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GROUP METHODS FOR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this research paper, we obtain the equivalent symmetries of non-homogeneous second order delay differential equations with variable coefficients. Group methods have been used to do this. The approach followed by us to obtain a Lie type invariance condition for the second order delay differential equation is by using Taylor's theorem for a function of more than one variable. This Lie type invariance condition established by us in this paper, will be used to obtain the determining equations of the second order delay differential equation. We study certain cases under which the delay differential equation admits infinitesimal generators. Further, by performing symmetry analysis of this delay differential equation, the complete group classification for it has been made.

Keywords: Delay differential equation, determining equations, Lie group, Lie invariance condition, splitting equation, symmetries.

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1. Introduction

Differential equations are extremely important in science and engineering. We mostly encounter differential equations while modelling most of our physical phenomenon. Generalizations of differential equations, called functional differential equations, are those wherein the unknown function appears with various different arguments. The simplest of these are called delay differential equations. Functional differential equations find a wide range of applications in traffic flow problems, signal processing, control systems, heat transfer problems, population models, evolution of species, prey-predator models, study of epidemics, rolling of ships, various branches of engineering, etc (see [1]). For more details one can refer to [2,3,4].

Symmetries are the backbone in making and studying laws governing nature. In [5] it is pointed that the regularities of the laws which are accounted by symmetries are independent of some inessential circumstances. As a simple example, the re-doing of any experiments at different instances of time and places depends on invariance laws of nature

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under space translation, rotation and time translation. In 1918, the existence of conservation laws, which is an important implication of symmetry in Mathematics and Physics was observed by Nöether [6] in proving a relation between continuous symmetries and conservation laws. Even scientists like Issac Newton and Joannes Kepler used the laws of mechanics to study the orbits of planets as a symmetry principle.

By defining an operator equivalent to the canonical Lie-Bäcklund operator, in [7,8], authors discuss symmetry analysis of first order delay differential equations. In [9] the author uses the canonical Lie-Bäcklund operator to obtains equivalent symmetries of a second order delay differential equation. The determining equations obtained by [9] contain terms with double delay as well. The approach used in our paper does not result in any terms with double delay. In [10] the modern state of classical group methods are studied. The study involves the application of classical group analysis using the principle of factorization. Symmetry analysis and group methods are used to classify ordinary differential equations which are linear and of the second order in [11]. In [12] first order delay differential equations with constant coefficients are studied. The study involves defining a Lie-Bäcklund operator to obtain the infinitesimal generators. Recent research in [13], points to obtaining symmetries for neutral differential equations of first order. The determining equations are set up using Taylor's theorem for a function of more than one variable.

To introduce second order delay differential equations, we let $f : J \times D^4 \to \mathbb{R}$. Here, D is a set which is open in $\mathbb{R}$, J is an interval in $\mathbb{R}$. J will either be $[t_0, \beta)$ or (α, β), where $\alpha \leq t_0 \leq \beta$ and $r > 0$ is a constant. An expression for such a delay differential equation of second order can be conveniently written as,

$$x''(t) = f(t, x(t), x'(t), x'(t-r), x(t-r)), \tag{1}$$

where x is a real valued function defined on some open interval J in $\mathbb{R}$. The notation $x'(t-r)$ and $x''(t-r)$ mean $\frac{dx}{dt}(t-r)$ and $\frac{d^2x}{dt^2}(t-r)$ respectively. We consider equation (1) for $t_0 \leq t \leq \beta$ along with the initial function

$$x(t) = \phi(t), \text{for} \quad t_0 - r \leq t \leq t_0. \tag{2}$$

where ϕ is a given initial function and t_0 is some point in J for which (1) holds. We specify the delay point $t - r$, in order to completely determine the problem. We would find a one-parameter group such that the delay differential equation (1) is invariant under this group. As we require the group to carry one solution curve of the equation to another of the same equation, we call this as our admitted Lie group.

