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ANALYTIC SOLUTION OF NONLINEAR SINGULAR BVP WITH MULTI-ORDER FRACTIONAL DERIVATIVES IN ELECTROHYDRODYNAMIC FLOWS

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Abstract. In this study, a power series formula is proposed in order to introduce a new innovated numerical method called a newly Power Series Method (NPSM), beside with a construction of its error bound, to obtain approximate solutions of the standard fractional counterpart for a Boundary Value Problem (BVP) that appears in ElectroHydroDynamic (EHD) flows of the fluid. The solution for numerous fractional derivatives of both rational and irrational orders are numerically computed. Based on the residual error computation, the validity of the obtained results is verified. A high accuracy and a clear efficiency of the proposed method are revealed by discussing several numerical comparisons between such method and others.

Keywords: ElectroHydroDynamic flow, Caputo's fractional derivative, power series method, Residual error.

AMS Subject Classification: 26A33, 34B16, 41A10.

1. Introduction

Several definitions have been proposed to calculate the fractional-order derivative. The most well-known definitions are the Riemann-Liouville, Caputo [1], Atangana and Baleanu [2], Caputo-Fabrizio [3], conformable and Grünwald-Letnikov definitions [4]. It is worth noting that these definitions could provide different fractional-order derivatives for a certain function. Moreover, Some of the simple fractional initial value problem using Atangana and Baleanu sense does not have nonzero solution [5]. Among of all mentioned definitions; the most two popular definitions are the Riemann-Liouville and the Caputo fractional-order derivatives. In fact, the Caputo one is often preferred due to its excellent results in several simulations that handle many of the real world processes and systems in a form of fractional-order differential equations. It has an ability to accept the initial conditions in terms of integer-order values, unlike the Riemann-Liouville definition,

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which does not have this practical advantage, and hence it's mostly applied for theoretical considerations [6].

Generally, most researchers exhibit that the fractional-order differential equations have better nature of complex dynamics than the integer order equations. In fact, most numerical solutions in this area improves many significant problems which were handled before in the classical case, some of these method are: Predictor Corrector Method (PCM) [7], Homotopy Method [8, 9], Adomain Decomposition Method (ADM) [10], Homotopy Perturbation Method (HPM) [11], Fractional Finite Difference Method (FFDM) [12, 13], Variational Iteration Method (VIM) [14], and Differential Transform Method (DTM) [13]. However, finding approximate solutions of fractional derivatives using Power Series Method (PSM) has been investigated in several techniques such as Taylor Method (TM), Differential Transforms Method (DTM) [15], Residual Power Series Method (RPSM) [16], and others. The main different point between all of these techniques, is the manner of finding the unknown coefficients of the resultant series. Most of such methods could provide a good approximation for the solution in specific intervals near the initial point, but if one needs that solution lying far from this point, then the results will be not enough accurate sure. For this reason, constructing a new method that depends upon the time interval of the solution is, indeed, an urgent need.

In this work, a new approach has been proposed for finding the coefficients of a constructed power series which will give, as will be shown later on, an accurate solution for the considered problem in the general domain, even if the fractional-order γ and β are irrationals or satisfy the property $\gamma - \beta < 1$. As a first step of the NPSM a new power series formula is derived for the fractional-order derivatives, which will be presented in Section 3. Next, some other derived formulas related to such formula are substituted in the main problem, and then some collection points are used in order to obtain the coefficients of the constructed power series. Section 5 discusses the manner of deriving the error bound for the solution, while Section 6 demonstrates several numerical results which show the efficiency of the proposed approach, followed by Section 7 that summarizes the conclusion of this work.

2. Preliminaries

The ElectroHydroDynamic (EHD) flow of a fluid in an ion drag configuration in a circular cylindrical conduit is governed by the non-linear Boundary Value Problem (BVP):

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + Ha^2\left(1 - \frac{w}{1-\alpha w}\right) = 0, \quad 0 < r < 1, \tag{1}$$

subject to the boundary conditions:

$$w^{'}(0) = 0,\ w(1) = 0. \tag{2}$$

where $w(r)$ is the velocity of the fluid, r is the radial distance from the cylindrical conduit center, Ha is the Hartmann electric number, and where α is the magnitude of the power of non-linearity which related to the pressure gradient, the ion mobility, and the current density at the inlet of the conduit.

The EHD flow of a fluid has been discussed well by McKee et al. [17]. They explained the fact that the non-dimensional variable α dominates the degree of non-linearity in such equation. Further, they showed that such equation could be approximated by two different linear equations for very small values of this variable, or very large too [18]. Afterward, the existence and uniqueness for this problem were dealt by Paullet et al. [19].

The problem under investigation, in (1)-(2), was handled using several numerical methods like Homotopy Analysis Method (HAM) [20], Spectral Method (SM) [21], Chebyshev Spectral Collocation Method (CSCM) [22], Orthonormal Bernstein Polynomials Method (OBPM) [23], and many others. Recently, a generalized model of such problem was proposed by Alomari et al. [18]. It has been, indeed, extended in the fractional case as follows:

$$D^{\gamma}w + \frac{1}{r}D^{\beta}w + Ha^2\left(1 - \frac{w}{1-\alpha w}\right) = 0, \tag{3}$$

where D^{γ}, $1 < \gamma \leq 2$, and D^{β}, $0 < \beta \leq 1$, are the fractional derivatives in the Caputo's sense, with the property that $\gamma - \beta \geq 1$, and with considering the same boundary conditions in (2). Obviously, Eq.(1) is, just, a special case of Eq.(3), by setting the fractional orders γ and β equal to 2 and 1, respectively.

3. Fractional Power Series

In this section, we give a series formula with its error term for the Caputo fractional derivative [24].

