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DOUBLY CONNECTED PITCHFORK DOMINATION AND IT'S INVERSE IN GRAPHS

M. A. ABDLHUSEIN[1], M. N. AL-HARERE[2], §

Abstract. Let G be a finite, simple, undirected graph and without isolated vertices. A subset D of V is a pitchfork dominating set if every vertex $v \in D$ dominates at least j and at most k vertices of $V - D$, for any j and k integers. A subset D^{-1} of $V - D$ is an inverse pitchfork dominating set if D^{-1} is a dominating set. The domination number of G, denoted by $\gamma_{pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G. The inverse domination number of G, denoted by $\gamma^{-1}_{pf}(G)$ is a minimum cardinality over all inverse pitchfork dominating sets in G. In this paper, a special modified pitchfork dominations called doubly connected pitchfork domination and it's inverse are introduced when $j = 1$ and $k = 2$. Some properties and bounds are studied with respect to the order and the size of the graph. These modified dominations are applied and evaluated for several well known and complement graphs.

Keywords: Dominating set, pitchfork domination, inverse pitchfork domination, doubly connected.

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1. Introduction

Let $G = (V, E)$ be a graph without isolated vertices has a vertex set V of order n and an edge set E of size m. For any vertex $v \in V$, the degree of v is defined as the number of edges incident on v and denoted by $deg(v)$. The minimum and maximum degrees of vertices are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The subgraph of G induced by the vertices in D is denoted by $G[D]$. The complement $\overline{G}$ of a simple graph G with vertex set $V(G)$ is the graph in which two vertices are adjacent if and only if they are not adjacent in G. For graph theoretic terminology we refer to [9] and [16]. The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a detailed survey of domination one can see [10] and [11]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D, that is $N[D] = V$. A dominating set D is said to be

[1] University of Baghdad, College of Education for Pure Sciences, (Ibn Al-Haitham), Department of Mathematics, Iraq.
University of Thi-Qar, College of Education for Pure Sciences, Department of Mathematics, Iraq.
e-mail: mmhd@utq.edu.iq; ORCID: https://orcid.org/0000-0003-0620-4576.
[2] University of Technology - Department of Applied Sciences - Iraq.
e-mail: 100035@uotechnology.edu.iq; ORCID: https://orcid.org/0000-0003-1125-4439.
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a minimal dominating set if no proper subset of D is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set D of G. Ore [15] introduced the expression dominating set and domination number. The importance of domination in various applications, led to the appearance of different types domination according to the purpose used for see for example [4, 5, 6, 8, 12, 13, 14]. Cyman et. al. [7] introduced the concept of doubly connected domination and gave some of it's bounds and properties. A new model of domination in graphs called the pitchfork domination and it's inverse are introduced by Al-harere and Abdlhusein [1, 2, 3]. For a finite, simple, undirected graph and without isolated vertices, a subset D of V is a pitchfork dominating set if every vertex $v \in D$ dominates at least j and at most k vertices of $V - D$, for any integers j and k. A subset D^{-1} of $V - D$ is an inverse pitchfork dominating set if D^{-1} is a dominating set. The domination number of G, denoted by $\gamma_{pf}(G)$ is the minimum cardinality over all pitchfork dominating sets in G. The inverse domination number of G, denoted by $\gamma^{-1}_{pf}(G)$ is the minimum cardinality over all inverse pitchfork dominating sets in G. Here, a new parameter of domination in graphs called doubly connected pitchfork domination and it's inverse are introduced and applied on some graphs with several properties and bounds. We refer to some results from [1] and [3] which will be depend here.

Proposition 1.1. [3]*: Let $G = (V, E)$ be a graph having a maximum degree $\Delta(G) \leq 2$, then $\gamma(G) = \gamma_{pf}(G)$.*

Theorem 1.1. [1] *The cycle graph C_n; ($n \geq 3$) has an inverse pitchfork domination such that: $\gamma^{-1}_{pf}(C_n) = \gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 1.2. [1] *The path graph P_n; ($n \geq 2$) has an inverse pitchfork domination such that:*

$$\gamma^{-1}_{pf}(P_n) = \begin{cases} \frac{n}{3} + 1 & if\ n \equiv 0\,(mod\,3) \\ \lceil \frac{n}{3} \rceil & if\ n \equiv 1, 2\,(mod\,3) \end{cases}$$

where $\gamma^{-1}_{pf}(P_2) = 1$.