As compared to the existing literature, we follow a completely different procedure of extending the results of obtaining symmetries of ordinary differential equations found in [14,15]. Our approach establishes a Lie type invariance condition for delay differential equations of second order which is used to obtain symmetries of the linear delay differential equation of second order. Our study uses Taylor's theorem for a function of more than one variable to obtain a Lie type invariance condition for delay differential equations of second order. Using this obtained condition, its prolongation and extension, the determining equations are constructed. The splitting of the determining equations is done in a manner similar to that done for ordinary differential equations. We then solve this resultant over-determined system of partial differential equations to obtain the generators of the corresponding Lie group and hence the desired equivalent symmetries of the differential equation with delay. We shall finally see different cases of the equation under study and obtain the equivalent symmetries for each of these cases. Such group methods for delay differential equations are of immense help to engineers, applied mathematicians and physicists in studying the properties of delay differential equations which are not always solvable.

We need the following definitions.

Definition 1.1. *(Solution of delay differential equation (1) satisfying (2))*
A function $x : [t_0 - r, \beta_1) \to D$, *for some* $\beta_1 \in (t_0, \beta]$, *such that,*

1. x *is differentiable.*
2. $x(t) = \phi(t)$ *for* $t_0 - r \leq t \leq t_0$,
3. $x(t)$ *reduces equation (1) to an identity on* $t_0 \leq t \leq \beta_1$.

We understand $x'(t_0)$ *and* $x''(t_0)$ *will mean the right handed derivative of* x *at* t_0.

We formally define a one-parameter group of transformations as below:

Definition 1.2 (16). *Consider transformations given by,* $\bar{t}_i = g_i(t_j, \delta), i, j = 1, 2, \cdots, n$. *Let the transformations continuously depend on the parameter* δ. *For each* i, *let* g_i *be a smooth function of the variables* t_j. *Further let* g_i *have a Taylor series representation in* δ *which is convergent. These set of transformations are said to form a* Lie Group *or a one-parameter group of transformations, if:*

1. *(*Existence of Identity*) There is a value of the parameter* δ, *say* $\delta = 0$ *which corresponds to the identity transformation,*
   $t_i = g_i(t_j, 0), i, j = 1, 2, \cdots, n.$
2. *(*Existence of Inverse*) There is a value of the parameter* δ, *say* $-\delta$ *which corresponds to the inverse transformation,*
   $t_i = g_i(\bar{t}_j, -\delta), i, j = 1, 2, \cdots, n.$
3. *(*Closure Property*) Two transformations carried out in succession result into another transformation of the set.*
   $\bar{t}_i = g_i(t_j, \delta), i, j = 1, 2, \cdots, n,$ *and* $\hat{t}_i = g_i(\bar{t}_j, \epsilon), i, j = 1, 2, \cdots, n,$ *then* $\hat{t}_i = g_i(t_j, \delta + \epsilon), i, j = 1, 2, \cdots n.$

Remark 1.1. *The associativity law for groups follows from the* closure *property.*

Example 1.1. *The simplest example of a Lie group is the translational group given by* $\bar{t} = t + \delta$.

The order of carrying out the transformations does not matter. If the order of carrying out the transformations is immaterial, then the group is termed as abelian.

For the one-parameter group of transformations, $\bar{t} = f_1(t, x; \delta)$, $\bar{x} = f_2(t, x; \delta)$, where f_1 and f_2 are two functions having continuous partial derivatives in t and x and also having a Taylor series in δ which is convergent. The first order infinitesimals (or coefficients of infinitesimal transformation) ω and Υ are defined by $\omega(t, x) = \dfrac{\partial f_1(t, x; 0)}{\partial \delta}$ and $\Upsilon(t, x) = \dfrac{\partial f_2(t, x; 0)}{\partial \delta}$.

For ordinary differential equations of first order given by, $\dfrac{dx}{dt} = G(t, x)$, on expressing the differential equation as $\dfrac{d\bar{x}}{d\bar{t}} = G(\bar{t}, \bar{x})$, and then using invariance under the Lie group, we get

$$\Upsilon_t + (\Upsilon_x - \omega_t)G - \omega_x G^2 = \omega G_t + \Upsilon G_x. \qquad (3)$$

Equation (3) is called Lie's Invariance condition for ordinary differential equations of first order.

In case of ordinary differential equations of second order $\frac{d^2\bar{x}}{d\bar{t}^2} = K(\bar{t}, \bar{x}, \frac{d\bar{x}}{d\bar{t}})$, using invariance, we get the Lie Invariance condition as

$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)K - 3\omega_x x' K$$
$$= \omega K_t + \Upsilon K_x + \Upsilon_{[t]} K_{x'}, \quad (4)$$

where $\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.