Theorem 3.1. *Let $y \in C^{n+1}[a,b]$, $0 < \alpha < 1$, and $a \geq 0$, then for every $t \in (a,b]$ there exist $\xi \in (a,b)$ such that the fractional derivative $D_a^{\alpha}y(t)$ in Caputo sense can be written in the form with its reminder term as:*

$$D_a^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\left(\sum_{i=1}^{n}\frac{y^{(i)}(a)(t-a)^{i-\alpha}}{\Pi_{j=1}^{i}(j-\alpha)} + \frac{y^{(n+1)}(\xi)(t-a)^{n+1-\alpha}}{\Pi_{j=1}^{n+1}(j-\alpha)}\right). \tag{4}$$

Proof. The fractional derivative $D_a^{\alpha}y(t)$ in the Caputo sense for $0 < \alpha < 1$ is known as

$$D_a^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{y'(x)}{(t-x)^{\alpha}}dx, t > a \tag{5}$$

By applying the integration by parts to Eq. (5) we have

$$D_a^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{y'(a)(t-a)^{1-\alpha}}{1-\alpha} + \int_a^t \frac{(t-x)^{1-\alpha}y''(x)}{(1-\alpha)}dx\right). \tag{6}$$

In the same manner if we apply the integration by parts to the second part of Eq.(6) $(n-1)$-times, we have

$$\begin{aligned} D_a^{\alpha}y(t) &= \frac{1}{\Gamma(1-\alpha)}\left(\begin{array}{c} \frac{y'(a)(t-a)^{1-\alpha}}{1-\alpha} + \frac{y''(a)(t-a)^{2-\alpha}}{(1-\alpha)(2-\alpha)} + \ldots \\ + \frac{y^{(n)}(a)(t-a)^{n-\alpha}}{\Pi_{j=1}^{n}(j-\alpha)} + \int_a^t \frac{(t-x)^{n-\alpha}y^{(n+1)}(x)}{\Pi_{j=1}^{n}(j-\alpha)}dx) \end{array}\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\sum_{i=1}^{n}\frac{y^{(i)}(a)(t-a)^{i-\alpha}}{\Pi_{j=1}^{i}(j-\alpha)} + \int_a^t \frac{(t-x)^{n-\alpha}y^{(n+1)}(x)}{\Pi_{j=1}^{n}(j-\alpha)}dx\right). \end{aligned}$$

Since $y \in C^{n+1}[a,b]$ and $(t-x)^{n-\alpha}$ does not change its sign on $[a,t]$, and by the Weighted Mean-Value Theorem there exist $\xi \in (a,t)$ such that

$$\begin{aligned} \int_a^t \frac{(t-x)^{n-\alpha}y^{(n+1)}(x)}{\Pi_{j=1}^{n}(j-\alpha)}dx &= \frac{y^{(n+1)}(\xi)}{\Pi_{j=1}^{n}(j-\alpha)}\int_a^t (t-x)^{n-\alpha}dx \\ &= \frac{y^{(n+1)}(\xi)(t-a)^{n+1-\alpha}}{\Pi_{j=1}^{n+1}(j-\alpha)}. \end{aligned}$$

Therefor, equation 4 can be obtained.

□

Theorem 3.1 can be generalized for any $m-1<\alpha<m$ where m is positive integer:

Theorem 3.2. *Let $y \in C^{n+m}[a,b]$, $m-1<\alpha<m$, where m is positive integer, and $a \geq 0$, then for every $t \in (a,b]$, there exist $\xi \in (a,b)$ such that the fractional derivative $D_a^\alpha y(t)$ in Caputo sense can be written in the form with its reminder term as:*

$$D_a^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \sum_{i=1}^{n} \frac{y^{(m+i-1)}(a)(t-a)^{m-\alpha+i-1}}{\Pi_{j=1}^{i}(m-\alpha+j-1)}$$

$$+\frac{y^{(m+n)}(\xi)(t-a)^{m+n-\alpha-1}}{\Gamma(m-\alpha)\Pi_{j=1}^{n+1}(m-\alpha+j-1)}. \tag{7}$$

Proof. The fractional derivative $D_a^\alpha y(t)$ in the Caputo sense for $m-1<\alpha<m$ is known as

$$D_a^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{y^{(m)}(x)}{(t-x)^{m+\alpha-1}} dx, t>a \tag{8}$$

By applying the integration by parts to (8) we have:

$$D_a^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{y^{(m)}(a)(t-a)^{m-\alpha}}{m-\alpha} + \int_a^t \frac{(t-x)^{m-\alpha} y^{(m+1)}(x)}{(m-\alpha)} dx \right) \tag{9}$$

In the same manner if we apply the integration by parts to the second part of (9) $(n-1)$-times, we have

$$D_a^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \left(\begin{array}{c} \frac{y^{(m)}(t-a)^{m-\alpha}}{m-\alpha} + \frac{y^{(m+1)}(a)(t-a)^{m-\alpha+1}}{(m-\alpha)(m-\alpha+1)} + \dots \\ +\frac{y^{(m+n-1)}(a)(t-a)^{m-\alpha+n-1}}{\Pi_{j=1}^{n}(m-\alpha+j-1)} + \int_a^t \frac{(t-x)^{m-\alpha+n-1} y^{(m+n)}(x)}{\Pi_{j=1}^{n}(m-\alpha+j-1)} dx) \end{array} \right)$$

$$= \frac{1}{\Gamma(m-\alpha)} \left(\sum_{i=1}^{n} \frac{y^{(m+i-1)}(a)(t-a)^{m-\alpha+i-1}}{\Pi_{j=1}^{i}(m-\alpha+j-1)} + \int_a^t \frac{(t-x)^{m-\alpha+n-1} y^{(m+n)}(x)}{\Pi_{j=1}^{n}(m-\alpha+j-1)} dx \right). \tag{10}$$

Since $y \in C^{m+n}[a,b]$ and $(t-x)^{m-\alpha+n-1}$ does not change its sign on $[a,t]$, and by the Weighted Mean-Value Theorem there exist $\xi \in (a,t)$ such that

$$\int_a^t \frac{(t-x)^{m-\alpha+n-1} y^{(m+n)}(x)}{\Pi_{j=1}^{n}(m-\alpha+j-1)} dx = \frac{y^{(m+n)}(\xi)}{\Pi_{j=1}^{n}(m-\alpha+j-1)} \int_a^t (t-x)^{m-\alpha+n-1} dx$$

$$= \frac{y^{(m+n)}(\xi)(t-a)^{m-\alpha+n}}{\Pi_{j=1}^{n+1}(m-\alpha+j-1)}.$$

Therefor, equation 7 can be obtained.

