigma_G(u_i) + \sigma_G(u_k))\sigma_{G'}(v_s)\Big) \\ &= n_2^3 \sum_{u_iu_k\in E(G)} (\sigma_G(u_i) + \sigma_G(u_k))^2 + 4n_1^2m_1 \sum_{v_s\in V(G')} \sigma_{G'}^2(v_s) \\ &\quad +4n_1n_2 \sum_{u_iu_k\in E(G)} (\sigma_G(u_i) + \sigma_G(u_k)) \sum_{v_s\in V(G')} \sigma_{G'}(v_s) \\ &= n_2^3 HS(G) + 4n_1^2m_1 \sum_{v_s\in V(G')} \sigma_{G'}^2(v_s) + 8n_1n_2S_1(G)W(G'), \text{by the definitions of} \\ &\quad \text{Wiener index, first and hyper-status connectivity index.} \end{aligned}$$

Similarly,

$$\begin{aligned} A_2 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box G')} \Big(\sigma_{G\Box G'}((u_i,v_r)) + \sigma_{G\Box G'}((u_i,v_s))\Big)^2 \\ &= n_1^3 HS(G') + 4n_2^2 m_2 \sum_{u_i\in V(G)} \sigma_G^2(u_i) + 8n_1n_2 S_1(G')W(G). \end{aligned}$$

From (4), A_1 and A_2, we have

$$\begin{aligned} HS(G\Box G') &= n_2^3 HS(G) + n_1^3 HS(G') + 4n_1^2 m_1 \sum_{v_s\in V(G')} \sigma_{G'}^2(v_s) \\ &\quad +4n_2^2 m_2 \sum_{u_i\in V(G)} \sigma_G^2(u_i) + 8n_1n_2\Big(S_1(G)W(G') + S_1(G')W(G)\Big). \end{aligned}$$

It is known that [11] $S_1(P_n) = \frac{1}{3}n(n-1)(2n-1)$ and $S_1(C_n) = \frac{n^3}{2}$ when n is even, and $\frac{n(n^2-1)}{2}$ otherwise. Similarly, one can easily verify that $W(P_n) = \frac{n(n^2-1)}{6}$ and $W(C_n) = \frac{n^3}{8}$ when n is even, and $\frac{n(n^2-1)}{8}$ otherwise.

Example 3.1. *For a C_4-nanotubes $TC_4(m,n) = C_n \times C_m$, the hyper-status connectivity index is given by $HS(C_n\Box C_m) =$*

$$\begin{cases} \frac{1}{4}(m^3 + n^3(n^5+m^5)) + m^4n^4, & \textit{where } n \textit{ is even and } m \textit{ is even} \\ m^3 + n^3\left[\frac{n(n^2-1)^2}{4} + \frac{m(m^2-1)^2}{4}\right] + n^2m^2(n^2-1)(m^2-1), & \textit{where } n \textit{ is odd and } m \textit{ is odd} \\ \frac{1}{4}m^3n^5 + \frac{1}{2}n^3m(m^2-1)^2 + \frac{1}{4}m^3n^2 + n^4m^2(m^2-1), & \textit{where } n \textit{ is even and } m \textit{ is odd} \\ \frac{5}{4}m^3n(n^2-1) + \frac{1}{4}(m^5n^3 + n^8 + 4m^4n^2(n^2-1)), & \textit{where } n \textit{ is even and } m \textit{ is odd} \end{cases}$$

3.2. **Join.** The *join* $G+H$ of two graphs G and H is the union $G\cup H$ together with all the edges joining $V(G)$ and $V(H)$. From the structure of $G+H$, the distance between two vertices u and v of $G+H$ is

$$d_{G+H}(u,v) = \begin{cases} 0, & \text{if } u=v \\ 1, & \text{if } uv\in E(G) \text{ or } uv\in E(H) \text{ or } (u\in V(G) \text{ and } v\in V(H)) \\ 2, & \text{otherwise.} \end{cases}$$

Moreover, the degree of a vertex v in $V(G+H)$ is

$$d_{G+H}(v) = \begin{cases} d_G(v) + |V(H)|, & if\ v\in V(G) \\ d_H(v) + |V(G)|, & if\ v\in V(H). \end{cases}$$

Now we obtain the value for HS of join of two given graphs.