2. Lie type invariance condition for Delay Differential Equations of Second Order

In this section, we shall derive a Lie type invariance condition for delay differential equations of second order . We specify the term with delay, where the delayed function is given. This is done so that we can completely determine our second order differential equation with delay and thus fully determine our problem.

The following Lie type invariance condition in this paper which is established by us and uses Taylor's theorem is a novel approach as far as obtaining symmetries of the second order delay differential equation is concerned.

Theorem 2.1. *Let a function F be defined on $I \times D \times I - r \times D^3$, where D is a set which is open in $\mathbb{R}$, I is an open interval in $\mathbb{R}$ and $I - r = \{l - r : l \in I\}$. Then the Lie type invariance condition for*

$$\frac{d^2x}{dt^2} = F(t, x(t), t - r, x(t - r), x'(t), x'(t - r)). \qquad (5)$$

is given by

$$\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} =$$
$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''$$

where

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

$$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega), \quad \textit{where} \quad D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + x'' \frac{\partial}{\partial x'} + \cdots$$

Proof. Consider the infinitesimal form of the Lie group under which the second order differential equation with delay is invariant

$$\bar{t} = t + \delta\omega(t, x) + O(\delta^2) \quad \text{and} \quad \bar{x} = x + \delta\Upsilon(t, x) + O(\delta^2).$$

We then naturally define,
$\overline{t - r} = t - r + \delta\omega(t - r, x(t - r)) + O(\delta^2)$ and

$\overline{x(t-r)} = x(t-r) + \delta \Upsilon(t-r, x(t-r)) + O(\delta^2)$
Then,

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \frac{\frac{d\bar{x}}{dt}}{\frac{d\bar{t}}{dt}} \\
&= \big[\frac{dx}{dt} + (\Upsilon_t + \Upsilon_x x')\delta + O(\delta^2)\big]\big[1 - (\omega_t + \omega_x x')\delta + O(\delta^2)\big] \\
&= \frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2),
\end{aligned}$$

Using the notation, $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$, it is possible to express,

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \frac{dx}{dt} + (D_t(\Upsilon) - x' D_t(\omega))\delta + O(\delta^2) \\
&= \frac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2),
\end{aligned}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
Considering the second-order extended infinitesimals, we get

$$\begin{aligned}
\frac{d^2\bar{x}}{d\bar{t}^2} &= \frac{d}{d\bar{t}}(\frac{d\bar{x}}{d\bar{t}}) \\
&= \frac{\frac{d}{dt}\big[\frac{dx}{dt} + [D_t(\Upsilon) - D_t(\omega)x']\delta + O(\delta^2)\big]}{1 + \delta D_t(\omega) + O(\delta^2)} \\
&= \big(\frac{d^2x}{dt^2} + D_t(\Upsilon_{[t]})\delta + O(\delta^2)\big)(1 - \delta D_t(\omega) + O(\delta^2)) \\
&= \frac{d^2x}{dt^2} + (D_t(\Upsilon_{[t]}) - D_t(\omega)x'')\delta + O(\delta^2).
\end{aligned}$$

So that $\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega)$.
We have just seen that $\Upsilon_{[t]}$ depends on the variables t, x and x', and as a result the definition of D_t needs to be extended. Therefore let
$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots$ Expanding $\Upsilon_{[tt]}$, results in,

$$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.$$

With the notations,

$$\omega^r = \omega(t-r, x(t-r)) \quad \text{and} \quad \Upsilon^r = \Upsilon(t-r, x(t-r)),$$

we can express,

$$\begin{aligned}
&\frac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\
&\qquad = x'(t-r) + [(\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} \\
&\qquad - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}]\delta + O(\delta^2).
\end{aligned}$$

Let $\Upsilon^r_{[t]} = (\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}$.
For invariance, $\frac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \overline{x(t)}, \overline{t-r}, \overline{(x(t-r))}, \frac{d\bar{x}}{d\bar{t}}, \frac{d\bar{x}}{d\bar{t}}(\overline{t-r}))$.

