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FUZZY CONGRUENCE ON $M\Gamma-$GROUPS

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Abstract. In this paper, we consider an algebriac structure $M\Gamma-$ group, which is a generalization of both the concepts module over a nearring and a gamma nearring, introduced by Satyanarayana [12]. In this paper, we define a fuzzy congruence on $M\Gamma-$module and obtain the one-one correspondence between the fuzzy congruences and fuzzy ideals on $M\Gamma-$groups. Further, we establish various related results between the congruences and ideals of $M\Gamma-$groups.

Keywords: $M\Gamma-$group, congruence, nearring module.

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1. Introduction

Nearrings are generalized rings which are crucial in the nonlinear theory of group mappings. Nearrings are defined in a natural way. For a group $(G, +)$ (not necessarily abelian), the set $M(G) = \{f : G \to G\}$ together with component-wise addition and composition of mappings forms a nearring but not a ring. Nearrings does not require the commutativity of addition. An important type of nearrings obtained by considering the additive closure $E(G)$ consists of all sums (or differences) of endomorphisms, which generalizes the concept of an endomorphism ring of an abelian group to the non-abelian case. More formerly, we give the definition as follows.

Pilz [10] A non-empty set N with two binary operations $+$ and $\cdot$ is called a *nearring* if it satisfies the following axioms.

(1) $(N, +)$ is a group (not necessarily Abelian);
(2) $(N, \cdot)$ is a semigroup;
(3) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in N$.

Precisely speaking, it is a right nearring. Moreover, a nearring N is said to be a zero-symmetric nearring if $n \cdot 0 = 0$ for all $n \in N$ where 0 is the additive identity in N. The concept of $\Gamma-$nearring, a generalization of the concepts the nearring and the $\Gamma-$ring, which

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was introduced by Satyanarayana [12]. Let $(M, +)$ be a group (not necessarily abelian) and Γ, a non-empty set. Then M is said to be a $\Gamma-$nearring if there exists a mapping $M \times \Gamma \times M \to M$ (the image of (a, α, b) is denoted by $a\alpha b$), satisfying the following conditions:

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$;
(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Further, M is said to be zero-symmetric if $a\alpha 0 = 0$ for all $a \in M$ and $\alpha \in \Gamma$, where 0 is the additive identity in M.

It is clear that if M is a $\Gamma-$nearring, then the elements of Γ act as binary operations on M such that the system $(M, +, \gamma)$ is a nearring for all $\gamma \in \Gamma$. The relations between the concepts $\Gamma-$nearring and nearring were studied by Satyanarayana [13], [14]. Some characterizations of prime ideals and corresponding radical properties were studied by Satyanarayana [13], [14], Booth [2], [3]. Also the ideal theory of modules over $\Gamma-$nearrings was studied by Booth and Groenewald [4].

Throughout this paper, M stands for a zero-symmetric $\Gamma-$nearring. For standard definitions and preliminary results on nearrings we refer to Pilz [10], and Satyanarayana and Syam Prasad [22].

Definition 1.1. [22] *Let M be a $\Gamma-$nearring. An additive group G is said to be a $\Gamma-$nearring-module (or $M\Gamma-$group) if there exists a mapping $M \times \Gamma \times G \to G$ (denote the image of (m, α, g) by $m\alpha g$ for $m \in M, \alpha \in \Gamma, g \in G$) satisfying the conditions*

(1) $(m_1 + m_2)\alpha_1 g = m_1\alpha_1 g + m_2\alpha_1 g$
(2) $(m_1\alpha_1 m_2)\alpha_2 g = m_1\alpha_1(m_2\alpha_2 g)$ *for* $m_1, m_2 \in M, \alpha_1, \alpha_2 \in \Gamma$ *and* $g \in G$.

An additive subgroup H of G is said to be $M\Gamma-$subgroup if $m\alpha h \in H$ for all $m \in M$, $\alpha \in \Gamma$ and $h \in H$. (Note that (0) and G are trivial $M\Gamma-$subgroups).

A normal subgroup H of G is said to be a ideal of G if $m\alpha(g + h) - m\alpha g \in H$ for $m \in M, \alpha \in \Gamma, g \in G$ and $h \in H$.

For $M\Gamma-$groups G_1 and G_2, a group homomorphism $\theta : G_1 \to G_2$ is said to be $M\Gamma-$homomorphism if $\theta(m\alpha g) = m\alpha(\theta g)$ for all $m \in M, \alpha \in \Gamma$ and $g \in G_1$.

The ideals of an $M\Gamma-$group are defined to be the kernals of $M\Gamma-$homomorphisms.

The concept of fuzzy subset was introduced by Zadeh [23]. Let A be a non-empty set. A mapping $\mu : A \to [0, 1]$ is called the fuzzy subset of A. For any $t \in [0, 1]$, $\mu_t = \{x \in A \mid \mu(x) \geq t\}$ is called as a level subset of μ. For any two fuzzy sets μ, σ in A, we write $\mu \subseteq \sigma$ if $\mu(x) \leq \sigma(x)$ for all $x \in A$. (In this case, we also say that μ is a subset of σ). Let X and Y be two non-empty sets, $f : X \to Y$, μ and σ be fuzzy subsets of X and Y respectively. Then $f(\mu)$, the image of μ under f is a fuzzy subset of Y defined by

$$(f(\mu))(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

$f^{-1}(\sigma)$, the preimage of σ under f is a fuzzy subset of X defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$ for all $x \in X$.

Definition 1.2. [22] *A non-empty fuzzy subset μ of an $M\Gamma-$group G is called a fuzzy ideal of G if*

(1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(2) $\mu(-x) = \mu(x)$

(3) $\mu(y + x - y) = \mu(x)$

(4) $\mu(m\gamma(a + b) - m\gamma a) \geq \mu(b)$, *for all* $x, y \in G$ *and for all* $m \in M, \gamma \in \Gamma$.