Proposition 1.2. [3] *Let $G = K_n$ the complete graph with $n \geq 3$, then $\gamma_{pf}(K_n) = n - 2$.*

Proposition 1.3. [1] *The complete graph K_n has an inverse pitchfork domination if and only if $n = 3, 4$ and $\gamma^{-1}_{pf}(K_n) = n - 2$.*

Theorem 1.3. [3] *Let G be the complete bipartite graph, then:*

$$\gamma_{pf}(K_{n,m}) = \begin{cases} m, & if\, n = 2 \wedge m < 3 \quad or \quad n = 1 \wedge m > 2 \\ m - 1, & if\, n = 2, m \geq 3 \\ n + m - 4, & if\, n, m > 2. \end{cases}$$

Theorem 1.4. [1] *The complete bipartite graph $K_{n,m}$ has an inverse pitchfork domination if and only if $K_{n,m} \equiv K_{1,2}, K_{2,2}, K_{2,3}, K_{2,4}, K_{3,3}, K_{3,4}$ or $K_{4,4}$ such that:*

$$\gamma^{-1}_{pf}(K_{n,m}) = \begin{cases} 2 & for\ K_{1,2} \\ n + m - 4 & if\, n,\, m\, =\, 2,\, 3,\, 4 \end{cases}$$

Note 1.1. [1] *For any graph G of order n and pitchfork domination number γ_{pf}, if $\gamma_{pf}(G) > \frac{n}{2}$ then G has no inverse pitchfork domination.*

2. New Parameters of Pitchfork Domination

In this section, the doubly connected pitchfork domination and the inverse doubly connected pitchfork domination are defined. The study of their bounds and properties are given with an application of these modified dominations on some standard and complement graphs.

Definition 2.1. *Let $G = (V, E)$ be a finite simple and undirected graph without isolated vertices. A subset $D \subseteq V(G)$ is a doubly connected pitchfork dominating set if D is pitchfork dominating set of G and both $G[D]$ and $G[V - D]$ are connected subgraphs of G.*

Definition 2.2. *A subset $D \subseteq V(G)$ is a minimal doubly connected pitchfork dominating set if there is no doubly connected pitchfork dominating subset from it.*

Definition 2.3. *The minimum dominating set D denoted by $\gamma_{pf}^{cc}-$set is the smallest minimal doubly connected pitchfork dominating set of G. The doubly connected pitchfork domination number denoted $\gamma_{pf}^{cc}(G)$ is the minimum cardinality over all doubly connected pitchfork dominating sets in G.*

Definition 2.4. *Let $G = (V, E)$ be a graph with $\gamma_{pf}^{cc}-$set D, a subset $D^{-1} \subseteq V - D$ is an inverse doubly connected pitchfork dominating set with respect to D, if D^{-1} is a pitchfork dominating set of G and both $G[D^{-1}]$ and $G[V - D^{-1}]$ are connected subgraphs of G.*

Definition 2.5. *A set D^{-1} is a minimal inverse doubly connected pitchfork dominating set if there is no inverse doubly connected pitchfork dominating subset from it.*

Definition 2.6. *The minimum inverse doubly connected pitchfork dominating set D^{-1} denoted by $\gamma_{pf}^{-cc}-$set is the smallest minimal inverse doubly connected pitchfork dominating set of G. The inverse doubly connected pitchfork domination number denoted by $\gamma_{pf}^{-cc}(G)$ is the minimum cardinality over all inverse doubly connected pitchfork dominating sets in G.*

Observation 2.1. *Let G be a graph having a doubly connected pitchfork domination number $\gamma_{pf}^{cc}(G)$. Then we have:*

(1) *G be a connected graph.*
(2) *$|V(G)| \geq 2$.*
(3) *$\delta(G) \geq 1$ and $\Delta(G) \geq 1$.*

Lemma 2.1. [15] *If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.*

Theorem 2.1. *Every graph G of order $n \geq 3$ with end-vertex, has no doubly connected pitchfork domination. The converse need not to be true.*