Theorem 3.2. *Let G and G' be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then $HS(G+G') = HM(G) + HM(G') - M_1(G)(8n_1+3n_2-8) - M_1(G')(8n_2+3n_1-8) + 4(2n_1+n_2-2)^2m_1 + 4(2n_2+n_1-2)^2m_2 + n_1n_2(3n_1+3n_2-4)^2 - 4(3n_1+3n_2-4)(m_2n_1+m_1n_2) + 8m_1m_2$.*

Proof: Let u be a vertex in $V(G)$. Then from the structure of $G+G'$, we obtain:

$$\begin{aligned} \sigma_{G+G'}(u) &= \sum_{v\in V(G+G')} d_{G+G'}((u,v)) \\ &= \sum_{v\in V(G)\ u\neq v, uv\notin E(G)} (2) + \sum_{v\in V(G),\ u\neq v, uv\in E(G)} (1) + \sum_{v\in V(G')} (1) \\ &= 2n_1 + n_2 - 2 - d_G(u). \end{aligned} \quad (5)$$

Similarly, if v is a vertex of G', then $\sigma_{G+G'}(v) = 2n_2 + n_1 - 2 - d_G(v)$. Hence by the definition of HS, we have

$$\begin{aligned} HS(G+G') &= \sum_{uv\in E(G+G')} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &= \sum_{uv\in E(G)} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 + \sum_{uv\in E(G')} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &\quad + \sum_{u\in V(G)} \sum_{v\in V(G')} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &= A_1 + A_2 + A_3, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_1 &= \sum_{uv\in E(G)} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &= \sum_{uv\in E(G)} \Big(2n_1 + n_2 - 2 - d_G(u) + 2n_1 + n_2 - 2 - d_G(v)\Big)^2, \text{ by (5)} \\ &= \sum_{uv\in E(G)} \Big(2(2n_1 + n_2 - 2) - (d_G(u) + d_G(v))\Big)^2 \\ &= \sum_{uv\in E(G)} \Big(4(2n_1 + n_2 - 2)^2 + (d_G(u) + d_G(v)))^2 - 4(2n_1 + n_2 - 2)(d_G(u) + d_G(v)))\Big) \\ &= 4(2n_1 + n_2 - 2)^2 m_1 + HM(G) - 4(2n_1 + n_2 - 2)M_1(G), \text{ by the definitions of first and hyper Zagerb index of } G. \end{aligned}$$

Similarly,

$$\begin{aligned} A_2 &= \sum_{uv\in E(H)} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &= 4(2n_2 + n_1 - 2)^2 m_2 + HM(G') - 4(2n_2 + n_1 - 2)M_1(G'). \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{u\in V(G)} \sum_{v\in V(H)} \Big(\sigma_{G+G'}(u) + \sigma_{G+G'}(v)\Big)^2 \\ &= \sum_{u\in V(G)} \sum_{v\in V(G')} \Big(2n_1 + n_2 - 2 - d_G(u) + 2n_2 + n_1 - 2 - d_{G'}(v)\Big)^2 \\ &= \sum_{u\in V(G)} \sum_{v\in V(G')} \Big(3n_1 + 3n_2 - 4 - d_G(u) - d_{G'}(v)\Big)^2 \\ &= \sum_{u\in V(G)} \sum_{v\in V(G')} \Big((3n_1 + 3n_2 - 4)^2 + d_G^2(u) + d_{G'}^2(v) - 2(3n_1 + 3n_2 - 4)d_G(u) \\ &\quad -2(3n_1 + 3n_2 - 4)d_{G'}(v) + 2d_G(u)d_{G'}(v)\Big) \\ &= n_1 n_2 (3n_1 + 3n_2 - 4)^2 - 4(3n_1 + 3n_2 - 4)(m_1 n_2 + m_2 n_1) + n_2 M_1(G) \\ &\quad + n_1 M_1(G') + 8m_1 m_2. \end{aligned}$$

From (6), A_1,A_2 and A_3, we have

$$\begin{aligned} H(G+G') &= HM(G)+HM(G')-M_1(G)(8n_1+3n_2-8)-M_1(G')(8n_2+3n_1-8) \\ &\quad +4(2n_1+n_2-2)^2m_1+4(2n_2+n_1-2)^2m_2+n_1n_2(3n_1+3n_2-4)^2 \\ &\quad -4(3n_1+3n_2-4)(m_2n_1+m_1n_2)+8m_1m_2. \end{aligned}$$

It can be easily verified that $HM(C_n) = 16n$ and $HM(P_n) = 16n - 30$. Similarly, $M_1(C_n) = 4n$ and $M_1(P_n) = 4n - 6$. Using Theorem 3.2, we have the following examples.