This gives,

$$\frac{d^2x}{dt^2} + \Upsilon_{[tt]}\delta + O(\delta^2) = F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2),$$
$$t - r + \delta\omega^r + O(\delta^2), x(t-r) + \delta\Upsilon^r + O(\delta^2),$$
$$\frac{dx}{dt} + \delta\Upsilon_{[t]} + O(\delta^2), \frac{dx}{dt}(t-r) + \Upsilon^r_{[t]}\delta + O(\delta^2))$$
$$= F(t, x, t-r, x(t-r), x'(t), x'(t-r)) +$$
$$(\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)})\delta$$
$$+ O(\delta^2).$$

Comparing the coefficient of δ, we get

$$\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} =$$
$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''. \quad (6)$$

Equation (6) is the obtained Lie type invariance condition for second order delay differential equations. □

Definition 2.1. *We define a prolonged operator for the delay differential equation of second order as below:*
$\zeta = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)}.$

Definition 2.2. *The extended operator, for a second order delay differential equations, would then naturally be defined as:*

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)} + \Upsilon_{[tt]}\frac{\partial}{\partial x''}. \quad (7)$$

Defining, $\Delta = x''(t) - F(t, x(t), t-r, x(t-r), x'(t), x'(t-r)) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[tt]} - \omega F_t - \Upsilon F_x - \omega^r F_{t-r} - \Upsilon^r F_{x(t-r)} - \Upsilon_{[t]} F_{x'(t)} - \Upsilon^r_{[t]} F_{x'(t-r)}. \quad (8)$$

From equations (8) and (6), we compare them and obtain,
$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.$
The substitution $x'' = F$ into $\zeta^{(1)}\Delta = 0$, gives us our desired invariance condition for the second order delay differential equation which is $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$. We will be using this to get our determining equations.

3. Equivalent Symmetries of A Delay Differential Equation Of Second Order Which Is Non-Homogeneous

We now do a group classification of the linear second order delay differential equation

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = h(t) \quad (9)$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\rho(t)$ are the twice differentiable variable coefficients.
We prove the following well known elementary results which will be used by us:

Proposition 3.1. *Let $x_1(t)$ be any solution of the equation (9). Then by changing the variables, $\bar{t} = t$ and $\bar{x} = x - x_1(t)$, the delay differential equation given by equation (9), can be transformed to a delay differential equation which is homogeneous, namely,*

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = 0. \quad (10)$$

Proof. On substituting $t = \bar{t}$ and $x(t) = \bar{x} + x_1(\bar{t})$ in (9), and by noting that
$x_1''(t) + \alpha(t)x_1'(t) + \beta(t)x_1'(t-r) + \gamma(t)x_1(t) + \rho(t)x_1(t-r) = h(t)$,
we see that this proposition follows easily. □

Proposition 3.2. *The delay differential equation*

$$x''(t) + \alpha_1(t)x'(t) + \beta_1(t)x'(t-r) + \gamma_1(t)x(t) + \rho_1(t)x(t-r) = 0, \tag{11}$$

with $\alpha_1(t), \beta_1(t), \gamma_1(t)$ *and* $\rho_1(t)$ *twice differentiable functions with variable coefficients can be transformed to a delay differential equation in which the term corresponding to the ordinary derivative of the first order is absent.*

Proof. By making the change, $x = v(t)w(t)$, where $v(t) \neq 0$ is a twice differentiable function in t and with $w(t)$ satisfying $w(t) = exp(-\int^t \frac{\alpha_1(\mu)d\mu}{2}) + w_0$, where w_0 is an arbitrary constant, equation (11), can be reduced to
$v''(t) + \beta_2(t)v'(t-r) + \gamma_2(t)v(t) + \rho_2(t)v(t-r) = 0$, where $\beta_2(t) = \beta_1(t)\frac{w(t-r)}{w(t)}$,
$\gamma_2(t) = \frac{w''(t) + \alpha_1(t)w'(t) + \gamma_1(t)w(t)}{w(t)}$, and $\rho_2(t) = \frac{\beta_1(t)w'(t-r) + \rho_1(t)w(t-r)}{w(t)}$. □

Remark 3.1. *This proposition is widely used in the removal of the coefficient of the first order derivative, for ordinary differential equations of second order. It is important to note that the group classification of (9) will not be altered on account of this change.*

