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SOME FRACTIONAL ESTIMATES OF UPPER BOUNDS INVOLVING FUNCTIONS HAVING EXPONENTIAL CONVEXITY PROPERTY

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Abstract. The main objective of this article is to consider the class of exponentially convex functions. We derive a new integral identity involving Riemann-Liouville fractional integral. Utilizing this identity as an auxiliary result we obtain new fractional bounds involving the functions having exponential convexity property.

Keywords: Convex functions, exponential convex functions, Hermite-Hadamard inequality, Riemann-Liouville fractional integral.

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1. Introduction

We start with the well known Hermite-Hadamard inequality. Let $[a, b] \subset \mathbb{R}$ be a bounded closed interval with $a < b$. If $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then we have the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Equality holds in either side only for affine functions. It gives us an estimate of the (integral) mean value of a continuous convex functions. This result of Hermite and Hadamard is very simple in nature but very powerful. Interestingly both sides of the above integral inequality characterizes convex functions. For some interesting details and applications of Hermite-Hadamard's inequality, we refer readers to [3–8, 10, 14–22, 25, 26]. Theory of convexity played a vital role in the development of theory of inequalities. Other than Hermite-Hadamard's inequality there are many famous results known in the theory of inequalities which can be obtained using the functions having convexity property. Many researchers have used different novel and innovative ideas in obtaining new generalizations

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of classical inequalities. Sarikaya et al. [25] used elegantly the concepts of fractional calculus and obtained a fractional refinement of Hermite-Hadamard's inequality. This idea compelled many researchers to use fractional calculus concepts in theory of inequalities and gradually many new fractional analogues of classical inequalities have been obtained in the literature. For details, see [4, 8–14, 17–19, 23–27].
In recent years theory of convexity experienced a rapid development and consequently the classical concept of convexity has been extended and generalized in different directions. For details, see [1, 3, 5, 20, 21]. Recently the class of exponential convex functions has been introduced and studied.

Definition 1.1 ([28, 29]). *Let $I \subset \mathbb{R}$ be an interval. Throughout the paper, we will use a function $f : I \to \mathbb{R}$ that satisfies the inequality*

$$e^{f(tx+(1-t)y)} \leq te^{f(x)} + (1-t)e^{f(y)},$$

for every pair $x, y \in \mathbb{I}$ and every $t \in [0, 1]$. Since the above inequality represents the convexity of the function e^f, we could say that the function f is exp-convex (inspired by the term of log-convex function). Here we also note that terms exp-convex and exponentially convex do not represent one and the same.

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction and stochastic optimization, see [1, 3, 24] and the references therein.
The class of exponentially convex functions was mentioned by Antczak [2], Dragomir et al. [5] and Noor et al. [20].
The motivation of this article is to discuss some new fractional bounds involving the functions having exponential convexity property. In order to obtain main results of the article we derive a new fractional integral identity. We hope that the ideas and techniques of this article will inspire interested readers.
We now recall the definition of Riemann-Liouville fractional integrals.

Definition 1.2 ([23]). *Let $\alpha > 0$. The left- and right-hand side Riemann-Liouville fractional integrals of order α are given by*

$$J^{\alpha}_{u+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{u}^{x} (x-t)^{\alpha-1} f(t)dt,$$

and

$$J^{\alpha}_{v-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{v} (t-x)^{\alpha-1} f(t)dt,$$

where $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt$ is the gamma function.

2. New Fractional Integral Identity

In this section, we derive a new integral identity essentially using the concept of Riemann-Liouville fractional integral.

Lemma 2.1. *Let $\alpha > 0$ be a number and let $f : [a, b] \to \mathbb{R}$ be a function that is differentiable on (a, b), then*

$$
\begin{aligned}
\Upsilon_f(a,b,\alpha) = \frac{b-a}{16}\Bigg[& \int_0^1 t^{\alpha} e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)} f'\Big(t\frac{3a+b}{4}+(1-t)a\Big)dt \\
& + \int_0^1 (t^{\alpha}-1) e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)} f'\Big(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\Big)dt \\
& + \int_0^1 t^{\alpha} e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)} f'\Big(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\Big)dt \\
& + \int_0^1 (t^{\alpha}-1) e^{f\left(tb+(1-t)\frac{a+3b}{4}\right)} f'\Big(tb+(1-t)\frac{a+3b}{4}\Big)dt\Bigg], \qquad (1)
\end{aligned}
$$