□

4. Solution Procedure

The Weierstrass approximation theorem states that every continuous function f defined on a closed interval $[a,b]$ can be approximated by a polynomial as closely as desired which converge uniformly to f. To approximate the solution of the fractional differential equation

$$D^\gamma y(r) + \frac{1}{r} D^\beta y(r) + Ha^2(1 - \frac{y(r)}{1-\alpha y(r)}) = 0, \tag{11}$$

with boundary conditions $y'(0) = 0$, and $y(1) = 0$, where $0 < r < 1$, $0 < \beta < 1$, and $1 \leq \gamma \leq 2$, suppose that we can approximate $y(r)$ by the $(n+1)^{th}$ Taylor polynomial of $y(r)$ about $r = 0$, that is

$$y(r) \approx w(r) = c_0 + \sum_{i=1}^{n+1} c_i r^i, \tag{12}$$

note that $c_1 = 0$ since $y'(0) = 0$. Using theorem (3.1), in our case and according to our assumption of $y(r)$ in (12), we can find the exact value of $D_a^{\beta} w(r)$ by

$$\frac{1}{\Gamma(1-\beta)} \sum_{i=1}^{n_1} \frac{w^{(i)}(a) r^{i-\beta}}{\Pi_{j=1}^{i}(j-\beta)}, \tag{13}$$

and using theorem (3.2), we can find the exact value of $D_a^{\gamma} w(r)$ by

$$\frac{1}{\Gamma(2-\gamma)} \sum_{i=1}^{n_2} \frac{w^{(i+1)}(a) r^{i-\gamma+1}}{\Pi_{j=1}^{i}(j-\gamma+1)}, \tag{14}$$

where n_1 and n_2 are some integer numbers, we can choose n_1=n_2=$n+2$ since $y^{(k)}(0) = 0$ for all $k \geq n+2$. To avoid the singularity of (11) at $r = 0$, we can multiply equation (11) by r, then substitute (12), (13) and (14) into (11), that is

$$rD^{\gamma} w(r) + D^{\beta} w(r) + rHa^2 (1 - \frac{w(r)}{1 - \alpha w(r)}) = 0, \tag{15}$$

To find the $(n+1)$ unknowns coefficients: $c_0, c_2, ..., c_{n+1}$, we generate a system of $(n+1)$ nonlinear algebraic equations, which can be solved numerically, for that, we can divided the interval $[0,1]$ into n sub-intervals, for simplification, we can choose $r_i = ih$, where $h = 1/n$, the resulting system is

$$r_i D^{\gamma} w(r_i) + D^{\beta} w(r_i) + Ha^2 r_i (1 - \frac{w(r_i)}{1 - \alpha w(r_i)}) = 0, i = 0, 2, 3, ..., n, \tag{16}$$

and since $y(1) = 0$, the equation $w(1) = 0$ is added to (16). By substituting $c_i's$ into (12) we obtain the solution $w(r)$ which approximate $y(r)$ with residual error $res(r)$, where the residual error can be computed using $w(r)$ and the Caputo definition for $D^{\gamma} w(r)$ and $D^{\beta} w(r)$. The residual error can be written as:

$$res(r) = rD^{\gamma} w(r) + D^{\beta} w(r) + rHa^2 (1 - \frac{w(r)}{1 - \alpha w(r)}).$$

Note that this process can be generalized to generate a power series of several variables [25] and to solve problems in fractional differential equations with high accuracy results.

5. Error Estimate

In this section, we provide some error estimate for the NPSM solution. Firstly, we rewrite the modeled as

$$f(r, w, D^{\gamma} w, D^{\beta} w) := r(1 - \alpha w) D^{\gamma} w + (1 - \alpha w) D^{\beta} w + rHa^2 (1 - \alpha w) - rHa^2 w.$$

Let $\tilde{w}$ be the approximate solution given in section 4 with

$$w = \tilde{w} + R_0, D^{\gamma} w = D^{\gamma} \tilde{w} + R_{\gamma}, D^{\beta} w = D^{\beta} \tilde{w} + R_{\beta}, \tag{17}$$

and $\tilde{f} = f(r, \tilde{w}, D^{\gamma}\tilde{w}, D^{\beta}\tilde{w})$. According to theorems 1 and 2, we assume $w, D^{\gamma}\tilde{w}$, and $D^{\beta}\tilde{w}$ are bounded by $K_0, K_{\gamma}, K_{\beta}$ respectively. With some simplifications the bound error can be written as

$$f - \tilde{f} = r(1-\alpha w)R_{\gamma} - r\alpha R_0 D^{\gamma}\tilde{w} + (1-\alpha w)R_{\beta} - \alpha R_0 D^{\beta}\tilde{w} - rHa^2R_0(1+\alpha),$$

and hence,

$$|f - \tilde{f}| \leq |H_1| + |H_2| + |H_3| + |H_4| + |H_5|, \tag{18}$$

where $H_1 = Ha^2rR_0(1+\alpha)$, $H_2 = R_{\gamma}r(1+\alpha K_0)$, $H_3 = R_{\beta}(1+\alpha K_0)$, $H_4 = \alpha r R_0 K_{\gamma}$ and $H_5 = \alpha K_{\beta} R_0$. Apply the infinite norm for (18) along the interval $[0, 1]$

$$||f - \tilde{f}|| \leq ||H_1|| + ||H_2|| + ||H_3|| + ||H_4|| + ||H_5||, \tag{19}$$

then the bound for (19) can be computed for each part of the right hand side of (19), for the first and second parts bound of $||H_1||$ and $||H_2||$ we have

$$\begin{aligned} ||H_1|| = \int_0^1 Ha^2rR_0(1+\alpha)dr &\leq \int_0^1 Ha^2 r\frac{Mr^{n+3}}{(n+2)!}(1+\alpha)dr \\ &= (1+\alpha)\frac{MHa^2}{(n+2)!(n+4)} \\ &\leq \frac{MHa^2}{(n+2)!}(1+\alpha), \end{aligned}$$

$$\begin{aligned} ||H_2|| = \int_0^1 rR_{\gamma}(1+\alpha K_0)dr &\leq \int_0^1 \frac{M_{\gamma}(1+\alpha K_0)r^{n+4-\gamma}}{\Gamma(2-\gamma)\Pi_{j=1}^{n+3}(j+1-\gamma)}dr \\ &= \frac{M_{\gamma}(1+\alpha K_0)}{\Gamma(2-\gamma)(n+5-\gamma)\Pi_{j=1}^{n+3}(j+1-\gamma)} \\ &\leq \frac{M_{\gamma}}{(n+2)!}(1+\alpha K_0), \end{aligned}$$

and the bound for $||H_3||$,$||H_4||$ and $||H_5||$ are $\frac{M_{\beta}}{(n+2)!}(1+\alpha K_0)$, $\frac{\alpha K_{\gamma} M}{(n+2)!}$, and $\frac{\alpha K_{\beta} M}{(n+2)!}$ respectively, where $M = Max_{\xi \in (0,1]} y^{(n+2)}(\xi)$, $M_{\beta} = Max_{\xi \in (0,1]} y^{(n+3)}(\xi)$, $M_{\gamma} = Max_{\xi \in (0,1]} y^{(n+4)}(\xi)$. Now, let $\mathfrak{M} = Max\{M, M_{\gamma}, M_{\beta}\}$ and $\mathfrak{K} = Max\{K_0, K_{\gamma}, K_{\beta}\}$, then

$$||f - \tilde{f}|| \leq \frac{\mathfrak{E}}{(n+2)!}, \text{ where } \mathfrak{E} = \mathfrak{M}(Ha^2(1+\alpha) + 1 + 4\alpha\mathfrak{K}).$$