Definition 1.3. [22]

Let μ *be a fuzzy normal subgroup of* G *and* $x \in G$. *Then the fuzzy subset* $x + \mu$ *of* G, *defined by* $(x + \mu)(y) = \mu(y - x)$ *for all* $y \in G$, *is called the fuzzy coset of* μ.

Proposition 1.4. [22] *Let* μ *be a fuzzy ideal of* G. *Then* $x + \mu = y + \mu$ *if and only if* $\mu(x - y) = \mu(0)$ *for all* $x, y \in G$.

2. Fuzzy congruence relations on $M\Gamma-$groups

It is well known that a congruence relation on a algebraic structure is an equivalence relation in which the underlined algebraic operations are preserved. In this section we define fuzzy congruence on $M\Gamma-$group which is analogue of the notion defined for module over nearrings.

Definition 2.1. *A relation* ρ *on* $M\Gamma-$*group* G *is called a congruence on* G *if* ρ *is an equivalence relation on* G *with* $(a, b) \in \rho$ *and* $(c, d) \in \rho$ *implies that* $(a + c, b + d) \in \rho$ *and* $(m\gamma a, m\gamma b) \in \rho$ *for all* $a, b, c, d \in G$ *and for all* $m \in M, \gamma \in \Gamma$.

Definition 2.2. *Let* G *be an* $M\Gamma-$*group. A non empty fuzzy relation* α *on* G *(that is, a mapping* $\alpha : G \times G \to [0, 1]$*) is called a fuzzy equivalence relation if*

(1) $\alpha(x, x) = \sup_{y,z \in G} \alpha(y, z)$ *for all* $x, y, z \in G$ *(fuzzy reflexive)*

(2) $\alpha(x, y) = \alpha(y, x)$ *for all* $x, y \in G$ *(fuzzy symmetric)*

(3) $\alpha(x, y) \geq \sup_{z \in G}(\min(\alpha(x, z), \alpha(z, y)))$ *for all* $x, y, z \in G$ *(fuzzy transitive)*

hold.

Definition 2.3. *A fuzzy equivalence relation* α *on an* $M\Gamma-$*group* G *is called a fuzzy congruence relation if*

(1) $\alpha(a + c, b + d) \geq \min\{\alpha(a, b), \alpha(c, d)\}$

(2) $\alpha(m\gamma a, m\gamma b) \geq \alpha(a, b)$ *for all* $a, b, c, d \in G, m \in M, \gamma \in \Gamma$.

Example 2.1. *Take* $M = (Z, +, .)$, *nearring of integers,* $G = (Z, +)$, *and* $\Gamma = \{\gamma\}$, *where* γ *is a usual multiplication of integers. Then* G *is an* $M\Gamma-$*group.*

Let

$$\alpha(x, y) = \begin{cases} 1, & if\, x = y, \\ 0.5, & if\, x \neq y \text{ and } x, y = 2n \text{ or } x, y = 2n + 1 \text{ for some } n \in Z. \\ 0, & otherwise. \end{cases}$$

This satisfies $\alpha(x, x) = \sup_{y,z \in G} \alpha(y, z)$ *for all* $y, z \in G$, $\alpha(x, y) = \alpha(y, x)$ *for all* $x, y \in G$, $\alpha(x, y) \geq \sup_{z \in G}\{\min\{\alpha(x, z), \alpha(z, y)\}\}$ $.\alpha(a + c, b + d) \geq \min\{\alpha(a, b), \alpha(c, d)\}$ *and* $\alpha(m_1\gamma g, m_2\gamma g) \geq \alpha(m_1, m_2)$ *for all* $g, m_1, m_2 \in M$ *and* $\gamma \in \Gamma$.

Example 2.2. *Take* $M = \{0, a, b, c\}$, $G = \{0, a, b, c\}$ *and* $\Gamma = \{\gamma_1, \gamma_2\}$ *with addition and multiplication operations as defined below. Then* G *is an* $M\Gamma-$*group.*

Here addition table defined for both M and G are as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\Gamma = \{\gamma_1, \gamma_2\}$ where γ_1 and γ_2 defined as follows:

γ_1	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

γ_2	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	a	a	a	c

Define

$$\alpha(x,y) = \begin{cases} 1, & if\, x = y \, and\, x \neq 0, y \neq 0, \\ 0.6, & if\, x = 0 \, or\, y = 0, \\ 0, & if\, x \neq y. \end{cases}$$

The above definition satisfies $\alpha(x,x) = \sup\limits_{y,z \in G} \alpha(y,z)$ for all $y, z \in G$,
$\alpha(x,y) = \alpha(y,x)$ *for all $x, y \in G$,*
$\alpha(x,y) \geq \sup\limits_{z \in G} \{\min\{\alpha(x,z), \alpha(z,y)\}\}$.
Further, $\alpha(a+c, b+d) \geq \min\{\alpha(a,b), \alpha(c,d)\}$ and
$\alpha(m_1\gamma_1 g, m_2\gamma_2 g) \geq \alpha(m_1, m_2)$ *for all $g, m_1, m_2 \in M$ and $\gamma_1, \gamma_2 \in \Gamma$.*

Theorem 2.4. *Let ρ be a relation on an $M\Gamma-$group G and λ_ρ be its characteristic function. Then ρ is a congruence relation on G if and only if λ_ρ is a fuzzy congruence on G.*

Proof. Suppose ρ is a congruence relation on G. We need to prove that $\lambda_\rho$is a fuzzy congruence on G.

(i) Since ρ is reflexive, we have $(x,x) \in \rho$ for all $x \in G$, so $\lambda_\rho(x,x) = 1 \geq \sup\limits_{y,z \in G} \lambda_\rho(y,z)$.