Proof. Suppose that v be an end-vertex of a graph G, then v is adjacent to only one support vertex say u. If G has a $\gamma_{pf}^{cc}-$set D then either $v \in D$ or $v \notin D$. If $v \in D$ then $u \notin D$ and $G[D]$ disconnected. If $v \notin D$ then $u \in D$ and $G[V - D]$ disconnected. Hence D isn't $\gamma_{pf}^{cc}-$set and G has no doubly connected pitchfork domination. The converse need not to be true see for example $C_n\,; n \geq 5$. □

Corollary 2.1. *Every graph G of order $n \geq 3$ with a doubly connected pitchfork domination, then G contains a cycle.*

Proof. Since $n \geq 3$, then $\delta(G) \geq 2$ and G has no end vertex according to Theorem (2.1). Therefor each vertex of G has a degree at least 2. Hence G has a cycle by Lemma (2.1). □

According to above Theorem (2.1), we give the following propositions:

Proposition 2.1. *For any graph H, the corona $H \odot K_1$ and $H \odot \overline{K}_n\,(n \geq 2)$ have no doubly connected pitchfork dominating sets.*

Proposition 2.2. *A tree T of order $n \geq 3$ has no doubly connected pitchfork dominating set.*

Theorem 2.2. *Every graph with a cut-vertex, has no doubly connected pitchfork dominating set. The converse need not to be true.*

Proof. Suppose that G be a graph with a cut-vertex u, then $G - u$ disconnected graph with at least two components W_1 and W_2. Suppose that G has a $\gamma_{pf}^{cc}-$set D then either $u \in D$ or $u \notin D$. If $u \notin D$ then $D \subseteq G - u$ and D contains some vertices from W_1 and another vertices from W_2, therefor $G[D]$ is disconnected which is contradiction. If $u \in D$ then $u \notin V - D$ and $V - D \subseteq G - u$ and $G[V - D]$ disconnected since it has vertices from W_1 and W_2 which is contradiction. Hence D isn't $\gamma_{pf}^{cc}-$set and G has no doubly connected pitchfork domination. The converse need not to be true, see for example $C_n\,; n \geq 5$. □

Proposition 2.3. *Every graph with a doubly connected pitchfork domination is a nonseparable (block) graph. The converse need not to be true.*

Proof. Let G be a graph with a doubly connected pitchfork domination. Then G is a nontrivial, connected, and has no cut vertices. Hence G is a nonseparable (block) graph. The converse isn't applied on a cycle graph C_n for $n \geq 5$. □

Theorem 2.3. *Let $G = (V,\, E)$ be a graph of order $n \geq 3$ with a doubly connected pitchfork domination, then G has no bridge.*

Proof. Let D be a $\gamma_{pf}^{cc}-$set of G, then for a contradiction suppose that G has a bridge $e \in E$, then $G - e$ is a disconnected graph contains at least two components W_1 and W_2. Where $e = u_1u_2$ such that $u_1 \in W_1$ and $u_2 \in W_2$. There are four cases as follows:
Case 1: If $u_1,\, u_2 \in D$, then e is contained in every $v_1 - v_2$ path of a vertices from $V - D$ such that $v_1 \in W_1$ and $v_2 \in W_2$. Hence $G[V - D]$ disconnected.
Case 2: If $u_1,\, u_2 \in V - D$, then e is contained in every $t_1 - t_2$ path of a vertices from D such that $t_1 \in W_1$ and $t_2 \in W_2$. Hence $G[D]$ disconnected.
Case 3: If $u_1 \in D$ and $u_2 \in V - D$, then for any vertex v_2 from W_2 ($v_2 \in D$) every $u_1 - v_2$ path contains e. Hence $G[D]$ disconnected.
Case 4: If $u_1 \in V - D$ and $u_2 \in D$, then for any vertex v_2 from W_2 ($v_2 \in V - D$) every $u_1 - v_2$ path contains e. Hence $G[V - D]$ disconnected.
All above cases give a contradiction with a doubly connected pitchfork domination since D is a $\gamma_{pf}^{cc}-$set of G. Thus G has no bridge. □

It is clear that $\gamma_{pf}^{cc}(G) = 1$ for P_2 and K_3. Now we discuss $\gamma_{pf}^{cc}(G)$ for any graph with degree $n \geq 4$.

Theorem 2.4. *Let $G(n, m)\,(n \geq 4)$, be a graph with a doubly connected pitchfork domination number $\gamma_{pf}^{cc}(G)$, then $2 \leq \gamma_{pf}^{cc}(G) \leq n - 1$.*

Proof. Suppose that $n \geq 4$ then D must be contains at least two vertices since if D has only one vertex then this vertex dominates three vertices which is contradict pitchfork domination definition. □

In the next Theorem, the relation between the size of graph and the doubly connected pitchfork domination number is given.