Example 3.2. *The suspension of a graph G is defined as $G + K_1$. The hyper-status connectivity index of $G+K_1$ is $HM(G)-(8n-5)M_1(G)+n(3n-1)^2+4m(4n^2-7n+2)$.*

Example 3.3. *The fan graph F_n on $n+1$ vertices is the suspension of P_n. The hyper-status connectivity index of $P_n + K_1$ is $25n^3 - 82n^2 + 121n - 68$.*

Example 3.4. *The wheel graph W_n on $n+1$ vertices is the suspension of C_n. The hyper-status connectivity index of $C_n + K_1$ is $25n^3 - 66n^2 + 45n$.*

Example 3.5. *The fan graph F_n on $n+1$ vertices is the suspension of P_n. The hyper-status connectivity index of $P_n + K_1$ is $25n^3 - 82n^2 + 121n - 68$.*

Example 3.6. *The hyper-status connectivity index of the cone graph $C_n+\overline{K_q}$ is $4n((2n+q-2)^2-(3q-4))+nq(3n+3q-4)(3n+3q-8)$.*

Example 3.7. *The complete bipartite graph $K_{p,q}$ is defined as join of $\overline{K_p}$ and $\overline{K_q}$. The hyper-status connectivity index of $K_{p,q}$ is $pq(3p+3q-4)^2$.*

3.3. **Composition.** The *composition* of two graphs G and H is denoted by $G[H]$. The vertex set of $G[H]$ is $V(G)\times V(H)$ and any two vertices (u_i, v_r) and (u_k, v_s) are adjacent if and only if $u_iu_k \in E(G)$ or $[u_i = u_k \textit{ and } v_rv_s \in E(H)]$. Now we obtain the hyper status connectivity index of $G[H]$.

Theorem 3.3. *Let G and G' be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then $HS(G[G']) = n_2^4HS(G) + 8n_2^2(n_2(n_2-1)S_1(G)) + n_1M_1(G') + 8n_2(4(n_2-1)m_1 - M_1(G'))W(G) + 4n_2^2m_2 \sum\limits_{u\in V(G)} (\sigma_G(u))^2 + (2m_1n_2 - 8n_1(n_1-1))M_1(G') + 16(n_2-1)^2(n_1m_2 + n_2^2m_1) + 8m_1m_2(m_2 - 4n_2(n_2-1))$.*

Proof: For the composition of two graphs, the degree of a vertex (u, v) of $G[G']$ is given by $d_{G[G']}((u, v)) = n_2d_G(u) + d_{G'}(v)$. Moreover, the distance between two vertices (u_i, v_r) and (u_k, v_s) of $G[G']$ is $d_{G[G']}((u_i, v_r), (u_k, v_s)) = \begin{cases} d_G(u_i, u_k) & u_i \neq u_k \\ 2 & u_i = u_k,\ v_rv_s \notin E(G') \\ 1 & u_i = u_k,\ v_rv_s \in E(G'). \end{cases}$

Let (u_i, v_r) be a vertex of $G[G']$. Then

$$\begin{aligned} \sigma_{G[G']}((u_i, v_r)) &= \sum_{(u_k,v_s)\in V(G[G'])} d_{G[G']}((u_i, v_r), (u_k, v_s)) \\ &= \sum_{(u_k,v_s)\in V(G[G']),\ u_i\neq u_k} d_G(u_i, u_k) + \sum_{(u_i,v_s)\in V(G[G'])} d_{G[G']}((u_i, v_r), (u_i, v_s)) \\ &= n_2\sigma_G(u_i) + d_{G'}(v_r) + 2(n_2 - 1 - d_{G'}(v_r)) \\ &= n_2\sigma_G(u_i) + 2(n_2 - 1) - d_{G'}(v_r). \qquad (7) \end{aligned}$$

