TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 289-301

POSITIVE SOLUTIONS FOR TWO-POINT CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS BY MONOTONE ITERATIVE SCHEME

ŞUAYIP TOPRAKSEVEN, §

Abstract. In this paper, two successively iterative schemes have been provided to show the existence of nontrivial solutions for nonlinear conformable fractional differential equation involving nonlocal boundary condition and a parameter. The iterative sequences begin with some constant. The fractional derivative in this study is based on the newly defined and so called "conformable fractional derivative". The corresponding Green's function that is singular at zero has been derived. Because of this singularity, the fixed point theorem can not be applied directly, thus a sequence of operators that are completely continuous is constructed and uniform convergence of these operators to the underlying operator is shown. Then a fixed point result on the order interval is applied. Nontrivial solutions of the problem and the positive solutions of the problem that are the limit of the iterative sequences constructed has been demonstrated.

Keywords: Successive iteration, Conformable Fractional Differential Equation, Boundary value problems, Order interval, Monotone iterative Schemes, Existence of solution

AMS Subject Classification: 34A08, 26A33

1. Introduction

A new and efficient way of modelling of many physical applications can be described by fractional differential equations. One of the important feature of differential equations is the theory of fractional differential equation (see [1],[2],[3]). During the time, there have been many attempt at defining the fractional derivative of a function. Some frequently used fractional derivatives are the Riemann–Liouville and Caputo fractional derivative. The Riemann-Lioville fractional derivatives are singular at zero. This leads to unusual initial conditions for fractional differential equations in the sense of Riemann-Lioville definition, thus it lacks of physical interpretation. On the other hand, the drawbacks of these definitions of fractional derivatives are that they do not obey some important rules of the classical calculus such as semi-group and commutative property and chain rule. In [4], a different definition of fractional derivative of a function has been given and it is called

Faculty of Engineering, Department of Computer Science, Artvin Çoruh University, 08100, Turkey.
e-mail: topraksp@artvin.edu.tr;ORCID: https://orcid.org/0000-0003-3901-9641.

§ Manuscript received: April 3, 2019; accepted: July 17, 2020.

"conformable fractional derivative". This definition agrees with the basic rules of the standard calculus and has have been further investigated by many authors, see [5] [6]. Several applications and a physical interpretation of conformable fractional derivative have been investigated and discussed in [7], [8], [9],[10],[11],[12],[13],[14],[15],[16],[17]. Recently, existence results for a class of nonlinear fractional differential equations have been extensively studied by many researcher, see [18],[19],[20],[21],[22],[23], [24],[25],[26], [27], [28], [29] and references therein. An efficient and important way of investigating nontrivial solutions of fractional boundary value problems is the monotone iterative method. This method has been used to prove the existence of the solutions in [18],[19],[20],[21].

In this work, positive solutions for the problem of the conformable differential equations involving nonlocal conditions at boundary and a parameter is discussed. To the best of knowledge, there are very little results for nontrivial solutions for boundary value problems of conformable fractional derivative. The difficulties in these conformable fractional differential equations arise from that the Green's function for conformable fractional differential equations is not continuous at the starting point. Therefore, one can not apply the monotone iterative method directly. In [22] and [23], the existence results have been studied for conformable fractional differential equations. However, their method require lower and upper solutions for the problem. In this study, a fixed point result is applied on the ordered interval to find non-zero solutions of the problem by constructing two monotone convergent sequences. The existence of positive solutions also has been proved. In the construction of monotone iterative schemes, lower and upper solutions are not needed. These sequences start with constants to approximate the solutions.

Consider the following nonlinear conformable fractional differential equation

$$\begin{cases} T_0^\alpha x(t) + f(t, x(t)) = 0 \\ x^{(k)}(0) = 0, \quad k = 0, 1, 2, \\ x(1) = a\int_0^1 x(\eta)d\eta \end{cases} \tag{1}$$

where $\alpha \in (3,4]$ and $0 < a \leq 3$ and T_0^α is the conformable derivative defined by Definition 2.1 below.

2. Preliminaries and Green's Function

In this section, some definitions, lemmas and conformable fractional calculus that are needed in the paper and Green's function associated with the problem (1) are introduced. In the sequel, $C[0,1]$ denotes the space of continuous functions on the interval $[0,1]$.

Definition 2.1. [4] *For a function $x : [0,\infty) \mapsto \mathbb{R}$, the conformable fractional derivative of order $\alpha \in (0,1]$ is given as*

$$(T_0^\alpha x)(t) = lim_{h\to 0}\frac{x(t + h(t-a)^{1-\alpha}) - x(t)}{h}$$

provided the limit exist. If this limit exist, we say that f has conformable differentiable. If $\lim_{t\to 0} T_0^\alpha x(t)$ exist, then $T_0^\alpha x(0)$ is defined and is equal to this limit.

Definition 2.2. [4] *The conformable fractional derivative of a function $x : [0,\infty) \mapsto \mathbb{R}$ of order $\alpha \in (n, n+1]$ is defined as*

$$(T_0^\alpha x)(t) = (T_0^\beta x^{(n)})(t)$$

where $\beta = \alpha - n$ and $x^{(n)}(x)$ exits.