6. Numerical Experiments

In this section, we present the numerical results for the solution of (1) and (2) in the standard and fractional cases. Firstly, let γ approach to 2 and β approach to 1, $\alpha = 0.5$ and vary $Ha^2 = 0.5, 1, 1.5, 2, 4, 10$ the numerical results are presented in tables 1 and 2 with its residual error. The present solution has $||Res.||_{\infty} \simeq 10^{-14}$ with 10-order of approximation while the HAM [20] solution has $||Res.||_{\infty} \simeq 0.014$ and OBCM [23] is 3.37×10^{-8} with 20-order of approximation. And for $\alpha = 1$ the results obtained in tables 3 and 4. The present solution has $||Res.||_{\infty} \simeq 10^{-13}$ with 10-order of approximation while the HAM [20] solution has $||Res.||_{\infty} \simeq 0.03$ and OBCM [23] is 1.94×10^{-6} with 20-order of approximation. We also plot the solution in figure 1. Secondly, the results for $\gamma = 1.3, \beta = 0.3$ with $\alpha = 0.5$ and $\alpha = 1$ introduce in tables 5-8 for several values of Ha^2. We noted that, for $\alpha = Ha^2 = 1$ DTM [18] gives the solution with $||Res.||_{\infty} \simeq 10^{-9}$ while the NPSM has 10^{-12}. The obtained solution for this case represent graphically in figure 2.

Finally, tables 9–12 give the NPSM solution in the case of irrational fractional derivatives $\gamma = \frac{\pi}{2}, \beta = \frac{\pi}{4}$ which is difficult to obtain by DTM since $\gamma - \beta < 1$. The residual errors reveal that the solution has accuracy with in 10^{-11}. NPSM solutions for this case is obtained in figure 2.

TABLE 1. The numerical solution for α=0.5,$\gamma \to 2$,$\beta \to 1$

	α=0.5, $Ha^2 = 0.5$		α=0.5, $Ha^2 = 1$		α=0.5, $Ha^2 = 1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1137456498	0.000	0.2070081525	0.000	0.2829765323	0.000
0.1	0.1126460196	0.000	0.2050839276	4.163×10^{-17}	0.2804593832	1.388×10^{-17}
0.2	0.1093424972	1.388×10^{-17}	0.1992933225	2.776×10^{-17}	0.2728695804	2.776×10^{-17}
0.3	0.1038212011	5.551×10^{-17}	0.1895826167	2.776×10^{-17}	0.2600921625	5.551×10^{-17}
0.4	0.0960590436	5.551×10^{-17}	0.1758625195	0.000	0.2419359038	1.110×10^{-16}
0.5	0.0860238038	8.327×10^{-17}	0.1580085509	5.551×10^{-17}	0.2181339695	0.000
0.6	0.0736742300	5.551×10^{-17}	0.1358616035	3.331×10^{-16}	0.1883450056	3.331×10^{-16}
0.7	0.0589601756	0.000	0.1092287140	2.220×10^{-16}	0.1521548162	0.000
0.8	0.0418227663	0.000	0.0778840714	1.110×10^{-16}	0.1090787970	2.220×10^{-16}
0.9	0.0221946029	2.220×10^{-16}	0.0415702941	5.551×10^{-16}	0.0585652904	2.220×10^{-16}
1.0	-7.2078×10^{-18}	1.110×10^{-16}	1.1282×10^{-17}	4.441×10^{-16}	2.6617×10^{-17}	6.661×10^{-16}

TABLE 2. The numerical solution for α=0.5,$\gamma \to 2$,$\beta \to 1$

	α=0.5, $Ha^2 = 2$		α=0.5, $Ha^2 = 4$		α=0.5, $Ha^2 = 10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.3447273110	0.000	0.4975662164	0.000	0.6289591491	0.000
0.1	0.3418045879	2.776×10^{-17}	0.4941746835	8.327×10^{-17}	0.6268697788	1.388×10^{-17}
0.2	0.3329725255	1.110×10^{-16}	0.4838197962	5.551×10^{-17}	0.6202616825	9.437×10^{-16}
0.3	0.3180394102	2.220×10^{-16}	0.4659554483	3.331×10^{-16}	0.6080686531	6.661×10^{-16}
0.4	0.2966857005	1.110×10^{-16}	0.4396555471	4.441×10^{-16}	0.5883338822	1.554×10^{-15}
0.5	0.2684642930	2.220×10^{-16}	0.4035930725	3.331×10^{-16}	0.5579308224	1.332×10^{-15}
0.6	0.2328014568	4.441×10^{-16}	0.3560167814	0.000	0.5121650676	3.331×10^{-15}
0.7	0.1889989421	2.220×10^{-16}	0.2947317659	1.998×10^{-15}	0.4442807901	4.885×10^{-15}
0.8	0.1362378182	4.441×10^{-16}	0.2170915578	4.441×10^{-16}	0.3449516050	1.332×10^{-14}
0.9	0.0735845853	6.661×10^{-16}	0.1200100085	8.882×10^{-16}	0.2019178321	3.109×10^{-14}
1.0	-4.6620×10^{-18}	4.441×10^{-16}	2.4286×10^{-17}	1.776×10^{-15}	-1.6653×10^{-16}	7.638×10^{-14}

TABLE 3. The numerical solution for α=1,$\gamma \to 2$,$\beta \to 1$

	α=1, $Ha^2 = 0.5$		α=1, $Ha^2 = 1$		α=1, $Ha^2 = 1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1132180551	0.000	0.2034157957	0.000	0.2729539294	0.000
0.1	0.1121272131	1.388×10^{-17}	0.2015523635	1.388×10^{-17}	0.2706076365	2.776×10^{-17}
0.2	0.1088494970	2.776×10^{-17}	0.1959401295	0.000	0.2635190692	8.327×10^{-17}
0.3	0.1033693935	2.776×10^{-17}	0.1865136787	5.551×10^{-17}	0.2515401417	5.551×10^{-17}
0.4	0.0956612384	0.000	0.1731653025	1.110×10^{-16}	0.2344273267	5.551×10^{-17}
0.5	0.0856894991	8.327×10^{-17}	0.1557469617	1.110×10^{-16}	0.2118466959	1.110×10^{-16}
0.6	0.0734091652	5.551×10^{-17}	0.1340730012	0.000	0.1833811418	4.441×10^{-16}
0.7	0.0587662401	5.551×10^{-17}	0.1079235740	1.110×10^{-16}	0.1485398095	4.441×10^{-16}
0.8	0.0416983252	5.551×10^{-17}	0.0770487015	2.220×10^{-16}	0.1067696130	4.441×10^{-16}
0.9	0.0221352877	2.220×10^{-16}	0.0411728656	2.220×10^{-16}	0.0574685112	2.220×10^{-16}
1.0	-1.0638×10^{-18}	0.000	-1.5476×10^{-17}	2.220×10^{-16}	2.4658×10^{-17}	4.441×10^{-16}