From the structure of $G[G']$ and definition of HS, we have

$$\begin{aligned} HS(G[G']) &= \sum_{(u_i,v_r)(u_k,v_s)\in E(G[G'])} \Big(\sigma_{G[G']}((u_i,v_r)) + \sigma_{G[G']}((u_k,v_s))\Big)^2 \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(G')} \Big(\sigma_{G[G']}((u_i,v_r)) + \sigma_{G[G']}((u_i,v_s))\Big)^2 \\ &\quad + \sum_{u_iu_k\in E(G)} \sum_{v_r\in V(G')} \sum_{v_s\in V(G')} \Big(\sigma_{G[G']}((u_i,v_r)) + \sigma_{G[G']}((u_k,v_s))\Big)^2 \\ &= A_1 + A_2, \ where \end{aligned} \tag{8}$$

$$\begin{aligned} A_1 &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big(\sigma_{G[G']}((u_i,v_r)) + \sigma_{G[G']}((u_i,v_s))\Big)^2 \\ &= \sum_{u_i\in v(G)} \sum_{v_rv_s\in E(G')} \Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_{G'}(v_r) + n_2\sigma_G(u_i) + 2(n_2-1) - d_{G'}(v_s)\Big)^2, \text{ by (7)} \\ &= \sum_{u_i\in v(G)} \sum_{v_rv_s\in E(G')} \Big(2n_2\sigma_G(u_i) + 4(n_2-1) - (d_{G'}(v_r) + d_{G'}(v_s)\Big)^2 \\ &= \sum_{u_i\in v(G)} \sum_{v_rv_s\in E(G')} \Big(4n_2^2\sigma_G^2(u_i) + 16(n_2-1)^2 + (d_{G'}(v_r) + d_{G'}(v_s))^2 + 16n_2(n_2-1)\sigma_G(u_i) \\ &\quad -4n_2\sigma_G(u_i)(d_{G'}(v_r) + d_{G'}(v_s)) - 8(n_2-1)(d_{G'}(v_r) + d_{G'}(v_s))\Big) \\ &= 4n_2^2m_2 \sum_{u_i\in V(G)} \sigma_G^2(u_i) + 16(n_2-1)^2n_1m_2 + n_1HM(G') + 32n_2(n_2-1)W(G)m_1 \\ &\quad -8n_2W(G)M_1(G') - 8(n_2-1)n_1M_1(G'), \text{ by the definitions of Wiener index,} \\ &\quad \text{first and hyper Zagreb index.} \end{aligned}$$

$$\begin{aligned} A_2 &= \sum_{u_iu_k\in E(G)} \sum_{v_r\in V(G')} \sum_{v_s\in V(H)} \Big(\sigma_{G[G']}((u_i,v_r)) + \sigma_{G[G']}((u_k,v_s))\Big)^2 \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_r\in v(G')} \sum_{v_s\in v(H)} \Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_{G'}(v_r) + n_2\sigma_G(u_k) + 2(n_2-1) - d_{G'}(v_s)\Big)^2 \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_r\in v(G')} \sum_{v_s\in v(G')} \Big(n_2(\sigma_G(u_i) + \sigma_G(u_k)) + 4(n_2-1) - (d_{G'}(v_r) + d_{G'}(v_s)\Big)^2 \\ &= \sum_{u_iu_k\in E(G)} \sum_{v_r\in v(G')} \sum_{v_s\in v(G')} \Big(n_2^2(\sigma_G(u_i) + \sigma_G(u_k))^2 + 16(n_2-1)^2 + (d_{G'}(v_r) + d_{G'}(v_s))^2 \\ &\quad +8n_2(n_2-1)(\sigma_G(u_i) + \sigma_G(u_k)) - 2n_2(\sigma_G(u_i) + \sigma_G(u_k))(d_{G'}(v_r) + d_{G'}(v_s)) \\ &\quad -8(n_2-1)(d_{G'}(v_r) + d_{G'}(v_s))\Big) \\ &= n_2^4HS(G) + 16(n_2-1)^2n_2^2m_1 + m_1(2n_2M_1(G') + 8m_1(m_2^2) + 8n_2^3(n_2-1)S_1(G) \\ &\quad -8n_2^2m_2S_1(G) - 32(n_2-1)n_2m_1m_2, \text{ by the definitions of first Zagreb index,} \\ &\quad \text{first and hyper status connectivity index.} \end{aligned}$$