Definition 2.3. [4] *Let $\alpha \in (n, n+1]$ for $n \in \mathbb{N}$. Then the conformable fractional integral of a function $x : [0, \infty) \mapsto \mathbb{R}$ is given as*

$$(\mathcal{I}_0^\alpha x)(t) = \frac{1}{n!}\int_0^t (t-s)^n (s-0)^{\alpha-n-1} x(s)\, ds.$$

Lemma 2.1. [4] *If $T_0^\alpha x \in C[0,1]$ and*

$$T_0^\alpha x(t) = 0, \quad \alpha \in (n, n+1],$$

then one has

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n, \quad c_i \in \mathbb{R}.$$

Lemma 2.2. [4] *If $x \in C[0,1]$, then one has*

$$(T_0^\alpha \mathcal{I}_0^\alpha x)(t) = x(t), \quad \alpha > 0 \quad \text{for} \quad t > 0.$$

Lemma 2.1 and Lemma 2.2 lead to

Lemma 2.3. *If $x \in C[0,1]$ and $T_0^\alpha x \in C[0,1] \cap L^1(0,1)$, then*

$$I_0^\alpha T_0^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n, \quad c_i \in \mathbb{R},$$

where n is the largest integer less than α.

A nonempty closed and convex subset P of a Banach space X is called a cone if the following are true: (1) for any nonnegative μ, $\mu P \subset P$; (2) $P \cap (-P) = \{0\}$. The cone P is normal if there is a $\mu > 0$ so that

$$\|g + h\| \geq \mu, \quad \forall g, h \in P, \quad \|g\| = \|h\| = 1.$$

For any $x_1, x_2 y \in X$, define the order interval $[x_1, x_2 y]$ by

$$[x_1, x_2 y] = (x_1 + P) \cap (y + P) = \{z \in X : x \leq z \leq y\}.$$

Here, $x \leq y$ means that $y - x \in P$. Clearly, $[x, y]$ is a convex set. Observe that in a Banach space ordered by cone, one has

$$0 \leq x_n, \quad x_n \to x \quad \text{implies} \quad 0 \leq x.$$

Consequently,

$$x_n \leq y_n, \quad x_n \to x, \quad y_n \to y \quad \text{implies} \quad x \leq y.$$

If $T : X \to X$ is a continuous map, then T is said to be monotone when $u \geq v$ implies $T(u) \geq T(v)$.

Lemma 2.4. [30] *Let Y be a Banach space ordered by a normal cone $P \subset Y$. Assume that $T : [w, z] \to Y$ is completely continuous and monotone operator such that $w \leq Tw, z \geq Tz$. Then T has a minimal and a maximal fixed points w_*, w^*, respectively, so that $w \leq w_* \leq w^* \leq z$. Moreover, $w_* = \lim_{n\to\infty} T^n w, w^* = \lim_{n\to\infty} T^n z$.*

Lemma 2.5. [31] *Let Y be a Banach space and $T : Y \to Y$ be an operator. If $T_n : Y \to Y$ for $n = 1, 2, \ldots$ are completely continuous operators and T_n converges uniformly to T (so that $\|T_n - T\| \to 0$ as $n \to \infty$), then T is also a completely continuous operator.*

Now, an equivalent integral equation for the boundary value problem of the fractional differential equation (1) will be presented by the help of the associated Green's function. Unlike the fractional boundary value problem (FBVP) in the sense of Rieman-Lioville or Caputo, the Green's function for conformable fractional differential equations is singular and this singularity makes the problem challenging.

Lemma 2.6. *Given* $h \in C[0,1] \cap L^1(0,1), \quad \alpha \in (3,4]$, *the linear FBVP*

$$\begin{cases} T_0^\alpha x(t) + h(t) = 0, \\ x^{(k)}(0) = 0, \quad k = 0,1,2, \\ x(1) = a \int_0^1 x(\eta) d\eta, \end{cases} \tag{2}$$

is equivalent to the following integral equation

$$x(t) = \int_0^1 G(t,\eta) h(\eta) d\eta,$$

where the Green's function is given as

$$G(t,\eta) = \frac{\eta^{\alpha-4}}{24p(0)} \begin{cases} 4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3, & 0 < \eta \le t \le 1, \\ 4t^3(1-\eta)^3 - at^3(1-\eta)^4, & 0 \le t \le \eta \le 1, \end{cases} \tag{3}$$

where $0 < a \le 3$ *and* $p(\eta) = 1 - \dfrac{a}{4}(1-\eta)$. *Note that,* $G(t,\eta)$ *is smooth on* $[0,1] \times (0,1]$ *and singular at* $\eta = 0$.

Proof. Applying the conformable integral operator $\mathcal{I}_0^\alpha$ to the both side of the problem (2) and using Lemma 2.3, an equivalent integral equation can be found as

$$x(t) = -I_0^\alpha h(t) + c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

for some constants c_0, c_1, c_2, c_3. As a result, the solution of the problem (2) is

$$x(t) = -\frac{1}{6} \int_0^t (t-\eta)^3 \eta^{\alpha-4} h(\eta) d\eta + c_0 + c_1 t + c_2 t^2 + c_3 t^3.$$

We have $c_0 = c_1 = c_2 = 0$ by the boundary conditions $u(0) = u'(0) = u''(0) = 0$. Moreover, one gets

$$x(1) = c_3 - \frac{1}{6} \int_0^1 (1-\eta)^3 \eta^{\alpha-4} h(\eta) d\eta. \tag{4}$$