(ii) $\lambda_\rho(x,y) = 1 \Leftrightarrow (x,y) \in \rho \Leftrightarrow (y,x) \in \rho$ (Since ρ is symmetric) $\Leftrightarrow \lambda_\rho(y,x) = 1$.

Also $\lambda_\rho(x,y) = 0 \Leftrightarrow (x,y) \notin \rho \Leftrightarrow (y,x) \notin \rho \Leftrightarrow \lambda_\rho(y,x) = 0$

(iii)If $\lambda_\rho(x,y) = 1$, then it is clear. Suppose $\lambda_\rho(x,y) = 0$. Then $(x,y) \notin \rho$ implies $(x,z) \notin \rho$ or $(z,y) \notin \rho$ for all $z \in G \Rightarrow \lambda_\rho(x,z) = 0$ or $\lambda_\rho(z,y) = 0$ for all $z \in G$.

Now, $\lambda_\rho(x,y) = \min\{\lambda_\rho(x,z), \lambda_\rho(z,y)\} = 0 = \sup\limits_{z \in G}\{0\} = \{0\}$.

Therefore $\lambda_\rho(x,y) = \sup\limits_{z \in G}\{\min\{\lambda_\rho(x,z), \lambda_\rho(z,y)\}\}$.

Thus we proved that λ_ρ is fuzzy equivalence on G.

Let $a, b, c, d \in M$ and $m \in M$.

(i) Suppose $\lambda_\rho(a,b) = 1, \lambda_\rho(c,d) = 1$ then $(a,b) \in \rho, (c,d) \in \rho$ implies $(a+c, b+d) \in \rho$. This means $\lambda_\rho(a+c, b+d) = 1 = \min\{1,1\} = \min\{\lambda_\rho(a,b), \lambda_\rho(c,d)\}$.

Suppose $\lambda_\rho(a,b) = 1, \lambda_\rho(c,d) = 0$. Then $(a,b) \in \rho$ and $(c,d) \notin \rho$.

Now $\lambda_\rho(a+c, b+d) \geq 0 = \min\{1,0\} = \min\{\lambda_\rho(a,b), \lambda_\rho(c,d)\}$

(ii) $\lambda_\rho(x,y) = 1 \Leftrightarrow (x,y) \in \rho \Leftrightarrow (m\gamma x, m\gamma y) \in \rho \Leftrightarrow \lambda_\rho(m\gamma x, m\gamma y) = 1$

Suppose $\lambda_\rho(x,y) = 0$.

Now $\lambda_\rho(m\gamma x, m\gamma y) \geq 0 = \lambda_\rho(x, y)$ for all $x, y \in G, m \in M$ and $\gamma \in \Gamma$.

Hence λ_ρ is a fuzzy congruence on G.

Conversely, suppose that λ_ρ is a fuzzy congruence on G. We have to show that ρ is a congruence relation on G.

(i) $1 \geq \sup_{y,z\in G} \lambda_\rho(y, z) = \lambda_\rho(x, x) \Rightarrow (x, x) \in \rho$, for all $x \in G$. Therefore ρ is reflexive.

(ii) $(x, y) \in \rho \Leftrightarrow \lambda_\rho(x, y) = 1 \Leftrightarrow \lambda_\rho(y, x) = 1 \Leftrightarrow (y, x) \in \rho$.

(iii) $(x, y) \in \rho$ and $(y, z) \in \rho \Leftrightarrow \lambda_\rho(x, y) = 1$ and $\lambda_\rho(y, z) = 1 \Leftrightarrow 1 = \min\{\lambda_\rho(x, y), \lambda_\rho(y, z)\}$.

Now $\lambda_\rho(x, z) \geq \sup_{z\in G}\{\min\{\lambda_\rho(x, y), \lambda_\rho(y, z)\}\} = \sup_{z\in G}\{1\} = 1$.

This implies $(x, z) \in \rho$. Therefore ρ is transitive.

(iv) Suppose $(x, y), (z, w) \in \rho$.

Now $\lambda_\rho(x + z, y + w) \geq \min\{\lambda_\rho(x, y), \lambda_\rho(z, w)\} = \min\{1, 1\} = 1$. Therefore $(x + z, y + w) \in \rho$

(v) Let $m \in M$ and $(x, y) \in \rho$.

Now $\lambda_\rho(m\gamma x, m\gamma y) \geq \lambda_\rho(x, y)$.

This implies $(m\gamma x, m\gamma y) \in \rho$. Hence ρ is a congruence relation on G □

Definition 2.5. *Let α be a fuzzy relation on an $M\Gamma-$group G for each $t \in [0, 1]$, the set $\alpha_t = \{(a, b) \in G \times G \mid \alpha(a, b) \geq t\}$ is called a level relation of α.*

Theorem 2.6. *Let α be a fuzzy relation on $M\Gamma-$group G. Then α is a fuzzy congruence on G if and only if α_t is a congruence on G for each $t \in im\ \alpha$ (image of α).*

Proof. Suppose α is a fuzzy congruence relation on G. We have to show that α_t is a congruence relation on G.

Let $t \in im\ \alpha$. Now $\alpha_t = \{(a, b) \in G \times G \mid \alpha(a, b) \geq t\}$. Since α is fuzzy reflexive, we have $\alpha(x, x) = \sup_{y,z\in G} \alpha(y, z) \geq t$ for all $x, y, z \in G$, and so $(x, x) \in \alpha_t$.. Therefore α_t is reflexive.

Suppose $(x, y) \in \alpha_t \Rightarrow \alpha(x, y) \geq t$ then $\alpha(y, x) \geq t \Rightarrow (y, x) \in \alpha_t$. Hence α_t is symmetric.