Theorem 2.5. *Let $G = (n, m)$ be any graph has a doubly connected pitchfork domination, then:*

$$\gamma_{pf}^{cc}(G) + n - 2 \leq m \leq \binom{n}{2} + (\gamma_{pf}^{cc}(G))^2 + (2-n)\,\gamma_{pf}^{cc}(G).$$

Proof. Let D be $\gamma_{pf}^{cc}-$ set of G , then:
Case 1: To prove the lower bound, since $G[D]$ and $G[V-D]$ be a connected graphs. Then let $m_1 = m(G[D]) = |D|-1 = \gamma_{pf}^{cc}-1$ and $m_2 = m(G[V-D]) = |V-D|-1 = n-\gamma_{pf}^{cc}-1$ to be G has as few edges as possible. Now by the definition of doubly connected pitchfork domination, there is at least one edge from every vertex in D to $V-D$. Then $m_3 = |D| = \gamma_{pf}^{cc}$. Therefor in general $m = m_1 + m_2 + m_3 \geq \gamma_{pf}^{cc} - 1 + n - \gamma_{pf}^{cc} - 1 + \gamma_{pf}^{cc}$ which is the lower bound.
Case 2: To prove the upper bound, suppose that $G[D]$ and $G[V-D]$ are two complete subgraphs to be G have maximum number of edges where the number of edges of D and $V-D$ equal to m_1 and m_2 respectively, then:

$$\begin{aligned} m_1 &= \frac{|D||D-1|}{2} = \frac{\gamma_{pf}^{cc}(\gamma_{pf}^{cc}-1)}{2} \\ m_2 &= \frac{|V-D||V-D-1|}{2} = \frac{(n-\gamma_{pf}^{cc})(n-\gamma_{pf}^{cc}-1)}{2} \end{aligned}$$

Now by the definition of doubly connected pitchfork domination, there exist at most two edges from every vertex of D to $V-D$, then the number of edges from D to $V-D$ equal to $2|D| = 2\gamma_{pf}^{cc}(G) = m_3$. Hence the number of edges of G equal to

$$\begin{aligned} m &= m_1 + m_2 + m_3 \\ &= \frac{1}{2}\left[(\gamma_{pf}^{cc})^2 - \gamma_{pf}^{cc}\right] + \frac{1}{2}\left[n^2 - n\gamma_{pf}^{cc} - n - n\gamma_{pf}^{cc} + (\gamma_{pf}^{cc})^2 + \gamma_{pf}^{cc}\right] + 2\gamma_{pf}^{cc} \\ &= (\gamma_{pf}^{cc})^2 - n\gamma_{pf}^{cc} + 2\gamma_{pf}^{cc} + \frac{n^2-n}{2} \end{aligned}$$

Which is the upper bound. □

The following some theorems are given depending on the fact that on the ordinary dominating set may be one vertex dominates all other vertices of G. While in the pitchfork dominating set every vertex in D dominates at most two vertices. Hence the order of $\gamma_{pf}(G)$ is equal or more than $\gamma(G)$.

Theorem 2.6. *Let $G = (n, m)$ be a graph with a doubly connected pitchfork domination, then:*

$$\gamma(G) \leq \gamma_{pf}(G) \leq \gamma_{pf}^{cc}(G).$$

Proof. From the definition of doubly connected pitchfork domination, every doubly connected pitchfork dominating set is a pitchfork dominating set and every pitchfork dominating set is a dominating set. □

Theorem 2.7. *Let $G = (n, m)$ be a graph with a doubly connected pitchfork domination, then:*

$$\gamma^c(G) \leq \gamma_{pf}^c(G) \leq \gamma_{pf}^{cc}(G)$$

Proof. From the definition of doubly connected pitchfork domination, every doubly connected pitchfork dominating set is a connected pitchfork dominating set and every connected pitchfork dominating set is a connected dominating set. □

Theorem 2.8. *Let $G = (n, m)$ be a graph with doubly connected pitchfork domination, then:*

$$\gamma^{cc}(G) \le \gamma^{cc}_{pf}(G).$$

Proof. From the definition of doubly connected pitchfork domination, every doubly connected pitchfork dominating set is a doubly connected dominating set. □