where

$$
\Upsilon_f(a,b,\alpha) = \frac{1}{2}\left[e^{f(\frac{3a+b}{4})} + e^{f(\frac{a+3b}{4})}\right] - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}
\left[J^{\alpha}_{(\frac{3a+b}{4})^-} e^{f(a)} + J^{\alpha}_{(\frac{a+b}{2})^-} e^{f(\frac{3a+b}{4})} + J^{\alpha}_{(\frac{a+3b}{4})^-} e^{f(\frac{a+b}{2})} + J^{\alpha}_{b^-} e^{f(\frac{a+3b}{4})}\right].
$$

Proof. Consider

$$
\Upsilon_f(a,b,\alpha) = \frac{b-a}{16}\sum_{n=1}^{4} I_n,
$$

where

$$I_1 = \int_0^1 t^{\alpha} e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)} f'\Big(t\frac{3a+b}{4}+(1-t)a\Big)dt$$
$$= \frac{4}{b-a} e^{f(\frac{3a+b}{4})} - \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{(\frac{3a+b}{4})^-} e^{f(a)},$$

$$I_2 = \int_0^1 (t^{\alpha}-1) e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)} f'\Big(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\Big)dt$$
$$= \frac{4}{b-a} e^{f(\frac{3a+b}{4})} - \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{(\frac{a+b}{2})^-} e^{f(\frac{3a+b}{4})},$$

$$I_3 = \int_0^1 t^{\alpha} e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)} f'\Big(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\Big)dt$$
$$= \frac{4}{b-a} e^{f(\frac{a+3b}{4})} - \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{(\frac{a+3b}{4})^-} e^{f(\frac{a+b}{2})}$$

and

$$I_4 = \int_0^1 (t^{\alpha}-1) e^{f\left(tb+(1-t)\frac{a+3b}{4}\right)} f'\Big(tb+(1-t)\frac{a+3b}{4}\Big)dt$$
$$= \frac{4}{b-a} e^{f(\frac{a+3b}{4})} - \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{b^-} e^{f(\frac{a+3b}{4})}.$$

Multiplying above integrals by $\frac{b-a}{16}$ and adding, then we get equality (1). The proof is complete. □

Theorem 2.1. *Let $\alpha > 0$ be a number and let $f : [a,b] \to \mathbb{R}$ be a function that is differentiable on (a,b). If the function $|f|$ is exp-convex and $|f'|$ is convex, then*

$$\Big|\Upsilon_f(a,b,\alpha)\Big| \le \frac{b-a}{96(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[12|e^{f(a)}f'(a)| - (2\alpha^3+11\alpha^2+18\alpha-2)$$
$$\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big| - \Big(2\alpha^3+3\alpha^2+4\alpha-12\Big)\Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)$$
$$-\Big(2\alpha^3+4\alpha-12\Big)\Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big) - 2\alpha(\alpha^2+3\alpha+2)|e^{f(b)}f'(b)|$$
$$+6(\alpha+1)\Big\{\Delta_1(a,b)+\Delta_3(a,b)\Big\} - \alpha(\alpha^2+6\alpha+5)\{\Delta_2(a,b)+\Delta_4(a,b)\}\Bigg], \tag{2}$$

where

$$\Delta_1(a,b) = \Big\{|e^{f(a)}f'\Big(\frac{3a+b}{4}\Big)| + |e^{f(\frac{3a+b}{4})}f'(a)|\Big\},$$
$$\Delta_2(a,b) = \Big\{|e^{f(\frac{a+b}{2})}f'\Big(\frac{3a+b}{4}\Big)| + |e^{f(\frac{3a+b}{4})}f'\Big(\frac{a+b}{2}\Big)|\Big\},$$
$$\Delta_3(a,b) = \Big\{|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+3b}{4}\Big)| + |e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+b}{2}\Big)|\Big\},$$

$$\Delta_4(a,b)=\left\{|e^{f(b)}f'\Big(\frac{a+3b}{4}\Big)|+|e^{f(\frac{a+3b}{4})}f'(b)|\right\}.$$