From (8), A_1 and A_2, we have

$$\begin{aligned} HS(G[G']) &= n_2^4HS(G) + 8n_2^2(n_2(n_2-1)S_1(G)) + n_1M_1(G') + 8n_2(4(n_2-1)m_1 \\ &\quad -M_1(G'))W(G) + 4n_2^2m_2 \sum_{u\in V(G)} (\sigma_G(u))^2 + (2m_1n_2 - 8n_1(n_1-1))M_1(G') \\ &\quad +16(n_2-1)^2(n_1m_2 + n_2^2m_1) + 8m_1m_2(m_2 - 4n_2(n_2-1)). \end{aligned}$$

Example 3.8. *The closed fence graph is defined as the composition of C_n and K_2. So, from Theorem 3.3, the hyper-status connectivity index of closed fence graph is given by*

$$HS(C_n[K_2]) = \begin{cases} 5n^5 + 40n^3 - 16n^2 + 26n - 12, & \text{if } n \text{ is even} \\ 5n^5 + 18n^3 + 8n^4 - 24n^2 + 27n, & \text{if } n \text{ is odd.} \end{cases}$$

References

[1] Ashrafi, A. R., Hamzeh, A., and Hossein-Zadeh, S. (2011), Calculation of some topological indices of splice and link graphs, J. Appl. Math. & Informatics, 29, 327 - 335.

[2] Ashrafi, A.R., Doslić, T. and Hamzeh, A. (2010), The Zagreb coindices of graph operations, Discrete Appl. Math., 158, 1571-1578.

[3] Doslić, T. (2005), Splices, links and their degree weighted Wiener polynomials, Graph Theory Notes N. Y., 48, 47-55.

[4] Devillers, J., Balaban, A.T. Eds.,(1999), Topological indices and related descriptors in QSAR and QSPR, Gordon and Breach, Amsterdam, The Netherlands.

[5] Dobrynin, A.A., Entringer, R. and Gutman, I. (2001), Wiener index of trees: Theory and applications, Acta Appl. Math., 66, 211-249.

[6] Harary, F. (1959), Status and contrastatus, Sociometry, 22, 23-43.

[7] Pattabiraman, K., Paulraja, P. (2012), On some topological indices of the tensor product of graphs, Discrete Appl. Math., 160, 267-279.

[8] Pattabiraman, K., Paulraja, P. (2012), Wiener and vertex PI indices of the strong product of graphs, Discuss. Math. Graph Thoery, 32, 749-769.

[9] Pattabiraman, K., Paulraja, P. (2011), Wiener index of the tensor product of a path and a cycle, Discuss. Math. Graph Thoery, 31, 737-751.

[10] Pattabiraman, K., Santhakumar, A. (2019), Computing first status-connectivity indices of composite graphs, Appl. Math. Scientific Comput., 479-487.

[11] Ramane, H. S., Yalnaik, A. S. (2016), Status connectivity indices of graphs and its applications to the boiling point of benzenoid hydrocarbons, J. Appl. Math. Comput., DOI 10.1007/s12190-016-1052-5.

[12] Shirdel, G.H., Rezapour, H. and Sayadi, A.M. (2013), The hyper-Zagreb index of graph operations, Iran. J. Math. Chem., 4, 213-220.

[13] Todeschini, R., Consonni, V. (2000), Handbook of molecular descriptors, Weinheim: Wiley-VCH.

[14] Todeschini, R., Consonni, V. (2009), Molecular descriptors for chemoinformatics, Vols. I and II, Weinheim: Wiley-VCH.

[15] Veylaki, M., Nikmehr, M.J. and Tavallaee, H.A. (2015), The third and hyper-Zagreb coindices of some graph operations, J. Appl. Math. Comput.,doi:10.1007/s12190-015-0872-z.

[16] Yeh, Y.N., Gutman, I. (1994), On the sum of all distances in composite graphs, Discrete Math., 135, 359-365.

[17] Wiener, H. (1947), Structural determination of the paraffin boiling points, J. Amer. Chem. Soc., 69, 17-20.

K. Pattabiraman graduated from Madras University, Chennai, India in 2002. He received his M.Sc., M.Phil. and Ph.D. in Mathematics from Annamalai University in year 2004, 2006 and 2012, respectively. He has been a faculty member of Mathematics in Government Arts College(A), Kumbakonam, India since 2006. His research interests focus mainly on graph theory.

A. Santhakumar received his Master degree in the year 2006 and M.Phil. degree in 2009 from Annamalai University. He presently doing his Ph.D. degree in Annamalai University. He is currently working as an assistant professor in CK College of Engineering and Technology, Cuddalore, India.