TABLE 4. The numerical solution for α=1,$\gamma \to 2$,$\beta \to 1$

	α=1, $Ha^2=2$		α=1, $Ha^2=4$		α=1, $Ha^2=10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.3254542675	0.000	0.4330995227	0.000	0.4920638661	0.000
0.1	0.3228595625	1.388×10^{-17}	0.4307207048	1.388×10^{-17}	0.4912399886	1.249×10^{-16}
0.2	0.3149903436	2.776×10^{-17}	0.4233626884	8.327×10^{-17}	0.4885803749	8.327×10^{-17}
0.3	0.3015926358	0.000	0.4103494689	3.886×10^{-16}	0.4832872919	2.776×10^{-17}
0.4	0.2822477445	2.220×10^{-16}	0.3905250974	3.331×10^{-16}	0.4737778868	6.106×10^{-16}
0.5	0.2563799395	0.000	0.3622216688	2.220×10^{-16}	0.4572525435	4.885×10^{-15}
0.6	0.2232685130	1.110×10^{-16}	0.3232382992	1.110×10^{-15}	0.4290267299	1.554×10^{-14}
0.7	0.1820649157	2.220×10^{-16}	0.2708526765	6.661×10^{-16}	0.3816909972	3.864×10^{-14}
0.8	0.1318151686	2.220×10^{-16}	0.2018850520	8.882×10^{-16}	0.3043801183	1.048×10^{-13}
0.9	0.0714869637	6.661×10^{-16}	0.1128234858	5.773×10^{-15}	0.1827207841	2.283×10^{-13}
1.0	2.3256×10^{-17}	4.441×10^{-16}	2.7755×10^{-17}	1.954×10^{-14}	4.4408×10^{-16}	5.222×10^{-13}

TABLE 5. The numerical solution for α=0.5, $\gamma = 13/10$,$\beta = 3/10$

	α=0.5, $Ha^2=0.5$		α=0.5, $Ha^2=1$		α=0.5, $Ha^2=1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1914916664	0.000	0.3394465367	0.000	0.4480371775	0.000
0.1	0.1840565025	6.939×10^{-18}	0.3282049857	1.388×10^{-17}	0.4358627492	9.714×10^{-17}
0.2	0.1711062212	3.469×10^{-17}	0.3082133607	1.527×10^{-16}	0.4136474113	4.163×10^{-16}
0.3	0.1558742920	5.551×10^{-17}	0.2840237442	1.985×10^{-15}	0.3858145097	4.940×10^{-15}
0.4	0.1386790399	1.540×10^{-15}	0.2558692654	4.052×10^{-15}	0.3521906383	2.470×10^{-14}
0.5	0.1197007358	6.800×10^{-16}	0.2237900399	2.212×10^{-14}	0.3123819196	6.256×10^{-14}
0.6	0.0990271481	1.590×10^{-14}	0.1876968341	7.716×10^{-14}	0.2658423752	1.075×10^{-13}
0.7	0.0767091307	3.109×10^{-14}	0.1474529259	9.542×10^{-14}	0.2119684553	2.032×10^{-13}
0.8	0.0527652123	1.023×10^{-13}	0.1028820866	8.777×10^{-13}	0.1501200606	3.595×10^{-13}
0.9	0.0272100903	1.479×10^{-13}	0.0538128082	3.204×10^{-12}	0.0796824631	2.399×10^{-12}
1.0	0.0000000000	4.372×10^{-13}	-5.684×10^{-14}	5.931×10^{-12}	0.0000000000	5.746×10^{-12}

TABLE 6. The numerical solution for α=0.5,$\gamma = 13/10$,$\beta = 3/10$

	α=0.5, $Ha^2=2$		α=0.5, $Ha^2=4$		α=0.5, $Ha^2=10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.5239383948	0.000	0.6428663622	0.000	0.6663904225	0.000
0.1	0.5126701617	1.596×10^{-16}	0.6385746562	9.541×10^{-17}	0.6664759961	1.247×10^{-18}
0.2	0.4914954885	2.220×10^{-16}	0.6294205565	8.119×10^{-16}	0.6661878084	2.806×10^{-16}
0.3	0.4638910157	2.581×10^{-15}	0.6152269010	1.818×10^{-15}	0.6655618484	7.464×10^{-16}
0.4	0.4290928645	6.023×10^{-15}	0.5935981653	3.053×10^{-15}	0.6639719230	1.701×10^{-15}
0.5	0.3860495414	1.665×10^{-14}	0.5609486217	8.216×10^{-15}	0.6598664922	4.515×10^{-15}
0.6	0.3334885538	3.147×10^{-14}	0.5121675997	5.329×10^{-14}	0.6490661002	5.477×10^{-14}
0.7	0.2700235865	1.354×10^{-14}	0.4404168989	2.478×10^{-13}	0.6206372890	9.048×10^{-14}
0.8	0.1942109497	3.278×10^{-13}	0.3372618723	1.110×10^{-12}	0.5473804732	8.755×10^{-13}
0.9	0.1046544008	2.171×10^{-12}	0.1934063377	3.252×10^{-12}	0.3711636371	1.739×10^{-12}
1.0	2.842×10^{-14}	6.349×10^{-12}	0.0000000000	7.162×10^{-12}	2.842×10^{-14}	4.953×10^{-12}

7. Conclusion

The solution of electrohydrodynamic flow in a circular cylindrical Conduit was successfully obtained for both integer and non-integer derivatives in terms of power series. Comparing with previous results the present algorithm has higher accuracy and it is applicable especially in cases that have not been resolved previously when γ-$\beta \leq 1$ and for the irrational order fractional derivatives. The results introduced in both tables and figures.