Additionally, we compute

$$\begin{aligned} \int_0^1 x(\eta) d\eta &= -\frac{1}{6} \int_0^1 \int_0^t (t-\eta)^3 \eta^{\alpha-4} h(\eta) d\eta dt + c_3 \int_0^1 \eta^3 d\eta \\ &= -\frac{1}{6} \int_0^1 \int_\eta^1 (t-\eta)^3 \eta^{\alpha-4} h(\eta) dt d\eta + c_3 \int_0^1 \eta^3 d\eta \\ &= -\frac{1}{24} \int_0^1 (1-\eta)^4 \eta^{\alpha-4} h(\eta) d\eta + \frac{c_3}{4} \end{aligned} \tag{5}$$

Form (4) and (5), one obtains

$$c_3 = \frac{1}{24p(0)} \int_0^1 \Big(4(1-\eta)^3 \eta^{\alpha-4} - a(1-\eta)^4 \Big) \eta^{\alpha-4} h(\eta) d\eta.$$

Thus, the problem (2) has a unique solution:

$$x(t) = -\frac{1}{6} \int_0^t (t-\eta)^3 \eta^{\alpha-4} h(\eta) d\eta + \frac{1}{24p(0)} \int_0^1 t^3 \Big(4(1-\eta)^3 \eta^{\alpha-4} - a(1-\eta)^4 \Big) \eta^{\alpha-4} h(\eta) d\eta.$$

Equivalently, the solution can be written as

$$
\begin{aligned}
x(t) &= -\frac{1}{6}\int_0^t (t-\eta)^3\eta^{\alpha-4}h(\eta)d\eta + \frac{1}{24p(0)}\int_0^t t^3\Big(4(1-\eta)^3\eta^{\alpha-4} - a(1-\eta)^4\Big)\eta^{\alpha-4}h(\eta)d\eta \\
&+ \frac{1}{24p(0)}\int_t^1 t^3\Big(4(1-\eta)^3\eta^{\alpha-4} - a(1-\eta)^4\Big)\eta^{\alpha-4}h(\eta)d\eta \\
&= \frac{1}{24p(0)}\Bigg[\int_0^t \Big(4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\Big)\eta^{\alpha-4}h(\eta)d\eta \\
&+ \int_t^1 \Big(4t^3(1-\eta)^3 - at^3(1-\eta)^4\Big)\eta^{\alpha-4}h(\eta)d\eta\Bigg]
\end{aligned}
$$

Thus, the desired result is proved. ■

Next, some important properties of the Green's function $G(t,\eta)$ that will be used in this paper will be given in the following lemma.

Lemma 2.7.

$$(A1)\quad G(t,\eta) \ge \frac{a}{6(4-a)}t^3(1-\eta)^3\eta^{\alpha-3}, \quad \forall t,\eta\in[0,1], \tag{6}$$

$$(A2)\quad G(t,\eta) \le \left[\frac{1}{2} + \frac{a}{6(4-a)}\right]t^3(1-\eta)^3\eta^{\alpha-4}, \quad \forall t,\eta\in(0,1], \tag{7}$$

$$(A3)\quad G(t,\eta) \le \left[\frac{1}{2} + \frac{a}{6(4-a)}\right](1-\eta)^3\eta^{\alpha-3}, \quad \forall t,\eta\in(0,1], \tag{8}$$

$$(A4)\quad G(t,\eta) > 0, \quad \forall t,\eta\in(0,1). \tag{9}$$

Proof. If $\eta \leqslant t$, one obtains

$$
\begin{aligned}
G(t,\eta) &= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\right] \\
&= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3\Big\{1-\frac{a}{4}(1-\eta)\Big\} - 4p(0)(t-\eta)^3\right] \\
&= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3p(\eta) - 4p(0)(t-\eta)^3\right] \\
&= \eta^{\alpha-4}\left\{\frac{1}{24}\left[4t^3(1-\eta)^3 - 4(t-\eta)^3\right] + \frac{p(\eta)-p(0)}{24p(0)}4t^3(1-\eta)^3\right\} \\
&\ge \eta^{\alpha-4}\left\{\frac{1}{24}\left[4t^3(1-\eta)^3\eta(1-t)\right] + \frac{\eta(1-p(0))}{24p(0)}4t^3(1-\eta)^3\right\} \\
&\ge \eta^{\alpha-4}\frac{1-p(0)}{24p(0)}4t^3\eta(1-\eta)^3 \\
&= \frac{a}{6(4-a)}t^3(1-\eta)^3\eta^{\alpha-3},
\end{aligned}
$$

which proves (A1) when $\eta \leqslant t$.

$$\begin{aligned}
G(t,\eta) &= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\right] \\
&= \frac{1}{24p(0)}\eta^{\alpha-4}\left[\big(1-\frac{a}{4}+\frac{a}{4}\big)4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\right] \\
&\leq \frac{1}{24p(0)}\eta^{\alpha-4}\left[12p(0)\int_{t-\eta}^{t(1-\eta)} s^2 ds + at^3(1-\eta)^3(1-(1-\eta))\right] \\
&\leq \frac{1}{24p(0)}\eta^{\alpha-4}\left[12p(0)t^2(1-\eta)^2\eta(1-t) + at^3(1-\eta)^3\right] \\
&= \Big(\frac{1}{2}+\frac{a}{6(4-a)}\Big)t^3(1-\eta)^3\eta^{\alpha-4},
\end{aligned}$$