Proof. Using Lemma 2.1, the property of modulus and the given hypothesis of the theorem, we have

$$\Big|\Upsilon_f(a,b,\alpha)\Big|\leq\frac{b-a}{16}\sum_{i=1}^{n}H_i \tag{3}$$

$$\begin{aligned}
H_1 &= \int_0^1 t^{\alpha}\Big|e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)}f'\Big(t\frac{3a+b}{4}+(1-t)a\Big)\Big|dt\\
&\leq \int_0^1 t^{\alpha}\Big[t\Big|e^{f(\frac{3a+b}{4})}\Big|+(1-t)|e^{f(a)}|\Big]\Big[t\Big|f'\Big(\frac{3a+b}{4}\Big)\Big|+(1-t)|f'(a)|\Big]dt\\
&= \int_0^1 t^{\alpha}\Big[t^2\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big|+(1-t)^2\Big|e^{f(a)}f'(a)\Big|\\
&+t(1-t)\Big\{\Big|e^{f(\frac{3a+b}{4})}f'(a)\Big|+\Big|e^{f(a)}f'\Big(\frac{3a+b}{4}\Big)\Big|\Big\}\Big]dt\\
&= \int_0^1 t^{\alpha}\Big[t^2\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big|+(1-t)^2\Big|e^{f(a)}f'(a)\Big|+t(1-t)\Delta_1(a,b)\Big]dt\\
&= \frac{(\alpha^2+3\alpha+2)\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big|+2\Big|e^{f(a)}f'(a)\Big|+(\alpha+1)\Delta_1(a,b)}{(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned} \tag{4}$$

$$\begin{aligned}
H_2 &= \int_0^1 (t^{\alpha}-1)\Big|e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)}f'\Big(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\Big|\Big)dt\\
&\leq \frac{-\alpha}{6(\alpha+1)(\alpha+2)(\alpha+3)}\Big[2(\alpha^2+3\alpha+2)|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)|\\
&+2(\alpha^2+6\alpha+11)|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)|+(\alpha^2+6\alpha+5)\Delta_2(a,b)\Big],
\end{aligned} \tag{5}$$

$$\begin{aligned}
H_3 &= \int_0^1 t^{\alpha}\Big|e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)}f'\Big(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\Big)\Big|dt\\
&\leq \frac{(\alpha^2+3\alpha+2)\Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\Big|+2\Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\Big|+(\alpha+1)\Delta_3(a,b)}{(\alpha+1)(\alpha+2)(\alpha+3)}
\end{aligned} \tag{6}$$

and

$$\begin{aligned} H_4 &= \int_0^1 (t^\alpha - 1)\left|e^{f\left(tb+(1-t)\frac{a+3b}{4}\right)} f'\left(tb+(1-t)\frac{a+3b}{4}\right|\right)dt \\ &\leq \frac{-\alpha}{6(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[2(\alpha^2+3\alpha+2)|e^{f(b)}f'(b)| \\ &+2(\alpha^2+6\alpha+11)|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)| + (\alpha^2+6\alpha+5)\Delta_4(a,b)\Bigg], \end{aligned} \tag{7}$$

Substituting (4), (5), (6) and (7) in (3), we get the desired inequality (2). □

Corollary 2.1. *If we choose $\alpha = 1$, then under the assumption of Theorem 2.1, we have a new result*

$$\begin{aligned} &\left|\frac{1}{2}\Big[e^{f(\frac{3a+b}{4})} + e^{f(\frac{a+3b}{4})}\Big] - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right| \leq \frac{b-a}{2304}\Bigg[12\Big\{|e^{f(a)}f'(a)| - |e^{f(b)}f'(b)| \\ &+\Delta_1(a,b) - \Delta_2(a,b) + \Delta_3(a,b) - \Delta_4(a,b)\Big\} - 29\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big| \\ &+3\Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big) + 6\Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\Bigg]. \end{aligned}$$