TABLE 7. The numerical solution for α=1, $\gamma = 13/10$,$\beta = 3/10$

	α=1, $Ha^2 = 0.5$		α=1, $Ha^2 = 1$		α=1, $Ha^2 = 1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1899546759	0.000	0.3277763009	0.000	0.4139205333	0.000
0.1	0.1827237321	3.816×10^{-16}	0.3179624214	1.388×10^{-17}	0.4053067730	2.741×10^{-16}
0.2	0.1700843760	5.412×10^{-16}	0.3001937930	6.939×10^{-17}	0.3888691327	7.910×10^{-16}
0.3	0.1551503131	2.498×10^{-15}	0.2781979704	5.967×10^{-16}	0.3670600400	3.747×10^{-16}
0.4	0.1382153712	1.024×10^{-14}	0.2520198732	6.911×10^{-15}	0.3391782484	4.163×10^{-16}
0.5	0.1194455904	4.019×10^{-14}	0.2215733052	3.320×10^{-14}	0.3043876323	1.288×10^{-14}
0.6	0.0989226696	1.515×10^{-13}	0.1866984795	8.665×10^{-14}	0.2618102129	7.239×10^{-14}
0.7	0.0766974518	3.896×10^{-13}	0.1472360154	2.703×10^{-13}	0.2106449132	1.950×10^{-13}
0.8	0.0527941062	9.459×10^{-13}	0.1030364394	5.028×10^{-13}	0.1502295478	5.540×10^{-13}
0.9	0.0272374994	3.122×10^{-12}	0.0539985263	9.909×10^{-14}	0.0801106273	2.592×10^{-12}
1.0	0.0000000000	6.922×10^{-12}	-5.684×10^{-14}	7.692×10^{-13}	0.0000000000	5.746×10^{-12}

TABLE 8. The numerical solution for α=1, γ =13/10,β =3/10

	α=1, $Ha^2 = 2$		α=1, $Ha^2 = 4$		α=1, $Ha^2 = 10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.4599147383	0.000	0.4982464112	0.000	0.5004117242	0.000
0.1	0.4539706290	1.735×10^{-17}	0.4975651132	8.882×10^{-17}	0.4998914309	1.071×10^{-16}
0.2	0.4418976701	5.065×10^{-16}	0.4959784157	1.776×10^{-16}	0.4999621304	1.323×10^{-15}
0.3	0.4244838301	3.414×10^{-15}	0.4928112172	0.000	0.4999578760	4.151×10^{-15}
0.4	0.4001620370	1.490×10^{-14}	0.4865203657	3.553×10^{-16}	0.4999112897	2.784×10^{-14}
0.5	0.3670021499	4.285×10^{-14}	0.4740145102	2.665×10^{-15}	0.4996555216	1.116×10^{-13}
0.6	0.3228568945	1.628×10^{-13}	0.4494973693	7.994×10^{-15}	0.4983983712	5.340×10^{-13}
0.7	0.2656147972	2.806×10^{-13}	0.4031722563	1.306×10^{-14}	0.4922988256	2.749×10^{-12}
0.8	0.1934655611	4.338×10^{-13}	0.3213522936	3.553×10^{-14}	0.4631868887	1.179×10^{-11}
0.9	0.1051435206	4.259×10^{-13}	0.1898984354	1.487×10^{-13}	0.3443605534	3.171×10^{-11}
1.0	0.0000000000	5.571×10^{-13}	7.105×10^{-15}	3.393×10^{-13}	1.136×10^{-13}	7.022×10^{-11}

TABLE 9. The numerical solution for α=0.5, γ =$\pi/2$,β =$\pi/4$

	α=0.5, $Ha^2 = 0.5$		α=0.5, $Ha^2 = 1$		α=0.5, $Ha^2 = 1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1414800191	0.000	0.2557478916	0.000	0.3464052875	0.000
0.1	0.1389522463	3.469×10^{-18}	0.2515257156	4.857×10^{-17}	0.3411876461	1.388×10^{-17}
0.2	0.1329054647	4.163×10^{-17}	0.2413653530	1.665×10^{-16}	0.3285423361	2.776×10^{-17}
0.3	0.1241823766	3.747×10^{-16}	0.2265642849	1.665×10^{-16}	0.3099073113	5.551×10^{-17}
0.4	0.1130499921	1.860×10^{-15}	0.2074374484	1.665×10^{-16}	0.2854703783	1.277×10^{-15}
0.5	0.0996463615	8.771×10^{-15}	0.1840679077	6.495×10^{-15}	0.2551017127	7.494×10^{-15}
0.6	0.0840447257	1.621×10^{-14}	0.1564169320	1.604×10^{-14}	0.2184932731	2.104×10^{-14}
0.7	0.0662798055	6.195×10^{-14}	0.1243701959	8.649×10^{-14}	0.1752193058	9.753×10^{-14}
0.8	0.0463599357	7.019×10^{-14}	0.0877601902	2.870×10^{-14}	0.1247683054	6.078×10^{-14}
0.9	0.0242754699	2.432×10^{-13}	0.0463824577	9.115×10^{-14}	0.0665685083	3.718×10^{-13}
1.0	-1.776×10^{-15}	4.171×10^{-13}	-3.552×10^{-15}	4.902×10^{-14}	-3.552×10^{-15}	8.481×10^{-13}

Error bound of the NPSM solution was constructed and revealed that the solution has accuracy of $O(1/n!)$. So, we can conclude that the NPSM is a reliable technique for this kind of problems and it can apply without any linearization of discretization.

TABLE 10. The numerical solution for α=0.5, $\gamma=\pi/2$,$\beta=\pi/4$

	α=0.5, $Ha^2=2$		α=0.5, $Ha^2=4$		α=0.5, $Ha^2=10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.4173866612	0.000	0.5732489905	0.000	0.6594503310	0.000
0.1	0.4117152179	4.163×10^{-17}	0.5684805256	5.551×10^{-17}	0.6584591853	1.317×10^{-15}
0.2	0.3978589181	3.886×10^{-16}	0.5563838305	2.914×10^{-16}	0.6555739471	7.980×10^{-16}
0.3	0.3771709785	1.305×10^{-15}	0.5371921696	8.327×10^{-17}	0.6500654567	5.551×10^{-17}
0.4	0.3495917209	2.831×10^{-15}	0.5095839928	1.638×10^{-15}	0.6400890312	9.714×10^{-17}
0.5	0.3146642925	3.331×10^{-16}	0.4714554646	1.216×10^{-14}	0.6222030654	1.887×10^{-15}
0.6	0.2716878169	1.804×10^{-14}	0.4199692602	4.480×10^{-14}	0.5902264473	7.216×10^{-15}
0.7	0.2197866402	7.871×10^{-14}	0.3515630665	1.076×10^{-13}	0.5334642377	3.508×10^{-14}
0.8	0.1579518845	7.372×10^{-14}	0.2619987130	1.311×10^{-13}	0.4344878933	3.075×10^{-14}
0.9	0.0850798971	2.587×10^{-13}	0.1465022313	3.531×10^{-13}	0.2675704738	7.150×10^{-14}
1.0	0.0000000000	1.113×10^{-12}	-3.552×10^{-15}	1.018×10^{-12}	0.0000000000	9.814×10^{-14}