By virtue of the propositions above, we shall obtain the symmetries of the equation below, namely,

$$x''(t) + \rho(t)x(t-r) + \beta(t)x'(t-r) + \gamma(t)x(t) = 0. \tag{12}$$

Our term with delay is specified as,

$$t^r = g(t) = t - r. \tag{13}$$

Applying operator $\zeta^{(1)}$ which was established in equation (7) to the equation (13), we obtain,

$$\omega^r = \omega. \tag{14}$$

Applying operator $\zeta^{(1)}$ defined by (7) to (12), we get,

$$\begin{aligned} &\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)(-\beta(t)x'(t-r) - \gamma(t)x \\ &\quad - \rho x(t-r)) - 3\omega_x x'(-\beta(t)x'(t-r) - \gamma(t)x - \rho x(t-r)) \\ &\quad = -\big[\omega(\beta'(t)x'(t-r) + \gamma'(t)x + \rho'(t)x(t-r)) \\ &\quad + \gamma(t)\Upsilon + \rho(t)\Upsilon^r + \beta(t)(\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) - \omega^r_{x(t-r)}(x'(t-r)^2))\big]. \end{aligned} \tag{15}$$

Differentiate (15) with respect to $x'(t-r)$ twice, we get,
$\omega^r_{x(t-r)} = 0$, which can be integrated to yield,

$$\omega(t,x) = A(t) \tag{16}$$

Differentiate equation (15) with respect to $x(t-r)$ twice, we get,
$\rho(t)\Upsilon^r_{x(t-r)x(t-r)} + \beta(t)\Upsilon^r_{(t-r)(x(t-r))(x(t-r))}$
$+ \beta(t)(\Upsilon^r_{x(t-r)x(t-r)x(t-r)} - \omega^r_{(t-r)(x(t-r))(x(t-r))})x'(t-r) = 0.$
Splitting the equation with respect to $x'(t-r)$, and noting that $\beta(t) \neq 0$ we obtain, $\Upsilon_{xxx} = \omega_{txx} = 0$, which can be integrated to yield,

$$\Upsilon(t,x) = \frac{1}{2}B(t)x^2 + C(t)x + H(t). \tag{17}$$

On substitution of the equations (16) and (17) in equation (15), we get,

$$\begin{aligned}
&\frac{1}{2}B''(t)x^2 + C''(t)x + H''(t) + (2(B'(t)x + C'(t)) - A''(t))x' + B(t)(x'(t-r))^2 \\
&\quad + (B(t)x + C(t) - 2A'(t))(-\beta(t)x'(t-r) - \gamma(t)x - \rho(t)x(t-r)) \\
&= -\big[A(t)(\beta'(t)x'(t-r) + \gamma'(t)x + \rho'(t)x(t-r)) + \gamma(t)(\frac{1}{2}B(t)x^2 + C(t)x + H(t)) \\
&\quad + \rho(t)(\frac{1}{2}B(t-r)x^2(t-r) + C(t-r)x(t-r) + H(t-r)) \\
&\quad + \beta(t)[(\frac{1}{2}B'(t-r)x^2(t-r) + C'(t-r)x(t-r) + H'(t-r)) \\
&\quad + (B(t-r)x(t-r) + C(t-r) - A'(t-r))x'(t-r)]\big]. \qquad (18)
\end{aligned}$$

Splitting equation (18) with respect to x^2, we get,

$$B''(t) = B(t)\gamma(t). \qquad (19)$$

Splitting equation (18) with respect to x, we get,

$$C''(t) = -\gamma'(t)A(t) - 2\gamma(t)A'(t) - 2B'(t). \qquad (20)$$

Splitting equation (18) with respect to $x'(t-r)$, we get,

$$A''(t) = 2C'(t). \qquad (21)$$

Splitting equation (18) with respect to $(x'(t-r))^2$, we get,

$$B(t) = 0. \qquad (22)$$

As a consequence of $B(t) = 0$, equation (17), reduces to

$$\Upsilon(t,x) = C(t)x + H(t). \qquad (23)$$

and equation (20) reduces to, $C''(t) = -\gamma'(t)A(t) - 2\gamma(t)A'(t)$.