Suppose $(x, y) \in \alpha_t, (y, z) \in \alpha_t \Rightarrow \alpha(x, y) \geq t$ and $\alpha(y, z) \geq t$.

Now $\min\{\alpha(x, y), \alpha(y, z)\} \geq t \Rightarrow \sup_{y\in G}\{\min\{\alpha(x, y), \alpha(y, z)\}\} \geq t$ so α_t is transitive and hence α_t is a equivalence relation on G.

Next we verify that α_t is a congruence relation.

Take $(a, b), (c, d) \in \alpha_t \Rightarrow \alpha(a, b) \geq t, \alpha(c, d) \geq t$

Then, $\alpha(a + c, b + d) \geq \min\{\alpha(a, b), \alpha(c, d)\} \geq \min\{t, t\} = t \Rightarrow (a + c, b + d) \in \alpha_t$.

Take $m \in M, a, b \in G$. Now $\alpha(m\gamma a, m\gamma b) \geq \alpha(a, b) \geq t$.

Therefore, $(m\gamma a, m\gamma b) \in \alpha_t$.

Conversely, suppose that α_t is a congruence relation on G. We need to verify that α is a fuzzy congruence relation on G.

Since $(x, x) \in \alpha_t$ for all $x \in G, t \in im\ \alpha$, we have $\alpha(x, x) \geq t$. Put $t = \alpha(0, 0)$. Then $\alpha(x, x) = \sup_{y,z\in G} \alpha(y, z)$ for all $x, y, z \in G$, and so α is fuzzy reflexive.

$\alpha(x, y) = t \Leftrightarrow (x, y) \in \alpha_t \Leftrightarrow (y, x) \in \alpha_t$ (since α_t is symmetric) $\Leftrightarrow \alpha(y, x) = t$, and so α is fuzzy symmetric.

Now if $\alpha(x, y) = t, \alpha(y, z) = t$, then $(x, y) \in \alpha_t, (y, z) \in \alpha_t$ (since α_t is transitive) $(x, z) \in \alpha_t$. This implies $\alpha(x, z) = t$, and so α is fuzzy transitive. □

Proposition 2.7. *Let α be a fuzzy congruence on an $M\Gamma-$group G and μ_α be a fuzzy subset of G, defined by $\mu_\alpha(a) = \alpha(a, 0), a \in G$. Then μ_α is a fuzzy ideal of G.*

Proof. We have, $\mu_\alpha(0) = \alpha(0,0) = \sup_{x,y\in G} \alpha(x,y) \neq 0$ (since α is non-empty, we have μ_α is non-empty).

For $a, b \in G, \mu_\alpha(a+b) = \alpha(a+b, 0) \geq \min\{\alpha(a,0), \alpha(b,0)\} = \min\{\mu_\alpha(a), \mu_\alpha(b)\}$

$$\begin{aligned}\mu_\alpha(-a) &= \alpha(-a,0)\\ &= \alpha(-a+0, -a+a)\\ &\geq \min\{\alpha(-a,-a), \alpha(0,a)\}\\ &= \alpha(0,a) = \alpha(a,0) = \mu_\alpha(a).\end{aligned}$$

In a similar way, $\mu_\alpha(a) \geq \mu_\alpha(-a)$. Therefore $\mu_\alpha(-a) = \mu_\alpha(a)$.

Also $\mu_\alpha(a+b-a) = \alpha(a+b-a, 0) = \alpha(a+b-a, a+0-a) \geq \alpha(b,0) = \mu_\alpha(b)$.

This proves μ_α is a fuzzy normal subgroup of G. For any $a, b \in G$, $\gamma \in \Gamma$ and $m \in M$,

$$\begin{aligned}\mu_\alpha(m\gamma(a+b) - m\gamma a) &= \alpha(m\gamma(a+b) - m\gamma a, 0)\\ &= \alpha(m\gamma(a+b) - m\gamma a, m\gamma a - m\gamma a)\\ &\geq \min\{\alpha(m\gamma(a+b), m\gamma a), \alpha(-m\gamma a, -m\gamma a)\} \text{ (since } \alpha \text{ is reflexive)}\\ &= \alpha(m\gamma(a+b), m\gamma a) \geq \alpha(a+b, a) \text{(since } \alpha \text{ is fuzzy congruence)}\\ &\geq \min\{\alpha(a,a), \alpha(b,0)\}\\ &= \alpha(b,0) = \mu_\alpha(b).\end{aligned}$$

Therefore $\mu_\alpha\{m\gamma(a+b) - m\gamma a\} \geq \mu_\alpha(b)$.

Hence μ_α is a fuzzy ideal of G. □

Remark 2.8. *If μ is a fuzzy ideal of $M\Gamma-$group G, then $\mu(z-y) = \mu(-y+z)$ for all $z, y \in G$.*

Proposition 2.9. *Let μ be a fuzzy ideal of an $M\Gamma-$group G. If α_μ be the fuzzy relation on G, defined by $\alpha_\mu(x,y) = \mu(x-y)$ for $x, y \in G$, then α_μ is a fuzzy congruence on G.*

Proof. Since $\mu \neq \phi$, we have $\alpha_\mu \neq \phi$. Let $x \in G$.

Now $\alpha_\mu(x,x) = \mu(x-x) = \mu(0) \geq \mu(y-z) == \alpha_\mu(y,z)$, for all $y, z \in G$

Therefore $\alpha_\mu(x,x) = \sup_{y,z\in G} \alpha(y,z)$. So α_μ is fuzzy reflexive.

We have, $\alpha_\mu(x,y) = \mu(x-y) = \mu(-(x-y)) = \mu(y-x) = \alpha_\mu(y,x)$. This shows that α_μ is fuzzy symmetric.