TABLE 11. The numerical solution for α=1, $\gamma=\pi/2$,$\beta=\pi/4$

	α=1, $Ha^2=0.5$		α=1, $Ha^2=1$		α=1, $Ha^2=1.5$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.1406934160	0.000	0.2501802379	0.000	0.3304896396	0.000
0.1	0.1381992846	2.082×10^{-17}	0.2461950390	2.776×10^{-17}	0.3259298693	5.551×10^{-17}
0.2	0.1322283409	1.249×10^{-16}	0.2365698097	1.110×10^{-16}	0.3147782905	1.527×10^{-16}
0.3	0.1236040553	2.220×10^{-16}	0.2224676037	6.384×10^{-16}	0.2981102773	1.665×10^{-16}
0.4	0.1125813022	5.274×10^{-16}	0.2041180599	2.998×10^{-15}	0.2758816786	1.388×10^{-15}
0.5	0.0992880226	8.327×10^{-16}	0.1815322363	5.745×10^{-15}	0.2477598321	1.227×10^{-14}
0.6	0.0837892362	6.245×10^{-15}	0.1546120225	1.973×10^{-14}	0.2132620840	3.436×10^{-14}
0.7	0.0661135992	2.248×10^{-14}	0.1231988270	1.243×10^{-14}	0.1718266007	1.675×10^{-13}
0.8	0.0462659242	1.943×10^{-14}	0.0870994008	1.671×10^{-13}	0.1228586062	2.669×10^{-13}
0.9	0.0242359569	2.118×10^{-14}	0.0461052400	1.779×10^{-13}	0.0657694078	6.221×10^{-13}
1.0	0.0000000000	2.519×10^{-13}	3.552×10^{-15}	4.796×10^{-14}	0.0000000000	1.497×10^{-12}

TABLE 12. The numerical solution for α=1, $\gamma=\pi/2$,$\beta=\pi/4$

	α=1, $Ha^2=2$		α=1, $Ha^2=4$		α=1, $Ha^2=10$	
r	Num.	$\|Res.\|$	Num.	$\|Res.\|$	Num.	$\|Res.\|$
0.0	0.3865231919	0.000	0.4766972597	0.000	0.4992711200	0.000
0.1	0.3820604562	4.857×10^{-17}	0.4744976092	6.592×10^{-17}	0.4994391668	3.269×10^{-16}
0.2	0.3709735326	1.665×10^{-16}	0.4686142087	1.735×10^{-16}	0.4990344377	5.018×10^{-16}
0.3	0.3539837881	5.551×10^{-16}	0.4584559288	4.163×10^{-16}	0.4981107926	1.457×10^{-16}
0.4	0.3306243232	1.832×10^{-15}	0.4422783716	5.551×10^{-17}	0.4959440139	1.197×10^{-14}
0.5	0.3000610711	3.109×10^{-15}	0.4173610230	3.969×10^{-15}	0.4908146386	1.529×10^{-14}
0.6	0.2612443994	1.177×10^{-14}	0.3798923035	3.220×10^{-15}	0.4785535296	5.003×10^{-14}
0.7	0.2130059153	3.525×10^{-14}	0.3249937605	4.347×10^{-14}	0.4495247744	7.652×10^{-13}
0.8	0.1541413218	3.919×10^{-14}	0.2470484452	4.396×10^{-14}	0.3836203255	6.672×10^{-14}
0.9	0.0834906484	8.460×10^{-14}	0.1403684909	3.830×10^{-14}	0.2470198698	2.954×10^{-12}
1.0	0.0000000000	5.740×10^{-14}	0.0000000000	4.263×10^{-14}	8.526×10^{-14}	7.501×10^{-12}

REFERENCES

[1] Khashan, M. M., Amin, R., I Syam, M., (2019), A New Algorithm for Fractional Riccati Type Differential Equations by Using Haar Wavelet. Mathematics, 7 (6), 545.

[2] Baleanu, D., Sajjadi, S. S., Jajarmi, A., Asad, J. H., (2019), New features of the fractional Euler-Lagrange equations for a physical system within non-singular derivative operator. The European Physical Journal Plus, 134 (4), 181.

[3] Saad, K. M., Baleanu, D., Atangana, A., (2018), New fractional derivatives applied to the Korteweg-de Vries and Korteweg-de Vries-Burger's equations. Computational and Applied Mathematics, 37 (4), 5203-5216.

[4] Khalil R., Al-Horani M., Yousef A., Sababheh M., (2014), A New Definition of Fractional Derivative, Journal of Computational and Applied Mathematics, 264, 65-70.

[5] Al-Refai M., (2018), Comparison principles for differential equations involving Caputo fractional derivative with Mittag-Leffler non-singular kernel,Electronic Journal of Differential Equations, 2018, (36), 1-10.

[6] Dariusz W. Brzezinski, (2017), Comparison of Fractional Order Derivatives Computational Accuracy-Right Hand vs Left Hand Definition, Applied Mathematics and Nonlinear Sciences, 2 (1), 237-248.

[7] Diethelm, K., Ford, N. J., and Freed, A. D., (2002), A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics, 29 (1-4), 3-22.

[8] Alomari, A. K., Awawdeh, F., Tahat, N., Ahmad, F. B., Shatanawi, W., (2013), Multiple solutions for fractional differential equations: Analytic approach. Applied Mathematics and Computation, 219 (17), 8893-8903.

[9] Alomari, A. K., (2015), A novel solution for fractional chaotic Chen system. J. Nonlinear Sci. Appl, 8, 478-488.

[10] Hu, Y., Luo, Y., and Lu, Z., (2008), Analytical solution of the linear fractional differential equation by Adomian decomposition method. Journal of Computational and Applied Mathematics, 215 (1), 220-229.

[11] Abdulaziz, O., Hashim, I., and Momani, S., (2008), Application of homotopy-perturbation method to fractional IVPs. Journal of Computational and Applied Mathematics, 216 (2), 574-584.

[12] Albadarneha, R. B., Batihab, I. M., and Zurigatb, M., (2016), Numerical solutions for linear fractional differential equations of order 1¡α¡ 2 using finite difference method (ffdm). Int. J. Math. Comput. Sci, 16 (1), 103-111.