Theorem 2.9. *Let $G = (n, m)$ be a graph with doubly connected pitchfork domination, then:*

$$\gamma^{c}(G) \le \gamma^{cc}_{pf}(G)$$

Proof. From the definition of doubly connected pitchfork domination, every doubly connected pitchfork dominating set is a connected dominating set. □

Theorem 2.10. *Let $G(n, m)$ $(n \ge 4)$, be a graph with an inverse doubly connected pitchfork domination number $\gamma^{-cc}_{pf}(G)$, then $2 \le \gamma^{-cc}_{pf}(G) \le n - 2$.*

Proof. Let D be a doubly connected pitchfork dominating set in G, since D is a $\gamma^{cc}_{pf}-$set and $n > 3$, then $|D| \ge 2$ according to Theorem (2.4). Since $|D^{-1}| \ge |D|$ then $|D^{-1}| \ge 2$ at least and $|D^{-1}| \le n - 2$ at most. □

3. An Applications on Some Graphs

Here, the doubly connected pitchfork domination number and it's inverse are applied and evaluated for some standard and complement graphs such as: path, cycle, wheel, complete, complete bipartite graph and some complement graphs.

Proposition 3.1. *Let P_n be a path with $n \ge 2$, then:*
1. P_n has doubly connected pitchfork domination if and only if $n = 2$ such that $\gamma^{cc}_{pf}(P_2) = 1$.
2. P_n has an inverse doubly connected pitchfork domination if and only if $n = 2$ such that $\gamma^{-cc}_{pf}(P_2) = 1$.

Proof. If $n = 2$, then D has one vertex and $V - D = D^{-1}$ such that D is a $\gamma^{cc}_{pf}-$set and D^{-1} is a $\gamma^{-cc}_{pf}-$set. For $n \ge 3$, then P_n has no doubly connected pitchfork domination according to Theorem (2.1). □

Theorem 3.1. *Let C_n be the cycle graph, then:*
1. C_n has doubly connected pitchfork domination if and only if $n = 3, 4$ where $\gamma^{cc}_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.
2. C_n has an inverse doubly connected pitchfork domination if and only if $n = 3, 4$ such that $\gamma^{-cc}_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.
3. $\gamma^{cc}_{pf}(C_n) + \gamma^{-cc}_{pf}(C_n) = n$ if and only if $n = 4$.
4. $\gamma^{cc}_{pf}(C_n) + \gamma^{-cc}_{pf}(C_n) = n - 1$ if and only if $n = 3$.

Proof. If $n = 3$, let D consists of any vertex of C_3, then D is a pitchfork dominating set and both $G[D]$ and $G[V - D]$ are connected graphs. Therefor D is a $\gamma^{cc}_{pf}-$set. In the same way we choose D^{-1}. Hence $\gamma^{cc}_{pf}(C_3) = \gamma^{-cc}_{pf}(C_3) = 1$. If $n = 4$, let D consists of any two adjacent vertices, then D is a pitchfork dominating set and both $G[D]$ and $G[V - D]$ are connected graphs. So that $D^{-1} = V - D$. Hence $\gamma^{cc}_{pf}(C_4) = \gamma^{-cc}_{pf}(C_4) = 2$. Thus D is a $\gamma^{cc}_{pf}-$set and D^{-1} is a $\gamma^{-cc}_{pf}-$set with order two. For $n \ge 5$, then every chosen of a pitchfork dominating set D get $G[D]$ and $G[V - D]$ disconnected graphs. Hence C_n has no doubly connected pitchfork domination. □

Theorem 3.2. *Let K_n be a complete graph with $n \geq 3$, then:*
1. K_n has doubly connected pitchfork domination such that $\gamma_{pf}^{cc}(K_n) = \gamma_{pf}(K_n) = n-2$.
2. K_n has an inverse doubly connected pitchfork domination if and only if $n = 3, 4$ such that $\gamma_{pf}^{-cc}(K_n) = \gamma_{pf}^{-1}(K_n) = n-2$.
3. $\gamma_{pf}^{cc}(K_n) + \gamma_{pf}^{-cc}(K_n) = n$ if and only if $n = 4$.