Now $\alpha_\mu(x,y) = \mu(x-y) = \mu(x-z+z-y) \geq \min\{\mu(x-z), \mu(z-y)\}$ (since μ is a fuzzy ideal).

Therefore $\alpha_\mu(x,y) \geq \min\{\mu(x-z), \mu(z-y)\}$ for all $z \in G$.

So $\alpha_\mu(x,z) \geq \sup_{\{} z \in G\} \min\{\mu(x-z), \mu(z-y)\} = \sup_{\{} z \in G\} \min\{\alpha_\mu(x,z), \alpha_\mu(z,y)\}$.

This shows that α_μ is fuzzy transitive. Hence α_μ is a fuzzy equivalence on G.

Let $x, y, u, v \in G$. Now,

$$\begin{aligned}\alpha_\mu(x+u,y+v) &= \mu(x+u-(y+v))\\ &= \mu(x+u-v-y)\\ &= \mu(-y+x+u-v) \text{ (By Remark 2.8)}\\ &\geq \min\{\mu(-y+x),\mu(u-v)\}\\ &= \min\{\mu(x-y),\mu(u-v)\}\\ &= \min\{\alpha_\mu(x,y),\alpha_\mu(u,v)\}.\end{aligned}$$

For $m \in M$, consider $\alpha_\mu(m\gamma x, m\gamma y)$.
Now

$$\begin{aligned}\alpha_\mu(m\gamma x, m\gamma y) &= \mu(m\gamma x - m\gamma y)\\ &= \mu\{m\gamma(y-y+x) - m\gamma y\}\\ &\geq \mu(-y+x) = \mu(x-y) = \alpha(x,y).\end{aligned}$$

Hence α_μ is a fuzzy congruence on G. □

Theorem 2.10. *Let G be an $M\Gamma-$group. Then there exists an inclusion-preserving bijection from the set of all fuzzy ideals of G to the set of all fuzzy congruence on G.*

Proof. Let $FI(G) = \{\mu \mid \mu \text{ is a fuzzy ideal of } G\}$ and
$FC(G) = \{\alpha \mid \alpha \text{ a fuzzy congruence on } G\}$.
Define $f : FI(G) \to FC(G)$ by $f(\mu) = \alpha_\mu$ for $\mu \in FI(G)$ and $g : FC(G) \to FI(G)$ by $g(\alpha) = \mu_\alpha$ for $\alpha \in FC(G)$.
Now $(g \circ f)(\mu) = g(f(\mu)) = g(\alpha_\mu) = \mu_{\alpha\mu}$ and
$\mu_{\alpha\mu}(a) = \alpha_\mu(a,0) = \mu(a-0) = \mu(a)$ for all $a \in G$. Therefore $\mu_{\alpha\mu} = \mu$.
Thus $(g \circ f)(\mu) = \mu = Id_{FI(G)}(\mu)$. This shows that f is injective.
Let $\mu_1, \mu_2 \in FI(G)$ and $\mu_1 \subseteq \mu_2$.
Now $\alpha_{\mu_2}(x,y) = \mu_2(x-y) \geq \mu_1(x-y)$ (since $\mu_1 \subseteq \mu_2$) $= \alpha_{\mu_1}(x,y)$ for all $(x,y) \in G \times G$.
Therefore $\alpha_{\mu_1} \subseteq \alpha_{\mu_2}$. Hence $f(\mu_1) \subseteq f(\mu_2)$. This shows that f is inclusion preserving mapping.
Let $\alpha \in FC(G)$. Then μ_α is a fuzzy ideal of G (by Proposition 2.7).
Now by Proposition 2.9 α_{μ_α} is a fuzzy congruence on G.
Now $(f \circ g)(\alpha) = f(g(\alpha)) = f(\mu_\alpha) = \alpha_{\mu_\alpha}$.
Also $\alpha_{\mu_\alpha}(x,y) = \mu_\alpha(x-y) = \alpha(x-y,0) = \alpha(x,y)$.
Therefore $\alpha_{\mu_\alpha} = \alpha$. Hence $(f \circ g)(\alpha) = f(g(\alpha)) = \alpha = Id_{FC(G)}(\alpha)$ for all $\alpha \in FC(G)$.
Thus $f \circ g = Id_{FC(G)}$ which implies f is surjective. This implies f is an inclusion preserving bijection from $FI(G)$ to $FC(G)$. □

Proposition 2.11. *Let α be a fuzzy congruence relation on an $M\Gamma-$group G and μ_α be the fuzzy ideal induced by α. Let $t \in Im\alpha$. Then $(\mu_\alpha)_t = \{x \in G \mid \alpha(x,0) \geq t\}$ is the ideal induced by the congruence α_t*

Proof. Let $t \in Im\alpha$. Since $\alpha(0,0) \geq t$, we have $0 \in (\mu_\alpha)_t$.
Let $x, y \in (\mu_\alpha)_t$.
Now $\alpha(x-y,0) = \alpha(x,y) \geq \sup_{z\in G}\{\min\{\alpha(x,z),\alpha(z,y)\}\} \geq \sup_{z\in G}\{\alpha(x,0),\alpha(y,0)\} = t$.
This implies $x - y \in (\mu_\alpha)_t$. Take $m \in M, \gamma \in \Gamma, g \in G, x \in (\mu_\alpha)_t$.
Now,

$$\begin{aligned}\alpha(m\gamma(g+x)-m\gamma g,0) &= \alpha(m\gamma(g+x),m\gamma g)\\ &\geq \sup_{m\in M,\gamma\in\Gamma,x,g\in G}\{\min\{\alpha(m\gamma(g+x),\alpha(m\gamma g)\}\}\\ &\geq t.\end{aligned}$$

This implies $wm\gamma(g+x)-m\gamma g\in(\mu_\alpha)_t$. Therefore $(\mu_\alpha)_t$ is an ideal of G. □