[13] Albadarneh, R. B., Zerqat, M., Batiha, I. M., (2016), Numerical solutions for linear and non-linear fractional differential equations. International Journal of Pure and Applied Mathematics, 106 (3), 859-871.

[14] Molliq, R. Y., Noorani, M. S. M., Hashim, I., and Ahmad, R. R., (2009), Approximate solutions of fractional Zakharov-Kuznetsov equations by VIM. Journal of Computational and Applied Mathematics, 233(2), 103-108.

[15] Arikoglu, A., and Ozkol, I., (2007), Solution of fractional differential equations by using differential transform method. Chaos, Solitons and Fractals, 34 (5), 1473-1481.

[16] Alquran, M. (2014), Analytical solutions of fractional foam drainage equation by residual power series method. Mathematical sciences, 8 (4), 153-160.

[17] McKee, S., Watson, R., Cuminato, J. A., Caldwell, J., and Chen, M. S., (1997), Calculation of electrohydrodynamic flow in a circular cylindrical conduit. ZAMM Journal of Applied Mathematics and Mechanics/Zeitschrift fur Angewandte Mathematik und Mechanik, 77 (6), 457-465.

[18] Alomari, A. K., Erturk, V. S., Momani, S., and Alsaedi, A., (2019), An approximate solution method for the fractional version of a singular BVP occurring in the electrohydrodynamic flow in a circular cylindrical conduit. The European Physical Journal Plus, 134 (4), 158.

[19] Paullet, J. E., (1999), On the solutions of electrohydrodynamic flow in a circular cylindrical conduit. ZAMM Journal of Applied Mathematics and Mechanics/Zeitschrift fur Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics, 79 (5), 357-360.

[20] Mastroberardino, A., (2011), Homotopy analysis method applied to electrohydrodynamic flow. Communications in Nonlinear Science and Numerical Simulation, 16 (7), 2730-2736.

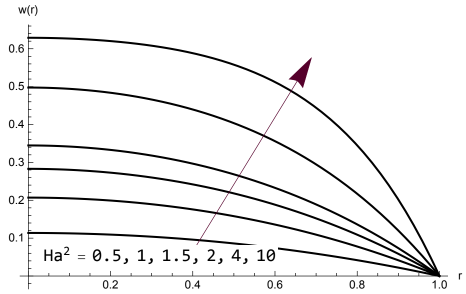
[21] Moghtadaei, M., Nik, H. S., and Abbasbandy, S., (2015), A spectral method for the electrohydrodynamic flow in a circular cylindrical conduit. Chinese Annals of Mathematics, Series B, 36 (2), 307-322.

[22] Beg, O. A., Hameed, M., and Beg, T. A., (2013), Chebyshev spectral collocation simulation of non-linear boundary value problems in electrohydrodynamics. International Journal for Computational Methods in Engineering Science and Mechanics, 14 (2), 104-115.

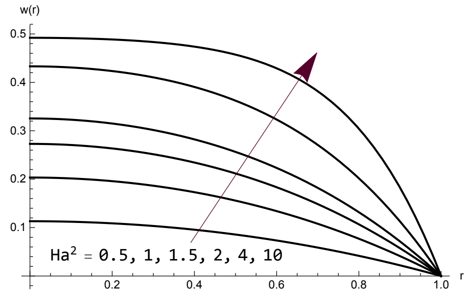
[23] Hosseini, E., Barid Loghmani, G., Heydari, M., and Wazwaz, A. M., (2017), A numerical study of electrohydrodynamic flow analysis in a circular cylindrical conduit using orthonormal Bernstein polynomials. Computational Methods for Differential Equations, 5 (4), 280-300.

[24] Almomanie, A. D., (2019), Numerical Solution for Fractional Differential Equation using Power Series, Dept. of Math., Al al-Bayt University, Almafraq, Jordan.

[25] Albadarneh, R. B., Shawagfeh, N. T., and Hammour, Z. S. A., (2012), General (n+ 1)-explicit finite difference formulas with proof. Applied Mathematical Sciences, 6 (20), 995-1009.

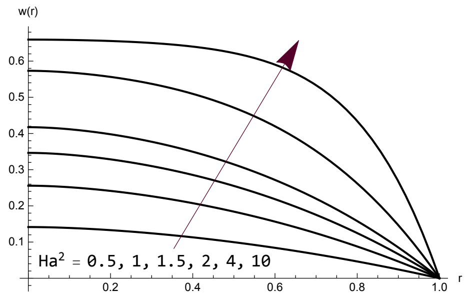


(A) α=0.5

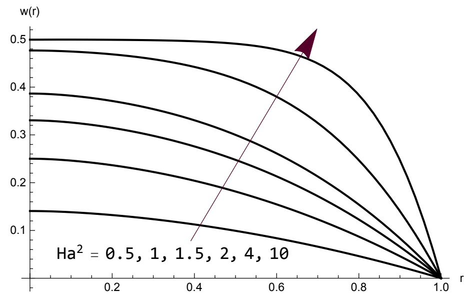


(B) α=1

FIGURE 1. Power series solution for the standard case γ =2, β =1 with vary Ha^2.

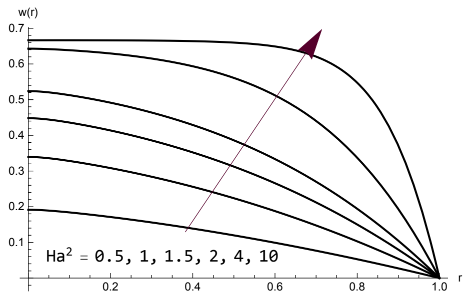


(A) α=0.5

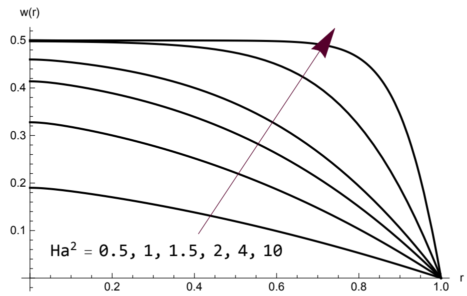


(B) α=1

FIGURE 2. Power series solution for γ =1.3, β =0.3 with vary Ha^2.



(A) α=0.5



(B) α=1

FIGURE 3. Power series solution for γ =$\pi/2$, β =$\pi/4$ with vary Ha^2.

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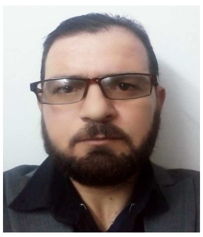

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