which proves (A2) $\eta \leqslant t$

$$\begin{aligned}
G(t,\eta) &= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\right] \\
&= \frac{1}{24p(0)}\eta^{\alpha-4}\left[\big(1-\frac{a}{4}+\frac{a}{4}\big)4t^3(1-\eta)^3 - at^3(1-\eta)^4 - 4p(0)(t-\eta)^3\right] \\
&\leq \frac{1}{24p(0)}\eta^{\alpha-4}\left[12p(0)\int_{t-\eta}^{t(1-\eta)} s^2 ds + at^3(1-\eta)^3(1-(1-\eta))\right] \\
&\leq \frac{1}{24p(0)}\eta^{\alpha-4}\left[12p(0)t^2(1-\eta)^2\eta(1-t) + a\eta(1-\eta)^3\right] \\
&= \Big(\frac{1}{2}+\frac{a}{6(4-a)}\Big)(1-\eta)^3\eta^{\alpha-3},
\end{aligned}$$

which proves (A3) when $\eta \leqslant t$.

If $t \leq \eta$, one has

$$\begin{aligned}
G(t,\eta) &= \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4\right] \\
&= \frac{1}{24p(0)}\eta^{\alpha-4}4t^3(1-\eta)^3\left[1-\frac{a}{4}(1-\eta)\right] = \frac{p(0)+p(\eta)-p(0)}{24p(0)}\eta^{\alpha-4}4t^3(1-\eta)^3 \\
&= \eta^{\alpha-4}\Big(\frac{1}{6}t^3(1-\eta)^3 + \frac{p(\eta)-p(0)}{6p(0)}t^3(1-\eta)^3\Big) \\
&\geq \frac{a}{6(4-a)}t^3(1-\eta)^3\eta^{\alpha-3},
\end{aligned}$$

which proves (A1) when $t \le \eta$.

$$G(t,\eta) = \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4\right]$$
$$\le \frac{1}{6p(0)}\eta^{\alpha-4}t^3(1-\eta)^3 \le \left[\frac{1}{2} + \frac{a}{6(4-a)}\right]t^3(1-\eta)^3\eta^{\alpha-4},$$

which proves (A2) when $t \le \eta$.

$$G(t,\eta) = \frac{1}{24p(0)}\eta^{\alpha-4}\left[4t^3(1-\eta)^3 - at^3(1-\eta)^4\right]$$
$$\le \frac{1}{6p(0)}\eta^{\alpha-4}t^3(1-\eta)^3 \le \left[\frac{1}{2} + \frac{a}{6(4-a)}\right](1-\eta)^3\eta^{\alpha-3},$$

which proves (A3) when $t \le \eta$.

Therefore, the proof is completed. ■

The Green's function satisfies the following bounds.

Lemma 2.8. *The following inequalities hold for the Green's function $G(t,\eta)$ defined by (3).*

$$\sigma_1(\eta)t^3 \le G(t,\eta) \le \sigma_2(\eta)t^3, \quad t,\eta \in (0,1],$$

where $\sigma_1(\eta) = \frac{a}{6(4-a)}(1-\eta)^3\eta^{\alpha-3}$ *and* $\sigma_2(\eta) = \left[\frac{1}{2} + \frac{a}{6(4-a)}\right](1-\eta)^3\eta^{\alpha-4}$.

Lemma 2.9. *The function $G(t,\eta)$ defined by (3) is continuous on $[0,1]\times(0,1]$ and satisfies*

$$\Big|G(t_2,\eta) - G(t_1,\eta)\Big| \le \eta^{\alpha-4}(t_2 - t_1), \qquad 0 \le t_1 \le t_2 \le 1.$$

Proof. If $0 \le t_1 \le t_2 \le 1$, then one obtains

$$\Big|G(t_2,\eta) - G(t_1,\eta)\Big| \le \frac{\eta^{\alpha-4}}{24p(0)}\Bigg(\Big|(t_2^3 - t_1^3)\Big[4(1-\eta)^3 - a(1-\eta)^4\Big]\Big|$$
$$+ \Big|4p(0)\Big((t_2-\eta)^3 - (t_1-\eta)^3\Big)\Big|\Bigg).$$

Let $g(\eta) := 4(1-\eta)^3 - a(1-\eta)^4$. Observe that $g'(\eta) = 4a(1-\eta)^3 - 12(1-\eta)^2 \le 0$ *for* $\eta \in [0,1]$. Thus, the function $g(\eta)$ is decreasing and $\max_{\eta\in[0,1]} g(\eta) = 4 - a = 4p(0)$. Moreover, by the help of the mean value theorem one has $(t_2-\eta)^3 - (t_1-\eta)^3 < 3(t_2 - t_1)$ and $t_2^3 - t_1^3 < 3(t_2 - t_1)$. Thus, the desired result follows

$$\Big|G(t_2,\eta) - G(t_1,\eta)\Big| \le \eta^{\alpha-4}(t_2 - t_1). \tag{10}$$

■

3. Existence of the Solutions

Consider the Banach space $C[0,1]$ equipped with the maximum norm, $\|z\| = \max_{0\le t\le 1}|z(t)|$. Let a closed cone $P \subset X$ be defined by

$$P = \{z \in C[0,1] : w(t) \ge 0, \quad 0 \le t \le 1\}.$$

Clearly, P is the normal cone of non-negative functions in $C[0,1]$. Define a partial order in this space as follows:

$$z_1, z_2 \in C[0,1], z_1 \leq z_2 \Longleftrightarrow z_1(t) \leq z_2(t) \quad \text{for} \quad t \in [0,1].$$