Theorem 2.2. *Let $\alpha > 0$ be a number and let $f : [a,b] \to \mathbb{R}$ be a function that is differentiable on (a,b). If the function $|f|$ is exp-convex and $|f'|^q$ is convex where $p^{-1} + q^{-1} = 1$, $q > 1$, then*

$$\begin{aligned} &\Big|\Upsilon_f(a,b,\alpha)\Big| \\ &\leq \frac{b-a}{6^{\frac{1}{q}}.16}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\Bigg[\left(\frac{\alpha}{1+p\alpha}\right)^{\frac{1}{p}}\Bigg\{\Bigg[2\Bigg(\Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big|^q + \Big|e^{f(a)}f'(a)\Big|^q\Bigg) + \Delta_5(a,b)\Bigg]^{\frac{1}{q}} \\ &+\Bigg[2\Bigg(\Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\Big|^q + \Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\Big|^q\Bigg) + \Delta_6(a,b)\Bigg]^{\frac{1}{q}}\Bigg\} \\ &+\left(\beta\Big(\alpha+1,\frac{1}{p}\Big)\right)\Bigg\{\Bigg[2\Bigg(\Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\Big|^q + \Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\Big|^q\Bigg) + \Delta_7(a,b)\Bigg]^{\frac{1}{q}} \\ &+\Bigg[2\Bigg(\Big|e^{f\left(b\right)}f'\Big(b\Big)\Big|^q + \Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\Big|^q\Bigg) + \Delta_8(a,b)\Bigg]^{\frac{1}{q}}\Bigg\}\Bigg], \end{aligned} \tag{8}$$

where

$$\Delta_5(a,b) = \Big|e^{f(\frac{3a+b}{4})}f'\Big(a\Big)\Big|^q + \Big|e^{f(\frac{3a+b}{4})}f'(a)\Big|^q, \tag{9}$$

$$\Delta_6(a,b) = \Big|e^{f(\frac{3a+b}{4})}f'\Big(\frac{a+b}{2}\Big)\Big|^q + \Big|e^{f(\frac{a+b}{2})}f'\Big(\frac{3a+b}{4}\Big)\Big|^q, \tag{10}$$

$$\Delta_7(a,b) = \Big|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+b}{2}\Big)\Big|^q + \Big|e^{f(\frac{a+b}{2})}f'(\frac{a+3b}{4})\Big|^q, \tag{11}$$

and

$$\Delta_8(a,b) = \left|e^{f(\frac{a+3b}{4})}f'\Big(b\Big)\right|^q + \left|e^{f\Big(b\Big)}f'(\frac{a+3b}{4})\right|^q. \tag{12}$$

Proof. Using Lemma 2.1, Hölder's inequality and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\Big|\Upsilon_f(a,b,\alpha)\Big| \\
&\le \frac{b-a}{16}\left[\left(\int_0^1 \Big(t^\alpha\Big)^p dt\right)^{\frac{1}{p}} \Big[\sum_{i=1}^{2} K_i^{\frac{1}{q}}\Big] + \left\{\left(\int_0^1 \Big(1-t^\alpha\Big)^p dt\right)^{\frac{1}{p}} \Big[\sum_{i=3}^{4} K_i^{\frac{1}{q}}\Big]\right]
\end{aligned} \tag{13}$$

$$\begin{aligned}
K_1 &= \int_0^1 \left|e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)} f'\Big(t\frac{3a+b}{4}+(1-t)a\Big)\right|^q dt \\
&\le \left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q \int_0^1 t^2 dt + \left|e^{f(a)}f'(a)\right|^q \int_0^1 (1-t)^2 dt + \left\{\left|e^{f(\frac{3a+b}{4})}f'(a)\right|^q\right. \\
&\left. + \left|e^{f\Big(a\Big)}f'\Big(\frac{3a+b}{4}\Big)\right|^q\right\} \int_0^1 t(1-t)dt \\
&= \frac{1}{6}\left[2\left(\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q + \left|e^{f(a)}f'(a)\right|^q\right) + \Delta_5(a,b)\right],
\end{aligned} \tag{14}$$

$$\begin{aligned}
K_2 &= \int_0^1 \left|e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)} f'\Big(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\Big)\right|^q dt \\
&\le \frac{1}{6}\left[2\left(\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q + \left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q\right) + \Delta_6(a,b)\right],
\end{aligned} \tag{15}$$