Using $B(t) = 0$, equation (18), simplifies to

$$\begin{aligned}
&C''(t)x + H''(t) + (2C'(t) - A''(t))x' + (C(t) - 2A'(t))(-\beta(t)x'(t-r) - \gamma(t)x - \rho(t)x(t-r)) \\
&= -\big[A(t)(\beta'(t)x'(t-r) + \gamma'(t)x + \rho'(t)x(t-r)) + \gamma(t)(C(t)x + H(t)) + \rho(t)(C(t-r)x(t-r) \\
&\quad + H(t-r)) + \beta(t)[(C'(t-r)x(t-r) + H'(t-r)) + (C(t-r) - A'(t-r))x'(t-r)]\big]. \qquad (24)
\end{aligned}$$

We split (24) with respect to $x(t-r)$, and get,

$$\rho(t)[C(t) - C(t-r)] = \rho'(t)A(t) + \beta(t)C'(t-r) + 2A'(t)\rho(t). \qquad (25)$$

Splitting equation (24) with respect to $x'(t-r)$, we get,

$$\beta(t)[C(t) - C(t-r)] = A(t)\beta'(t) + \beta(t)(2A'(t) - A'(t-r)). \qquad (26)$$

Since $\omega = \omega^r$, equation (26) becomes

$$\beta(t)[C(t) - C(t-r)] = A(t)\beta'(t) + \beta(t)A'(t). \qquad (27)$$

Splitting equation (24) with respect to constant term, we obtain,

$$H''(t) = -\beta(t)H'(t-r) - \gamma(t)H(t) - \rho(t)H(t-r). \qquad (28)$$

We see that, $H(t)$ is a solution to the second order homogeneous delay differential equation (12)

So far we have obtained from equations (14), (20), (21), (23), (25), (26) and (28)

$$\omega = \omega^r, \quad \Upsilon = C(t)x + H(t). \tag{29}$$

$$\omega_{tt} = 2C'(t), \quad C''(t) = -\gamma'(t)\omega - 2\gamma(t)\omega_t. \tag{30}$$

$$H''(t) = -\beta(t)H'(t-r) - \gamma(t)H(t) - \rho(t)H(t-r). \tag{31}$$

$$\beta(t)[C(t) - C(t-r)] = \omega\beta'(t) + \beta(t)\omega'(t). \tag{32}$$

$$\rho(t)[C(t) - C(t-r)] = \rho'(t)\omega(t) + \beta(t)C'(t-r) + 2\omega'(t)\rho(t). \tag{33}$$

Integrating equation (30), we get, $C(t) = \frac{\omega_t}{2} + c_1$, where c_1 is a constant.

Since $\omega = \omega^r$, we have, $C(t) = C(t-r)$. Hence, equation (32) gives,

$$\beta(t)\omega(t) = c_2, \tag{34}$$

here c_2 represents a constant which is arbitrary.

On writing equation (33) as

$$\begin{aligned} \rho'(t)\omega + 2\omega_t\rho(t) &= -\beta(t)C'(t-r) \\ &= -\frac{\beta(t)\omega_{tt}^r}{2} \\ &= -\frac{\beta(t)}{2}\omega_{tt}. \end{aligned} \tag{35}$$

We now study group classification of equation (9), by taking possible cases:

Theorem 3.1. *The delay differential equation (12), under the assumption that $\beta \neq 0$, $\rho \neq 0$ admits Lie group of dimension 3. Further, the generators of this group are given by* $\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = \frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{\beta(t)}\right)'\frac{\partial}{\partial x}, \quad \zeta_3^* = H(t)\frac{\partial}{\partial x}.$

Proof. Substituting $C(t) = \frac{\omega_t}{2} + c_1$, in equation (30), we get,

$\omega_{ttt} = -(2\gamma'(t)\omega + 4\gamma(t)\omega_t$ or $\omega\omega_{ttt} = -(2\gamma'(t)\omega^2 + 4\gamma(t)\omega\omega_t$.

Integrating this, we get,

$$\omega\omega_{tt} - \frac{\omega_t^2}{2} + 2\gamma(t)\omega^2 = c_3, \tag{36}$$

where c_3 is a constant.