Next, the existence results and two iterative schemes for the nonlinear fractional BVP (2) will be given. Given the continuous function $f : [0,1] \times [0,\infty) \to \mathbb{R}$, the following operators are introduced.

$$(T_n x)(t) := \int_{\frac{1}{n+1}}^{1} G(t,\eta) f\big(\eta, x(\eta)\big) d\eta, \quad n = 1, 2, \ldots, \tag{11}$$

$$(Tx)(t) := \int_{0}^{1} G(t,\eta) f\big(\eta, x(\eta)\big) d\eta. \tag{12}$$

Since $G(t,\eta)$ is not continuous at $\eta = 0$, the standard argument can not be applied to show that the operator T is completely continuous. Thus, first the operators T_n will be shown to be completely continuous. Then, complete continuity of T follows from Lemma 2.5.

From now on, the assumption below is supposed to be held true.

(A1) $f \in C([0,1] \times [0,\infty))$ and for $x \in [x_0, y_0]$ in $C[0,1]$, there exits a positive constant M so that

$$\max_{0 \leq t \leq 1, x_0 \leq x \leq y_0} |f(t,x)| = M.$$

Lemma 3.1. *Assume that (A1) holds. Then the operators T_n defined by (11) and the operator T defined by (12) are completely continuous operators.*

Proof. Let $[x_0, y_0]$ be the order interval in $C[0,1]$. For $u \in [x_0, y_0]$, the operators T_n are continuous in view of the continuity of $f(t,x)$ and $G(t,\eta)$ on the set $[0,1] \times (\frac{1}{1+n}, 1]$ for each $n \in \mathbb{N}$. It is proved that $T_n : [x_0, y_0] \to C[0,1]$ is completely continuous. First, the uniform boundedness of $T_n([x_0, y_0])$ in $C[0,1]$ will be shown. The assumption (A1) implies that there is a positive constant M so that $|f(t,x)| \leq M, \quad t \in [0,1]$. Using Lemma 2.8, it follows that, for $x \in [x_0, y_0]$

$$\begin{aligned}
\|T_n u\| &= \max_{0 \leq t \leq 1} \int_{\frac{1}{n+1}}^{1} G(t,\eta) \Big| f(\eta, u(\eta)) \Big| d\eta \leq \int_{\frac{1}{n+1}}^{1} \sigma_2(\eta) \Big| f(\eta, u(\eta)) \Big| d\eta \\
&\leq M \left[\frac{1}{2} + \frac{a}{6(4-a)} \right] \int_{\frac{1}{n+1}}^{1} (1-\eta)^3 \eta^{\alpha-4} d\eta \leq M \left[\frac{1}{2} + \frac{a}{6(4-a)} \right] \int_{\frac{1}{n+1}}^{1} \eta^{\alpha-4} d\eta \\
&= M \left[\frac{1}{2} + \frac{a}{6(4-a)} \right] \frac{1}{\alpha-3}.
\end{aligned}$$

On the other hand, for any $u \in [x_0, y_0]$ and $0 < t_1 \leq t_2 \leq 1/n$, by using Lemma 2.9 and the assumption (A1), one has

$$\begin{aligned}
\Big| (T_n u)(t_2) - (T_n u)(t_1) \Big| &\leq \int_{\frac{1}{n+1}}^{1} \Big| G(t_2,\eta) - G(t_1,\eta) \Big| |f(\eta, u(\eta))| d\eta \\
&\leq M \int_{\frac{1}{n+1}}^{1} \eta^{\alpha-4} d\eta \\
&\leq M (t_2 - t_1) \frac{1}{\alpha-3}.
\end{aligned}$$

For $u \in [x_0, y_0]$ and $0 < t_1 \leq \frac{1}{n} \leq t_2 \leq 1$, one has

$$\begin{aligned}\Big|(T_n u)(t_2) - (T_n u)(t_1)\Big| &\leq \int_{\frac{1}{n+1}}^{t_2} \Big|G(t_2,\eta) - G(t_1,\eta)\Big||f(\eta, u(\eta))|d\eta \\ &+ \int_{t_2}^{1} \Big|G(t_2,\eta) - G(t_1,\eta)\Big||f(\eta, u(\eta))|d\eta \\ &\leq M(t_2 - t_1)\Big(\int_{\frac{1}{n+1}}^{t_2} \eta^{\alpha-4} d\eta + \int_{t_2}^{1} \eta^{\alpha-4} d\eta\Big) \\ &\leq M(t_2 - t_1)\frac{1}{\alpha - 3}.\end{aligned}$$

For $u \in [x_0, y_0]$ and $\frac{1}{n} < t_1 \leq t_2 \leq 1$, one has

$$\begin{aligned}\Big|(T_n u)(t_2) - (T_n u)(t_1)\Big| &\leq \int_{\frac{1}{n+1}}^{t_1} \Big|G(t_2,\eta) - G(t_1,\eta)\Big||f(\eta, u(\eta))|d\eta \\ &+ \int_{t_1}^{t_2} \Big|G(t_2,\eta) - G(t_1,\eta)\Big||f(\eta, u(\eta))|d\eta \\ &+ \int_{t_2}^{1} \Big|G(t_2,\eta) - G(t_1,\eta)\Big||f(\eta, u(\eta))|d\eta \\ &\leq M(t_2 - t_1)\Big(\int_{\frac{1}{n+1}}^{t_1} \eta^{\alpha-4} d\eta + \int_{t_1}^{t_2} \eta^{\alpha-4} d\eta + + \int_{t_2}^{1} \eta^{\alpha-4} d\eta\Big) \\ &\leq M(t_2 - t_1)\frac{1}{\alpha - 3}.\end{aligned}$$

Thus, complete continuity of $T_n : ([x_0, y_0])$ follows from the Arzela–Ascoli theorem.