$$\begin{aligned}
K_3 &= \int_0^1 \left|e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)} f'\Big(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\Big)\right|^q dt \\
&\le \frac{1}{6}\left[2\left(\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q + \left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q\right) + \Delta_7(a,b)\right],
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
K_4 &= \int_0^1 \left|e^{f\left(tb+(1-t)\frac{a+3b}{4}\right)} f'\Big(tb+(1-t)\frac{a+3b}{4}\Big)\right|^q dt \\
&\le \frac{1}{6}\left[2\left(\left|e^{f\Big(b\Big)}f'\Big(b\Big)\right|^q + \left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q\right) + \Delta_8(a,b)\right].
\end{aligned} \tag{17}$$

Substituting (14), (15), (16) and (17) in (13), we get the desired inequality (8). This completes the proof. □

Corollary 2.2. *If we choose $\alpha = 1$, then under the assumption of Theorem 2.2, we have a new result*

$$\left|\frac{1}{2}\left[e^{f(\frac{3a+b}{4})}+e^{f(\frac{a+3b}{4})}\right]-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$
$$\leq \frac{b-a}{6^{\frac{1}{q}}.16}\Bigg[\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\Bigg\{\left[2\left(\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q+\left|e^{f(a)}f'(a)\right|^q\right)+\Delta_5(a,b)\right]^{\frac{1}{q}}$$
$$+\left[2\left(\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q+\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q\right)+\Delta_6(a,b)\right]^{\frac{1}{q}}\Bigg\}$$
$$+\left(\beta\Big(2,\frac{1}{p}\Big)\right)\Bigg\{\left[2\left(\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q+\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q\right)+\Delta_7(a,b)\right]^{\frac{1}{q}}$$
$$+\left[2\left(\left|e^{f\left(b\right)}f'\Big(b\Big)\right|^q+\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q\right)+\Delta_8(a,b)\right]^{\frac{1}{q}}\Bigg\}\Bigg].$$

Theorem 2.3. *Let $\alpha > 0$ be a number and let $f : [a, b] \to \mathbb{R}$ be a function that is differentiable on (a, b). If the function $|f|$ is exp-convex and $|f^{'}|^q$ is convex where $p^{-1} + q^{-1} = 1,\ q \geq 1$, then*

$$\left|\Upsilon_f(a,b,\alpha)\right|$$
$$\leq \frac{b-a}{16}\left(\frac{1}{6(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}$$
$$\Bigg[\left\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q+12\left|e^{f(a)}f'(a)\right|^q+6(\alpha+1)\Delta_5(a,b)\right\}^{\frac{1}{q}}$$
$$+\alpha\Bigg\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q+(\alpha^2+6\alpha+11)\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q$$
$$+(\alpha+5)\Delta_6(a,b)\Bigg\}^{\frac{1}{q}}+\alpha^{\frac{1}{p}}\Bigg\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q$$
$$+12\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q+6(\alpha+1)\Delta_7(a,b)\Bigg\}^{\frac{1}{q}}+\alpha^{1+\frac{1}{p}}\Bigg\{6(\alpha^2+3\alpha+2)\left|e^{f(b)}f'(b)\Big)\right|^q$$
$$+(\alpha^2+6\alpha+11)\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q+(\alpha+5)\Delta_8(a,b)\Bigg\}^{\frac{1}{q}}\Bigg], \tag{18}$$

where $\Delta_5(a,b)$, $\Delta_6(a,b)$, $\Delta_7(a,b)$ and $\Delta_8(a,b)$ are given in (9)-(12), respectively.

Proof. Using Lemma 2.1, the power mean inequality and the given hypothesis of the theorem, we have

$$\left|\Upsilon_f(a,b,\alpha)\right| \leq \left[\left(\int_0^1 t^\alpha dt\right)^{\frac{1}{p}}\sum_{i=1}^{2}N_i^{\frac{1}{q}}\right]+\left[\left(\int_0^1(1-t^\alpha)dt\right)^{\frac{1}{p}}\sum_{i=3}^{4}N_i^{\frac{1}{q}}\right], \tag{19}$$