Proposition 2.12. *Let μ be a fuzzy ideal of an $M\Gamma-$group G and α_μ be the fuzzy congruence induced by μ. Let $t\in Im\mu$. Then $(\alpha_\mu)_t$ is the congruence on G induced by μ_t.*

Proof. For $(x,y)\in G\times G, \alpha_\mu(x,y)=\mu(x-y)$.
Take $t\in Im\mu$. Then $\alpha_\mu(t)=\{(x,y)\mid \alpha_\mu(x,y)\geq t\}$.
Let β be the congruence on G induced by μ_t.
Here $(x,y)\in\beta\Leftrightarrow x-y\in\mu_t$.
Let $(x,y)\in(\alpha_\mu)_t\Rightarrow(\alpha_\mu)(x,y)\geq t$ implies $\mu(x-y)\geq t\Rightarrow x-y\in\mu_t\Rightarrow(x,y)\in\beta$.
Therefore $(\alpha_\mu)_t\subseteq\beta$.
Take $(x,y)\in\beta$. Then $x-y\in\mu_t$, implies $\mu(x-y)\geq t\Rightarrow(\alpha_\mu)(x,y)\geq t$ implies $(x,y)\in(\alpha_\mu)_t$. Therefore $\beta\subseteq(\alpha_\mu)_t$. Hence $(\alpha_\mu)_t=\beta$. □

Definition 2.13. *Let G be an $M\Gamma-$group and α be a fuzzy congruence on G. A fuzzy congruence β on G is said to be $\alpha-$invariant if $\alpha(x,y)=\alpha(u,v)$ implies that $\beta(x,y)=\beta(u,v)$ for all $(x,y),(u,v)\in G\times G$.*

Lemma 2.14. *Let G be an $M\Gamma-$group and μ be a fuzzy ideal of G. Let α be the fuzzy congruence on G induced by μ. Then the fuzzy relation (α/α) on G/μ, defined by*

$$(\alpha/\alpha)(x+\mu,y+\mu)=\alpha(x,y),$$

is a fuzzy congruence on G/μ.

Proof. Now $x+\mu=u+\mu$ and $y+\mu=v+\mu$,
implies $\mu(x-u)=\mu(0)$ and $\mu(y-v)=\mu(0)$ (By Proposition 1.4.).
Since $\alpha(x,u)=\mu(x-u)=\mu(0)$,
we have, $\alpha(x,u)=\sup_{p,q\in G}\alpha(p,q)$ and $\alpha(y,v)=\mu(y-v)=\mu(0)$.
Also $\alpha(y,v)=\sup_{p,q\in G}\alpha(p,q)$.
Now

$$\begin{aligned}\alpha(x,y) &\geq \min\{\alpha(x,u),\alpha(u,y)\}\\ &= \alpha(x,y)\\ &\geq \min\{\alpha(u,v),\alpha(v,y)\}\\ &= \alpha(u,v).\end{aligned}$$

In a similar way,

$$\begin{aligned}\alpha(u,v) &\geq \min\{\alpha(u,x),\alpha(x,v)\}\\ &= \alpha(x,v)\\ &\geq \min\{\alpha(x,y),\alpha(y,v)\}\\ &= \alpha(x,y).\end{aligned}$$

Therefore $\alpha(x,y)=\alpha(u,v)$.
Hence (α/α) is well defined.
We need to verify that (α/α) is a fuzzy congruence on G/μ.

For this, $(\alpha/\alpha)(x+\mu, x+\mu) = \alpha(x,x) = \sup_{y,z\in G} \alpha(y,z) = \sup_{y,z\in G} (\alpha/\alpha)(y+\mu)$.

Therefore (α/α) is fuzzy reflexive.

Now $(\alpha/\alpha)(x+\mu, y+\mu) = \alpha(x,y) = \alpha(y,x) = (\alpha/\alpha)(y+\mu, x+\mu)$. Therefore (α/α) is fuzzy symmetric.

We have,

$$\begin{aligned}(\alpha/\alpha)(x+\mu, y+\mu) &= \alpha(x,y)\\ &\geq \sup_{z\in G}\{\min(\alpha(x,z), \alpha(z,y))\}\\ &= \sup_{z+\mu\in G/\mu}\{\min((\alpha/\alpha)(x+\mu, z+\mu), (\alpha/\alpha)(z+\mu, y+\mu))\}.\end{aligned}$$

Next we show that (α/α) is fuzzy congruence.

(i) We have, for all $x, y, z, w \in G$,

$$\begin{aligned}(\alpha/\alpha)(((x+\mu)+(y+\mu)), ((z+\mu)+(w+\mu))) &= (\alpha/\alpha)((x+y+\mu), (z+w+\mu))\\ &= \alpha(x+y, z+w)\\ &\geq \min\{\alpha(x,z), \alpha(y,w)\}\\ &= \min\{(\alpha/\alpha)(x+\mu, z+\mu), (\alpha/\alpha)(y+\mu, w+\mu)\}.\end{aligned}$$

(ii) Let $m \in M, \gamma \in \Gamma$.

Now,

$$\begin{aligned}(\alpha/\alpha)(m\gamma(x+\mu), m\gamma(y+\mu)) &= (\alpha/\alpha)((m\gamma x+\mu), (m\gamma y+\mu))\\ &= \alpha(m\gamma x, m\gamma y)\\ &\geq \alpha(x,y)\\ &= (\alpha/\alpha)(x+\mu, y+\mu).\end{aligned}$$

Therefore (α/α) is a fuzzy congruence on G/μ. □

Theorem 2.15. *Let G be an $M\Gamma-$group and μ be a fuzzy ideal of G. Let α be the fuzzy congruence on G induced by μ. Then there exists a bijection between $FC_\alpha(G)$ of $\alpha-$invariant fuzzy congruence on G and $FC_{(\alpha/\alpha)}(G/\mu)$ of $(\alpha/\alpha)-$invariant fuzzy congruences on G/μ.*

Proof. Let β be an $\alpha-$invariant fuzzy congruence on G.