It is now shown that $T_n([x_0, y_0])$ converges uniformly to $T([x_0, y_0])$ and thus, complete continuity of $T([x_0, y_0])$ follows from Lemma 2.5. Observe that for $x \in [x_0, y_0]$, using the assumption (A1) and Lemma 2.8, one has

$$\begin{aligned}\|T_n u - Tu\| = \max_{0\leq t\leq 1} \Big|(T_n u)(t) - (Tu)(t)\Big| &\leq \max_{0\leq t\leq 1} \int_0^{\frac{1}{n+1}} G(t,\eta)\Big|f(\eta, u(\eta))\Big|d\eta \\ &\leq M\left[\frac{1}{2} + \frac{a}{6(4-a)}\right] \int_0^{\frac{1}{n+1}} (1-\eta)^3 \eta^{\alpha-4} d\eta \\ &\leq M\left[\frac{1}{2} + \frac{a}{6(4-a)}\right] \frac{1}{(n+1)^{\alpha-3}\alpha - 3}.\end{aligned}$$

Note that $\|T_n u - Tu\| \to 0$ as $n \to \infty$ since $\alpha \in (3, 4]$. This shows that $T_n([x_0, y_0])$ converges to $T([x_0, y_0])$ uniformly. ■

The existence result and monotone sequences will be given in the next theorem.

Theorem 3.1. *If there are two real numbers c_1, c_2 with $0 \leq c_1 < c_2$, and the assumptions below are met:*

$(A1')$ $f \in C((0,1) \times [c_1, c_2])$ *and* $f(t,x) \geq 0, \quad x \geq 0, t \in [0,1]$ *and* $\max_{(t,x(t)) \in (0,1)\times[c_1,c_2]} f(t,x(t)) \leq M.$

$(A2)$ $f(t,\hat{w}) \leq f(t,\bar{x}), t \in [0,1], c_1 \leq \hat{x} \leq \bar{x} \leq c_2.$

$(A3)$ $\int_0^1 \sigma_1(\eta) f(\eta, c_1\eta^3) d\eta \geq c_1$ *and* $\int_0^1 \sigma_2(\eta) f(\eta, c_2) d\eta \leq c_2.$

$(H4)$ $f(t,0) \not\equiv 0.$

then there are two positive solutions x^*, y^* of the problem (1) so that $0 < x^* \leq y^* \leq c_2, \quad t \in [0,1]$ as a limit of $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$ where the iterative schemes are given by for each $n = 0,1,2\ldots$.

$$x_0(t) = c_1 t^3, \quad x_{n+1}(t) = \int_0^1 G(t,\eta) f\big(\eta, x_n(\eta)\big) d\eta,$$
$$y_0(t) = c_2, \quad y_{n+1}(t) = \int_0^1 G(t,\eta) f\big(\eta, y_n(\eta)\big) d\eta.$$

Proof. Let $[x_0, y_0]$ be the order interval in $C[0,1]$. Observe that the fixed points of the equation $Tu = u$ are the solutions of the problem (1).

The complete continuity of $T([x_0, y_0])$ has been demonstrated in Lemma 3.1

The assumption (A2) assures that T is a monotone operator on order interval $[x_0, y_0]$.

For $t \in [0,1]$, using the assumptions (A1') and (A2) and Lemma 2.8, one has

$$x_1 = (Tx_0)(t) = \int_0^1 G(t,\eta) f(\eta, x_0(\eta)) d\eta = \int_0^1 G(t,\eta) f(\eta, c_1 t^3) d\eta$$
$$\geq t^3 \int_0^1 \sigma_1(\eta) f(\eta, c_1\eta^3) d\eta \geq c_1 t^3 = x_0(t)$$

which implies $x_0(t) \leq x_1(t), t \in [0,1]$ and using again the assumption (A2) one gets

$$x_2(t) = (Tx_1)(t) = \int_0^1 G(t,\eta) f(\eta, x_1(\eta)) d\eta \geq \int_0^1 G(t,\eta) f(\eta, x_0(\eta)) d\eta \geq x_1.$$

Similar argument shows that

$$y_1 = (Ty_0)(t) = \int_0^1 G(t,\eta) f(\eta, y_0(\eta)) d\eta = \int_0^1 G(t,\eta) f(\eta, c_2) d\eta$$
$$\leq t^3 \int_0^1 \sigma_2(\eta) f(\eta, c_2) d\eta \leq c_2 = y_0(t).$$