where

$$\begin{aligned}
N_1 &= \int_0^1 t^\alpha \left| e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)} f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt \\
&\le \int_0^1 \Bigg[t^{\alpha+2}\left| e^{f(\frac{3a+b}{4})} f'\left(\frac{3a+b}{4}\right)\right|^q + t^\alpha(1-t)^2\left| e^{f\left(a\right)} f'\left(a\right)\right|^q \\
&+t^{\alpha+1}(1-t)\left\{\left| e^{f(\frac{3a+b}{4})} f'\left(a\right)\right|^q + \left| e^{f\left(a\right)} f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\Bigg] dt \\
&= \int_0^1 \left[t^{\alpha+2}\left| e^{f(\frac{3a+b}{4})} f'\left(\frac{3a+b}{4}\right)\right|^q + t^\alpha(1-t)^2\left| e^{f\left(a\right)} f'\left(a\right)\right|^q + t^{\alpha+1}(1-t)\Delta_5(a,b)\right] dt \\
&= \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[(\alpha^2+3\alpha+2)\left| e^{f(\frac{3a+b}{4})} f'\left(\frac{3a+b}{4}\right)\right|^q \\
&+2\left| e^{f(a)} f'(a)\right|^q + (\alpha+1)\Delta_5(a,b)\Bigg], \qquad (20)
\end{aligned}$$

$$\begin{aligned}
N_2 &= \int_0^1 (1-t^\alpha)\left| e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)} f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \\
&= \frac{\alpha}{6(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[6(\alpha^2+3\alpha+2)\left| e^{f(\frac{a+b}{2})} f'\left(\frac{a+b}{2}\right)\right|^q \\
&+(\alpha^2+6\alpha+11)\left| e^{f(\frac{3a+b}{4})} f'\left(\frac{3a+b}{4}\right)\right|^q + (\alpha+5)\Delta_6(a,b)\Bigg], \qquad (21)
\end{aligned}$$

$$\begin{aligned}
N_3 &= \int_0^1 t^\alpha\left| e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)} f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \\
&= \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[(\alpha^2+3\alpha+2)\left| e^{f(\frac{a+3b}{4})} f'\left(\frac{a+3b}{4}\right)\right|^q \\
&+2\left| e^{f(\frac{a+b}{2})} f'\left(\frac{a+b}{2}\right)\right|^q + (\alpha+1)\Delta_7(a,b)\Bigg], \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
N_4 &= \int_0^1 (1-t^\alpha)\left| e^{f\left(tb+(1-t)\frac{a+3b}{4}\right)} f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt \\
&= \frac{\alpha}{6(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[6(\alpha^2+3\alpha+2)\left| e^{f(b)} f'(b)\right)\Bigg|^q \\
&+(\alpha^2+6\alpha+11)\left| e^{f(\frac{a+3b}{4})} f'\left(\frac{a+3b}{4}\right)\right|^q + (\alpha+5)\Delta_8(a,b)\Bigg]. \qquad (23)
\end{aligned}$$

Substituting (20), (21), (22) and (23) in (19). We get the inequality (18). □

Corollary 2.3. *If we choose $\alpha = 1$, then under the assumption of Theorem 2.3, we have a new result*

$$\frac{1}{2}\Big[e^{f(\frac{3a+b}{4})} + e^{f(\frac{a+3b}{4})}\Big] - \frac{1}{b-a}\int_a^b e^{f(x)}dx$$
$$\leq \frac{b-a}{16}\left(\frac{1}{288}\right)^{\frac{1}{q}}\left[\left\{36\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q + 12\left|e^{f(a)}f'(a)\right|^q + 12\Delta_5(a,b)\right\}^{\frac{1}{q}}\right.$$
$$+\left\{36\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q + 18\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q + 6\Delta_6(a,b)\right\}^{\frac{1}{q}}$$
$$+\left\{36\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q + 12\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q + 12\Delta_7(a,b)\right\}^{\frac{1}{q}}$$
$$\left.+\left\{36\left|e^{f(b)}f'(b)\Big)\right|^q + 18\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q + 6\Delta_8(a,b)\right\}^{\frac{1}{q}}\right].$$