Define $(\beta/\alpha)(x+\mu, y+\mu) = \beta(x,y)$ for all $x, y \in G$.

We need to verify that β/α is well defined.

Suppose $x+\mu = u+\mu$ *and* $y+\mu = v+\mu$. Then $\alpha(x,y) = \alpha(u,v)$ (by Lemma 2.14)

Since β is $\alpha-$invariant, we have $\beta(x,y) = \beta(u,v)$. Therefore β/α is well defined.

Now to show that β/α is an $(\alpha/\alpha)-$invariant fuzzy congruence on G/μ.

Suppose $(\alpha/\alpha)(x+\mu, y+\mu) = (\alpha/\alpha)(u+\mu, v+\mu) \Rightarrow \alpha(x,y) = \alpha(u,v)$.

Since β is $\alpha-$ invariant, we have,

$\beta(x,y) = \beta(u,v) \Rightarrow (\beta/\alpha)(x+\mu, y+\mu) = (\beta/\alpha)(u+\mu, v+\mu)$.

Therefore (β/α) is an (α/α) invariant fuzzy congruence on G/μ.

Define $f : FC_\alpha(G) \to FC_(\alpha/\alpha)(G/\mu)$ by $f(\beta) = \beta/\alpha$.

Suppose $\beta_1, \beta_2 \in FC_\alpha(G)$ such that $\beta_1(x,y) \neq \beta_2(x,y)$.

Now $(\beta_1/\alpha)(x+\mu, y+\mu) = \beta_1(x,y) \neq \beta_2(x,y) = (\beta_2/\alpha)(x+\mu, y+\mu)$.

Therefore θ is injective.

To prove f is surjective, let β' be an $(\alpha/\alpha)-$invariant fuzzy congruence on G/μ.

We define a fuzzy relation β on G as $\beta(x,y) = \beta'(x+\mu, y+\mu)$.

Now, (i)

$$\begin{aligned}\beta(x,x) &= \beta'(x+\mu, x+\mu) \\ &= \sup_{y+\mu, z+\mu \in G/\mu} \beta'(y+\mu, z+\mu) \\ &= \sup_{y,z\in G} \beta(y,z).\end{aligned}$$

(ii) $\beta(x,y) = \beta'(x+\mu, y+\mu) = \beta'(y+\mu, x+\mu) = \beta(y,x)$.
(iii) $\beta(x,y) \geq \sup_{z\in G} \{\min\{\beta(x,z), \beta(z,y)\}\}$.
Thus β is a fuzzy equivalence relation on G.

(iv) We have,

$$\begin{aligned}\beta(x+a, y+b) &= \beta'(x+a+\mu, y+b+\mu) \\ &= \beta'(x+\mu+a+\mu, y+\mu+b+\mu) \\ &\geq \min\left\{\beta'(x+\mu, y+\mu), \beta'(a+\mu, b+\mu)\right\} \\ &= \min\left\{\beta(x,y), \beta(a,b)\right\}.\end{aligned}$$

(v) For any $m \in M, x, y \in G$ *and* $\gamma \in \Gamma$,

$$\begin{aligned}\beta(m\gamma x, m\gamma y) &= \beta'(m\gamma x+\mu, m\gamma y+\mu) \\ &= \beta'(m\gamma(x+\mu), m\gamma(y+\mu)) \\ &\geq \beta'(x+\mu, y+\mu) \\ &= \beta(x,y).\end{aligned}$$

This shows that β is a fuzzy congruence relation on G.
It remains to prove that β is an $\alpha-$invariant. Suppose $\alpha(x,y) = \alpha(u,v)$.
Now,

$$\begin{aligned}(\alpha/\alpha)(x+\mu, y+\mu) &= (\alpha/\alpha)(u+\mu, v+\mu) \\ &\implies (\beta/\alpha)(x+\mu, y+\mu) \\ &= (\beta/\alpha)(u+\mu, v+\mu) \\ &\implies \beta(x,y) = \beta(u,v).\end{aligned}$$

Therefore β is $\alpha-$invariant.
Now, $(\beta/\alpha)(x+\mu, y+\mu) = \beta(x,y) = \beta'(x+\mu, y+\mu)$ for all $(x+\mu, y+\mu) \in G/\mu \times G/\mu$
Therefore $\beta' = (\beta/\alpha) = f(\beta)$. Hence f is surjective. □

Theorem 2.16. *Let G be an $M\Gamma-$group and μ be a fuzzy ideal of G. Let α be the fuzzy congruence on G induced by μ. Let $t = \sup\ im\alpha$. Then $G/\mu \cong G/\alpha_t$.*

Proof. Define $f : G/\mu \to G/\alpha_t$ by $f(x+\mu) = x\alpha_t$, where $x\alpha_t$ denotes the congruence class containing x of the congruence α_t.

We verify that f is well defined.