Then, by induction, one finds the iterative schemes $\{x_n\}$ and $\{y_n\}$ satisfying, $t \in [0,1]$

$$y_0(t) \geq y_1(t) \geq \cdots \geq y_n(t) \geq \ldots x_n(t) \geq \cdots \geq x_1(t) \geq x_0(t)$$

so that one has by induction, for each $n \in \mathbb{N}$,

$$y_{n+1} \leq y_n, \quad x_n \leq x_{n+1}.$$

Now, Lemma 2.4 implies that one has x^* and y^* that are minimal and maximal fixed points of T obeying $c_1 t^3 \leq x^*(t) \leq y^*(t) \leq c_2$. The assumption (A3) ensures non-zero solution when $c_1 > 0$. If $c_1 = 0$ and $f(t, 0) \not\equiv 0$, one has $\|x^*\| > 0$. By Lemma 2.7, one has

$$x^*(t) = (Tx^*)(t) = \int_0^1 G(t,\eta) f(\eta, x^*(\eta))d\eta$$
$$\geq \frac{a}{6(4-a)} t^3 \int_0^1 (1-\eta)^3 \eta^{\alpha-3} f(\eta, x^*(\eta))d\eta, \tag{13}$$
$$\|x^*\| = \max_{t\in[0,1]} \left|(Tx^*)(t)\right| \leq \left(\frac{1}{2} + \frac{a}{6(4-a)}\right) \int_0^1 (1-\eta)^3 \eta^{\alpha-3} f(\eta, x^*(\eta))d\eta. \tag{14}$$

From (13) and (14), it is found that

$$x^*(t) \geq \frac{at^3}{12-2a} \|x^*\| > 0.$$

Thus, the proof is completed. ■

Remark 3.1. *To compute the solution, the iterative sequences are started off with a simple function and a constant.*

4. A Numerical Example

The numerical results are given by iterative sequences in this section for the following FBVP

$$\begin{cases} T_0^{\frac{7}{2}} x(t) + 10t^3 + 5 + \frac{1}{25} x(t)(10 - x(t)) = 0, \\ x^{(k)}(0) = 0, \quad k = 0, 1, 2, \\ x(1) = a \int_0^1 x(s)\, ds. \end{cases} \tag{15}$$

Thus, $\alpha = \frac{7}{2}, a = 1/2, f(t,x) = 10t^3 + 5 + \frac{1}{25}x(t)(10 - x(t))$ in this problem. Let $c_1 = 0$ and $c_2 = 5$ for easy calculation. Note that the assumptions (A1') and (A2) for the function $f(t,x)$ on the set $[0,1] \times [0,5]$ hold and $f(t,0) = 10t^3 + 5, \quad f(t,5) = 10t^3 + 6$. In addition, a simple calculation reveals that $\sigma_1(\eta) = \dfrac{1}{42}(1-\eta)^3\eta^{1/2}, \quad \sigma_2(\eta) = \dfrac{23}{42}(1-\eta)^3\eta^{-1/2}$ and

$$\int_0^1 \sigma_1(\eta) f(\eta, 0) d\eta \approx 0.01309 > 0,$$
$$\int_0^1 \sigma_2(\eta)(10\eta^3 + 6)dx \approx 3.0611 < 5.$$

Thus, the condition (A2) on $[0,1] \times [0,5]$ is satisfied. Therefore, the problem has two non zero solutions w^*, z^* along with $0 < w^*(t) \leq z^*(t) \leq 5, 0 \leq t \leq 1$ and $w_n \to w^*, \quad z_n \to z^*$, where

$$w_0(t) = 0,$$
$$w_{n+1} = (Tw_n)(t),$$

and

$$z_0(t) = 5,$$
$$z_{n+1} = (Tz_n)(t),$$

by the help of Theorem 3.1. The first and second term of the sequence $w_n(t)$ by the MATLAB R2016a are given as follows

$$w_0(t) = 0,$$

$$w_1(t) = \frac{150016t^3 - 3360t^{13/2} + 144144}{9009}$$

$$w_2(t) = 7t^{7/2}((720153608192t^6)/42599947264875 - (24015424t^3)/135270135 + (8192t^7)/847875 + (8t^{7/2})/175 + (32768t^{10})/233107875 + (32768t^{13})/49139499123 - (595864t^{13/2})/23648625 - (2400256t^{19/2})/13816878075 - 32/7))/2 + (6812428601890695551286t^3)/40694766863326901325.$$

5. Conclusions

In this work, the existence of positive solutions for a class of conformable fractional equations with integral boundary conditions has been investigated. Using the monotone iterative method on a cone and some inequalities associated with the Green's function, two iterative sequences are constructed for approximating the solution. The corresponding Green's function for the conformable fractional differential equations is singular. Thus, the fixed point theorem can not be directly applied. To overcome this difficulty, the sequences of operators are defined and it is shown that they converge to the operator so that the fixed point theorem on this operator can be applied. Not only the existence of a positive solutions has been proved, but also monotone iterative schemes have been established.

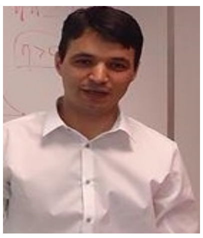
6. Acknowledgements

The author would like to thank the anonymous reviewers for their valuable and constructive comments and suggestions that helped to improve the manuscript.