Theorem 2.4. *Let $\alpha > 0$ be a number and let $f : [a,b] \to \mathbb{R}$ be a function that is differentiable on (a,b). If the function $|f|$ is exp-convex and $|f^{'}|^q$ is convex where $p^{-1} + q^{-1} = 1,\ q \geq 1$, then*

$$\left|\Upsilon_f(a,b,\alpha)\right| \leq \frac{b-a}{16}\left(\frac{1}{6(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}$$
$$\left[\left\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q + 12\left|e^{f(a)}f'(a)\right|^q + 6(\alpha+1)\Delta_5(a,b)\right\}^{\frac{1}{q}}\right.$$
$$+\alpha\left\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q + (\alpha^2+6\alpha+11)\left|e^{f(\frac{3a+b}{4})}f'\Big(\frac{3a+b}{4}\Big)\right|^q\right.$$
$$\left.+(\alpha+5)\Delta_6(a,b)\right\}^{\frac{1}{q}} + \left\{6(\alpha^2+3\alpha+2)\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q\right.$$
$$\left.+12\left|e^{f(\frac{a+b}{2})}f'\Big(\frac{a+b}{2}\Big)\right|^q + 6(\alpha+1)\Delta_7(a,b)\right\}^{\frac{1}{q}} + \alpha\left\{6(\alpha^2+3\alpha+2)\left|e^{f(b)}f'(b)\Big)\right|^q\right.$$
$$\left.\left.+(\alpha^2+6\alpha+11)\left|e^{f(\frac{a+3b}{4})}f'\Big(\frac{a+3b}{4}\Big)\right|^q + (\alpha+5)\Delta_8(a,b)\right\}^{\frac{1}{q}}\right]. \tag{24}$$

where $\Delta_5(a,b)$, $\Delta_6(a,b)$, $\Delta_7(a,b)$ and $\Delta_8(a,b)$ are given in (9)-(12), respectively.

Proof. From Lemma 2.1, Hölder's inequality and the given hypothesis of the theorem, we have

$$\left|\Upsilon_f(a,b,\alpha)\right|$$
$$\leq \frac{b-a}{16}\left[\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\sum_{i=1}^{2} N_i^{\frac{1}{q}}\right) + \left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\sum_{i=3}^{4} N_i^{\frac{1}{q}}\right)\right], \tag{25}$$

Substituting (20), (21), (22) and (23) in (25), we get the inequality (24). □

Corollary 2.4. *If we choose $\alpha = 1$, then under the assumption of Theorem 2.4, we have a new result*

$$\left|\frac{1}{2}\left[e^{f(\frac{3a+b}{4})}+e^{f(\frac{a+3b}{4})}\right]-\frac{1}{b-a}\int_a^b e^{f(x)}dx\right| \le \frac{b-a}{16}\left(\frac{1}{144}\right)^{\frac{1}{q}}$$
$$\left[\left\{36\left|e^{f(\frac{3a+b}{4})}f'\left(\frac{3a+b}{4}\right)\right|^q+12\left|e^{f(a)}f'(a)\right|^q+12\Delta_5(a,b)\right\}^{\frac{1}{q}}\right.$$
$$+\left\{36\left|e^{f(\frac{a+b}{2})}f'\left(\frac{a+b}{2}\right)\right|^q+18\left|e^{f(\frac{3a+b}{4})}f'\left(\frac{3a+b}{4}\right)\right|^q+6\Delta_6(a,b)\right\}^{\frac{1}{q}}$$
$$+\left\{36\left|e^{f(\frac{a+3b}{4})}f'\left(\frac{a+3b}{4}\right)\right|^q+12\left|e^{f(\frac{a+b}{2})}f'\left(\frac{a+b}{2}\right)\right|^q+12\Delta_7(a,b)\right\}^{\frac{1}{q}}$$
$$\left.+\left\{36\left|e^{f(b)}f'(b)\right)\right|^q+18\left|e^{f(\frac{a+3b}{4})}f'\left(\frac{a+3b}{4}\right)\right|^q+6\Delta_8(a,b)\right\}^{\frac{1}{q}}\right].$$