We have,

$$\begin{aligned} x+\mu=y+\mu &\implies \mu(x-y)=\mu(0)\\ &\implies \alpha(x,y)=\sup\ im\alpha \geq t\\ &\implies (x,y)\in\alpha_t\\ &\implies x\alpha_t=y\alpha_t\\ &\implies f(x+\mu)=f(y+\mu). \end{aligned}$$

Therefore f is well defined.
Next we verify that f is an $M\Gamma-$homomorphism.
We have,

$$\begin{aligned} f(x+\mu+y+\mu) &= f(x+y+\mu)\\ &=(x+y)\alpha_t\\ &=x\alpha_t+y\alpha_t\\ &=f(x+\mu)+f(y+\mu). \end{aligned}$$

Let $m\in M, \gamma\in\Gamma$ and $x+\mu\in G/\mu$.
Now, $f(m\gamma(x+\mu))=f(m\gamma x+\mu)=(m\gamma x)\alpha_t=m\gamma(x\alpha_t)=m\gamma f(x+\mu)$.
Therefore f is an $M\Gamma-$homomorphism.
Further,

$$\begin{aligned} f(x+\mu)=f(y+\mu) &\implies x\alpha_t=y\alpha_t\\ &\implies (x,y)\in\alpha_t\\ &\implies \alpha(x,y)=t\\ &\implies \mu(x-y)=t=\alpha(0,0)=\mu(0). \end{aligned}$$

This shows that $x+\mu=y+\mu$. So f is injective. Clearly f is surjective. Hence $G/\mu\cong G/\alpha_t$. □

3. Conclusions

The concept module over a gamma nearring (called as, $M\Gamma-$group) is a generalization of the concepts module over a ring, module over a nearring where in addition is not necessarily abelian. We have introduced fuzzy congruence on $M\Gamma-$group and obtained one-one correspondence between fuzzy substructures of $M\Gamma-$groups and corresponding congruence relations. This can be extended to left ideals / $M\Gamma-$subgroups / two sided ideals of $M\Gamma-$groups and related substructures.

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References

[1] B. Assaye, M. Alamneh, L. N. Mishra, Y. Mebrat, (2019), Dual skew Heyting almost distributive lattices, Applied Mathematics and Nonlinear Sciences, Vol. 4, Issue 1, pp. 151162.

[2] Booth, G. L., (1988), A note on Gamma nearrings, Stud. Sci. Math. Hungarica, 23 pp.471-475.

[3] Booth, G. L., (1990), Radicals of Gamma nearrings, Publ.Math.Debrecen,39 pp.223-230.

[4] Booth, G. L., Groenewald, N. J., (1992), Special Radicals of Nearring Modules, Quaestiones Mathematicae,15(2) pp.127-137.

[5] Deepmala, L. N. Mishra, (2015), Differential operators over modules and rings as a path to the generalized differential geometry, FACTA UNIVERSITATIS (NI Š) Ser. Math. Inform., ISSN No. 0352-9665, Vol. 30, No. 5, pp. 753-764.

[6] Deepmala, N. Subramanian, L. N. Mishra, (2016), The Topological groups of Triple Almost Lacunary χ^3 sequence spaces defined by a Orlicz function, Electronic Journal of Mathematical Analysis and Applications, Vol. 4(2), pp. 272- 280.

[7] Jun, Y. B., Kim, K. H., Ozturk M. A., (2001), Fuzzy Maximal ideals of Gamma Nearrings, Turk. J. of Mathematics, 25 pp.457- 463.

[8] Jun, Y. B., Kim, K. H., Yon, Y. H., (1999), Intuitionistic Fuzzy Ideals of Near-rings, Journal of Inst. of Math. and Comp. Sci. (Math.Ser.) 12(3) pp.221-228.

[9] Jun, Y. B., Sapanci, M., Ozturk, M. A., (1998), Fuzzy Ideals in Gamma Near-rings, Tr. J. of Mathematics, 22, pp.449-459.

[10] Pilz, G., (1983), Near-rings, North Holland.

[11] Salah, Abou-Zaid, (1991), On Fuzzy Subnear-rings and Ideals, Fuzzy sets and Systems, 44, pp.139-146.

[12] Satyanarayana, Bh., (1984), Contributions to Nearring Theory, Doctoral Thesis, Nagarjuna University.

[13] Satyanarayana, Bh., (1999), The f-Prime Radical in Γ - Nearrings, South East. Bull. Math., 23 pp.507-511.

[14] Satyanarayana, Bh., (1999), A Note on $\Gamma-$ Near-rings, Indian J. Mathematics, 41(3) pp.427-433.

[15] Syam Prasad, K.,(2000), Contributions to Near-ring Theory II, Doctoral Thesis, Nagarjuna University.

[16] Srinivas, K. B., Syam Prasad, K., Jagadeesha B., (2019), Interval valued L-fuzzy prime ideals, triangular norms and partially ordered groups: Soft Computing, 23 pp. 907 - 920.

[17] Syam Prasad K., Babushri Srinivas, K., Jagadeesha B., (2016), Interval Valued L-Fuzzy Cosets and Isomorphism Theorems, Afrika Mathematika (Springer), 27 (3) pp.393-408.

[18] Jagadeesha, B., Babushri Srinivas, K., Syam Prasad, K., (2015), Interval Valued L-Fuzzy Ideals based on t-norms and t-conorms, Journal of Intelligent Fuzzy Systems (IOS Press), Netherlands, 28(6) pp.2631-2641.

[19] Satyanarayana, Bh., Syam Prasad, K., Ram Prasad, J. L., (2009), Fuzzy Cosets of Prime Fuzzy Submodules, Journal of Fuzzy Mathematics, (USA), 17(3) pp.595-603.

[20] Satyanarayana, Bh., Syam Prasad, K., (2005), Fuzzy Cosets of Gamma Nearrings, Turkish Journal of Mathematics, (Turkey), 29 pp.11-22.

[21] Satyanarayana, Bh., Syam Prasad, K., TVP Kumar, (2004), On IFP N-Groups and Fuzzy IFP Ideals, Indian Journal of Mathematics, 46(1) pp.11-19.

[22] Satyanarayana, Bh., Syam Prasad, K., (2013), Nearrings, Fuzzy ideals and Graph theory, CRC press (Taylor and Francis, U.K., U.S.A).

[23] Zadeh L. A., (1965), Fuzzy sets, Information and control, 8, pp.338-353.

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