References

[1] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., (2006). Theory and Applications of Fractional Differential Equations, Elsevier, Boston.
[2] Miller, K.S. and Ross, B., (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York.
[3] Podlubny, I.,(1999). Fractional Differential Equations Academic Press, New York.
[4] Khalil, R., Horani, M.A., Yousef, A. and Sababheh, M., (2014). A new definition of fractional derivative, J. Comput. Appl. Math., 264, pp. 65–70.
[5] Abdeljawad, T., (2015). On conformable fractional calculus, J. Comput. Appl. Math., 279, pp. 57–66.
[6] Horani, M.A and Khalil, R., (2018). Total fractional differentials with applications to exact fractional differential equations, International Journal of Computer Mathematics, 95:6-7, pp. 1444-1452.
[7] Yang, S., Wang, L. and Zhang, S., (2018). Conformable derivative: application to non-Darcian flow in low-permeability porous media, Appl. Math. Lett., 79, pp. 105–110.
[8] Zhao, D. and Luo, M., (2017). General conformable fractional derivative and its physical interpretation, Calcolo 54, pp. 903–917.
[9] Usta, F. and Sarikaya, M.Z., (2018). On generalization of Pachpatte type inequalities for conformable fractional integral,TWMS Journal of Applied and Engineering Mathematics, 8(1), pp. 106-113
[10] Sarikaya, M.Z., Budak, H. and Usta, F.,(2019). On generalized the conformable fractional calculus,TWMS Journal of Applied and Engineering Mathematics, 9(4), pp. 792-799.
[11] Usta, F. and Sarikaya, M.Z., (2019). The analytical solution of Van der Pol and Lienard differential equations within conformable fractional operator by retarded integral inequalities, Demonstratio Mathematica, 52(1), pp. 204-212.
[12] Sarikaya, M.Z. and Usta, F., (2016). On comparison theorems for conformable fractional differential equations, International Journal of Analysis and Applications, 12(2), pp. 207-214.

[13] Usta, F. and Sarikaya, M.Z., (2018). On generalization conformable fractional integral inequalities, FILOMAT, 32(16), pp. 5519-5526.
[14] Usta, F. and Sarikaya, M.Z., (2017). Explicit bounds on certain integral inequalities via conformable fractional calculus, Cogent Mathematics, 4(1), pp. 1-8.
[15] Usta, F. and Sarikaya, M.Z., (2017). Some improvements of conformable fractional integral inequalities, International Journal of Analysis and Applications, 14(2), pp. 162-166.
[16] Usta, F., Budak, H., Tunc, Tuba. and Sarikaya, M.Z., (2018). New bounds for the Ostrowski type inequalities via conformable fractional calculus, Arabian Journal of Matheamtics, 7, pp. 317-328.
[17] Usta, F., (2016). Fractional type Poisson equations by radial basis functions Kansa approach, Journal of Inequalities and Special Functions, 7(4), pp. 143-149.
[18] Sun, Y. and Zhao, M., (2014). Positive solutions for a class of fractional differential equations with integral boundary conditions, Appl. Math. Lett., 34, pp. 17-21.
[19] Li, S., Zhang, X., Wu, Y. and Caccetta L., (2013). Extremal solutions for p-Laplacian differential systems via iterative computation, Appl. Math. Lett., 26, pp. 1151-1158.
[20] Zhang, X., Wang, L. and Sun, Q., (2014). Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, Appl. Math. Comput., 226, pp. 708-718.
[21] Cabada, A. and Wang, G., (2012). Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, J. Math. Anal. Appl., 389, pp. 403-411.
[22] Caballero, J. and Cabrera, I., Sadarangani K., (2012). Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, Abstr. Appl. Anal., 2012, pp. 1–11.
[23] Cabada, A. and Hamdi, Z., (2014). Nonlinear fractional differential equations with integral boundary value conditions, Appl. Math. Comput., 228, pp. 251-257.
[24] Zhao, X., Chai, C. and Ge, W., (2013). Existence and nonexistence results for a class of fractional boundary value problems, J. Appl. Math. Comput., 41, pp. 17-31.
[25] Zhang, X., Liu, L., Wu, Y. and Lu Y., (2013). The iterative solutions of nonlinear fractional differential equations, Appl. Math. Comput., 219, pp. 4680-4691.
[26] Al-Refai, M. and Hajji, M., (2011). Monotone iterative sequences for nonlinear boundary value problems of fractional order, Nonlinear Anal., 74, pp. 3531-3539.
[27] Yao, Q., (2010). Successively iterative method of nonlinear Neumann boundary value problems, Appl. Math. Comput., 217, pp. 2301-2306.
[28] Dong, X., Bai, Z. and Zhang, Q., (2017). Positive solutions to boundary value problems of p-Laplacian with fractional derivative. Bound. Value Probl., 2017,pp. 5.
[29] Zhong, W. and Wang, L., (2018). Positive solutions of conformable fractional differential equations with integral boundary conditions, Bound. Value Probl., 2018, pp. 137.
[30] Amann, H., (1976). Fixed point equations and nonlinear eigenvalue problems in order banach spaces, SIAM Rev., 18, pp. 620–709.
[31] Shilov, G., Chilov, G. AND Silverman, R., (1996). Elementary Functional Analysis, Dover Publications.



Suayip TOPRAKSEVEN received his BSc degree from Ömer Halisdemir University, Turkey in 2006 and MSc from University of Cincinnati, USA in 2010 and PhD (Applied Mathematics) from University of Cincinnati, USA in 2014. At present, he is working as an asst. professor in the Department of Computer Engineering at Artvin Çoruh University. He is interested in Approximation Theory, Numerical solution of differential equations, Finite element solution and Discontinuous Galerkin methods, Integral Equations, Fractional Calculus, Partial Differential Equations.