Theorem 2.5. *Let $\alpha > 0$ be a number and let $f : [a, b] \to \mathbb{R}$ be a function that is differentiable on (a, b). If the function $|f|$ is exp-concave and $|f^{'}|^q$ is concave where $p^{-1} + q^{-1} = 1$ with $q \ge 1$, then*

$$\left|\Upsilon_f(a,b,\alpha)\right| \le \frac{b-a}{16}\left[\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left\{\left|e^{f(\frac{7a+b}{8})}f'\left(\frac{7a+b}{8}\right)\right|+\left|e^{f(\frac{5a+3b}{8})}f'\left(\frac{5a+3b}{8}\right)\right|\right\}\right.$$
$$\left.+\left(\frac{1}{\alpha}\beta\left(\frac{1}{\alpha}+1,p\right)\right)^{\frac{1}{p}}\left\{\left|e^{f(\frac{3a+5b}{8})}f'\left(\frac{3a+5b}{8}\right)\right|+\left|e^{f(\frac{a+7b}{8})}f'\left(\frac{a+7b}{8}\right)\right|\right\}\right]. \tag{26}$$

Proof. Using Lemma 2.1, Hölder inequality and the given hypothesis of the theorem, we have

$$\left|\Upsilon_f(a,b,\alpha)\right| \le \frac{b-a}{16}\left[\left(\int_0^1\left(t^\alpha\right)^p\right)^{\frac{1}{p}}\sum_{i=1}^{2}S_i^{\frac{1}{q}}+\left(\int_0^1\left(1-t^\alpha\right)^p\right)^{\frac{1}{p}}\sum_{i=3}^{4}S_i^{\frac{1}{q}}\right]. \tag{27}$$

Using the exponential concavity of $|f^{'}|^q$ and the Jensen's integral inequality, we have

$$\begin{aligned} S_1 &= \int_0^1\left|e^{f\left(t\frac{3a+b}{4}+(1-t)a\right)}f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt \\ &\le \left|e^{f(\frac{7a+b}{8})}f'\left(\frac{7a+b}{8}\right)\right|^q, \end{aligned} \tag{28}$$

$$\begin{aligned} S_2 &= \int_0^1\left|e^{f\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)}f'\left(t\frac{3a+b}{4}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \\ &\le \left|e^{f(\frac{5a+3b}{8})}f'\left(\frac{5a+3b}{8}\right)\right|^q, \end{aligned} \tag{29}$$

$$S_3 = \int_0^1 \left| e^{f\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)} f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt$$
$$\leq \left|e^{f(\frac{3a+5b}{8})} f'\left(\frac{3a+5b}{8}\right)\right|^q, \tag{30}$$

and

$$S_4 = \int_0^1 \left| e^{f(tb+(1-t)\frac{a+3b}{4})} f'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt$$
$$\leq \left|e^{f(\frac{a+7b}{8})} f'\left(\frac{a+7b}{8}\right)\right|^q. \tag{31}$$

Substituting (28), (29), (30) and (31) in (27), we get the inequality (26). □

Corollary 2.5. *If we choose $\alpha = 1$, then under the assumption of Theorem 2.5, we have a new result*

$$\left|\frac{1}{2}\left[e^{f(\frac{3a+b}{4})} + e^{f(\frac{a+3b}{4})}\right] - \frac{1}{b-a}\int_a^b e^{f(x)}dx\right|$$
$$\leq \frac{b-a}{16}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left|e^{f(\frac{7a+b}{8})} f'\left(\frac{7a+b}{8}\right)\right| + \left|e^{f(\frac{5a+3b}{8})} f'\left(\frac{5a+3b}{8}\right)\right|\right\}\right.$$
$$\left.+\beta\left(\frac{1}{2},p\right)^{\frac{1}{p}}\left\{\left|e^{f(\frac{3a+5b}{8})} f'\left(\frac{3a+5b}{8}\right)\right| + \left|e^{f(\frac{a+7b}{8})} f'\left(\frac{a+7b}{8}\right)\right|\right\}\right].$$

3. Conclusions

In this paper, a new integral identity has been derived. Based on this identity, we have proved several new integral inequalities for exponentially convex functions via Riemann-Liouville fractional integral operators.

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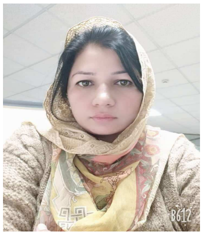

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