If $c_2 \neq 0$, then from equation (34),

$$\omega = \frac{c_2}{\beta(t)}. \tag{37}$$

From equation (29),

$$\Upsilon(t,x) = x\left(\frac{c_2}{2}\left(\frac{1}{\beta(t)}\right)' + c_1\right) + H(t). \tag{38}$$

Using equation (35), we obtain,

$2\left(\rho'(t) - 2\frac{\beta'(t)}{\beta(t)}\rho(t)\right) = \left(\beta''(t) - \frac{2(\beta'(t))^2}{\beta(t)}\right).$

The solution of this linear first order ordinary differential equation is

$\rho(t) = c_4\beta^2(t) + \frac{\beta'(t)}{2},$

where c_4 is an arbitrary constant.

From equation (36),

$\gamma(t) = \frac{1}{2}\left[c_5\beta^2(t) - \frac{3}{2}\left(\frac{\beta'(t)}{\beta(t)}\right)^2 + \frac{\beta''(t)}{\beta(t)}\right]$, where $c_5 = \frac{c_3}{c_2^2}$.

Since, $\omega = \omega^r$, $\beta(t) = \beta(t-r)$,

The infinitesimals (or as they are also known as the coefficients of the infinitesimal transformation) are obtained to be

$$\omega = \frac{c_2}{\beta(t)}, \quad \Upsilon = x\left(\frac{1}{2}\left(\frac{c_2}{\beta(t)}\right)' + c_1\right) + H(t). \tag{39}$$

The most general form of the infinitesimal generator of the Lie group is given by

$$\zeta^* = c_1 x\frac{\partial}{\partial x} + c_2\left(\frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{\beta(t)}\right)'\frac{\partial}{\partial x}\right) + H(t)\frac{\partial}{\partial x}, \tag{40}$$

where $H(t)$ solves equation (12).

For the case $c_2 = 0$, we get,

$$\omega = 0, \quad \Upsilon = c_1 x + H(t). \tag{41}$$

The most general form of the infinitesimal generator of the Lie group is given by

$$\zeta^* = (c_1 x + H(t))\frac{\partial}{\partial x}. \tag{42}$$

□

Theorem 3.2. *The delay differential equation (12), under the hypothesis that $\beta \neq 0$ and $\rho = 0$ admits to having a Lie group of dimension 3. The generators of this group are given by* $\zeta_1^* = \frac{\partial}{\partial t}$, $\quad \zeta_2^* = x\frac{\partial}{\partial x}$, $\quad \zeta_3^* = H(t)\frac{\partial}{\partial x}$.

Proof. Using equation (33), we see that, $C(t-r) = c_6$, where c_6 is the constant of integration. Using equation (34), we obtain, $\omega = \frac{c_2}{\beta(t)}$.

From equation (35), $\omega = c_7 t + c_8$, both c_7 and c_8 being arbitrary constants. From equation (36), $\gamma\omega^2 = c_9$, where $c_9 = \frac{1}{2}\left[c_3 + \frac{c_7^2}{2}\right]$, is an arbitrary constant.

Further, since $\omega = \omega^r$, we get $c_7 = 0$ and $\omega = c_8$.

If $c_8 \neq 0$, then $\gamma(t) = \frac{c_9}{c_8^2}$, $\beta(t) = \frac{c_2}{c_8}$.

The most general form of the infinitesimal generator of the Lie group is given by

$$\zeta^* = c_8\frac{\partial}{\partial t} + (c_6 x + H(t))\frac{\partial}{\partial x}. \tag{43}$$

If $c_8 = 0$, then $\omega = 0$ and $\Upsilon = c_6 x + H(t)$.

The most general form of the infinitesimal generator of the Lie group is given by

$$\zeta^* = (c_6 x + H(t))\frac{\partial}{\partial x}. \tag{44}$$

□

Theorem 3.3. *The delay differential equation (12), under the supposition that $\beta = 0$, $\rho \neq 0$ admits a Lie group of dimension 4. The generators of this group are given by* $\zeta_1^* = \frac{1}{\sqrt{\rho(t)}}\frac{\partial}{\partial t}$, $\quad \zeta_2^* = \left[\left(-\frac{\rho'(t)}{\rho^{3/2}(t)}\right)x\right]\frac{\partial}{\partial x}$, $\quad \zeta_3^* = x\frac{\partial}{\partial x}$, $\quad \zeta_4^* = H(t)\frac{\partial}{\partial x}$.

Proof. From equation (35), we get, $\omega = \sqrt{\frac{c_{10}}{\rho(t)}}$. Hence,

$$\begin{aligned} \Upsilon &= C(t)x + H(t) \\ &= \left(\frac{\omega_t}{2} + c_1\right)x + H(t) \\ &= \left(-\frac{\sqrt{c_{10}}}{4}\frac{\rho'(t)}{\rho^{3/2}(t)} + c_1\right)x + H(t). \end{aligned}$$

If $c_{10} \neq 0$, then from equation (36), $\gamma(t) = \frac{1}{2}\left[\frac{c_3}{c_{10}}\rho(t) + \frac{\rho''(t)}{2\rho(t)} - \frac{5}{8}\left(\frac{\rho'(t)}{\rho(t)}\right)^2\right]$.
The infinitesimal generator in this case is given by,

$$\zeta^* = \sqrt{\frac{c_{10}}{\rho(t)}}\frac{\partial}{\partial t} + \left[\left(-\frac{\rho'(t)\sqrt{c_{10}}}{4\rho^{3/2}(t)} + c_1\right)x + H(t)\right]\frac{\partial}{\partial x}. \tag{45}$$

If $c_{10} = 0$, then $\omega = 0, \Upsilon = c_1x + H(t)$.
Hence, the most general form of the infinitesimal generator of the Lie group is given by,

$$\zeta^* = (c_1x + H(t))\frac{\partial}{\partial x}. \tag{46}$$

□

4. An Illustrative Example

We shall apply symmetry analysis to classify a differential equation with delay arising in control systems studied in [17]. Consider the equation

$$x''(t) + \frac{b}{m}x'(t) + \frac{q}{m}x'(t-r) + \frac{k}{m}x(t) = 0 \tag{47}$$

where b denotes the damping coefficient, the angle of tilt x is measured from the normal upright position, m is the mass. Due to the control which induces a time lag, the resultant force is represented by $qx'(t-r)$. In the equation k is a positive constant. Following the approach given in the previous section, and keeping to the same notations, we see that, $\beta(t) = \frac{q}{m}$, a constant and $\rho(t) = 0$. Performing symmetry analysis of equation (47), we get, $\omega = c_{11}$, a constant and $\Upsilon = c_{12}x + E(t)$, where c_{12} is an arbitrary constant and $E(t)$ solves equation (47). Hence, the Lie group is generated by, $\zeta_1^* = \frac{\partial}{\partial t}$, $\zeta_2^* = x\frac{\partial}{\partial x}$ and $\zeta_3^* = E(t)\frac{\partial}{\partial x}$.
Furthermore, solving the system,
$\frac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}) = c_{11}$ and $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}) = c_{12}\bar{x} + E(\bar{t})$, together with the conditions that, $\bar{t} = t$ and $\bar{x} = x$, when $\delta = 0$, we obtain the delay differential equation given by (47) is invariant under the Lie group
$\bar{t} = t + c_{11}\delta$, $\bar{x} = \frac{1}{c_{12}}\left[e^{c_{12}\delta}(c_{12}x + E(t)) - E(t + c_{11}\delta)\right]$.
It is noteworthy to mention here that this model was actually used for anti rolling stabilization of ships before 1945 which is seen in [18].

5. Conclusions

In this paper, we have made a group classification of equation (12). Based on the different assumptions considered, we have summarized our results below:

(1) Under the conditions that $\beta \neq 0, \rho \neq 0$, the infinitesimal generator of the Lie group for the differential equation with delay given by (12) is given by (40).
(2) Under the conditions $\beta \neq 0, \rho = 0$, the infinitesimal generator of the Lie group for the differential equation with delay given by (12) is given by (43).
(3) Under the conditions $\beta = 0, \rho \neq 0$, the infinitesimal generator of the Lie group for the differential equation with delay given by (12) is given by (45).

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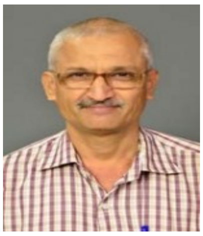

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