Proof. 1. Let D be a pitchfork dominating set in K_n. Since every vertex in D dominates at most two vertices, then $V-D$ contains only two vertices which are dominated by all the other vertices. Since every component of order 2 or more of complete graph is complete, therefor $G[D]$ and $G[V-D]$ are connected. Hence D be a $\gamma_{pf}^{cc}-$set in K_n with order $n-2$.
2. It is clear when $n = 3, 4$ then K_n has an inverse doubly connected pitchfork domination number equals to $n-2$ by similar technique of proof 1. But if $n \geq 5$ then K_n has no inverse pitchfork domination according to Note (1.1) since $\gamma_{pf}^{cc}(K_n) > \frac{n}{2}$.
3. It is clear from 1 and 2. □

Theorem 3.3. *Let $K_{n,m}$ be a complete bipartite graph with $n, m \geq 2$, then:*
1. $K_{n,m}$ has a doubly connected pitchfork domination if and only if $n, m \geq 2$ such that $\gamma_{pf}^{cc}(K_{n,m}) = \gamma_{pf}(K_{n,m})$.
2. $K_{n,m}$ has an inverse doubly connected pitchfork domination if and only if $n, m = 2, 3, 4$ such that $\gamma_{pf}^{-cc}(K_{n,m}) = \gamma_{pf}^{-1}(K_{n,m})$.
3. $\gamma_{pf}^{cc}(K_{n,m}) + \gamma_{pf}^{-cc}(K_{n,m}) = n+m$ for $K_{2,2}, K_{2,4}, K_{4,4}$.

Proof. Let A and B are two disjoint sets of vertices of $K_{n,m}$ such that $|A| = n$ and $|B| = m$.
1- Three cases are obtained as follows:
Case 1: If $n = m = 2$ then the $\gamma_{pf}^{cc}-$set D contains any vertex in A and any vertex in B.
Case 2: If $n = 2$ and $m \geq 3$, suppose that $A = \{v_1, v_2\}$ and let D contains one vertex of A such as v_1 and $m-2$ vertices of B, then v_1 will dominate two vertices. Therefore, all the $m-2$ vertices of B which are in D will dominate v_2. Hence, $\gamma_{pf}^{cc}(K_{n,m}) = m-1$.
Case 3: If $n, m > 2$, then D must be contain $n-2$ vertices of A and $m-2$ vertices of B where all the $n-2$ vertices will dominate the two vertices of B. Also, all $m-2$ vertices of B which are in D will dominate the two vertices of A that belong to $V-D$. Hence, $\gamma_{pf}^{cc}(K_{n,m}) = m+n-4$.
2- The proof is clear when $n, m \leq 4$ by similar technique of 1. If $m \geq 5$, since D contains two vertex of A and $m-2$ vertices of B by 1, then if D^{-1} contains the other two vertices of A, it will dominate all the $m-2$ vertices of B that belong to D, but $m-2 \geq 3$ which is a contradiction. Hence, $K_{n,m}$ has no inverse doubly connected pitchfork domination.
3. Since $D^{-1} = V-D$ in $K_{2,2}, K_{2,4}, K_{4,4}$. □

Theorem 3.4. *Let W_n be the wheel graph of order $n+1$, then:*
1. W_n has doubly connected pitchfork domination such that:

$$\gamma_{pf}^{cc}(W_n) = \begin{cases} 2, & if\, n = 3 \\ n-2, & if\, n \geq 4 \end{cases}$$

2. W_n has an inverse doubly connected pitchfork domination if and only if $n = 3, 4$ such that $\gamma_{pf}^{-cc}(W_n) = \gamma_{pf}^{cc}(W_n)$.
3. $\gamma_{pf}^{cc}(W_n) + \gamma_{pf}^{-cc}(W_n) = n+1$ if and only if $n = 3$.
4. $\gamma_{pf}^{cc}(W_n) + \gamma_{pf}^{-cc}(W_n) = n$ if and only if $n = 4$.

Proof. 1. If $n = 3$ let D consist of any two adjacent vertices of the cycle. Then every vertex of D dominates the two remaining vertices with $G[D]$ and $G[V-D]$ are connected. Hence D is $\gamma_{pf}^{cc}-$set of W_3 with order two. If $n \geq 4$ let D contains all the vertices except

the central vertex and another any two adjacent vertices of the cycle say u_1, u_2. Then every vertex of D dominates the central vertex, except the vertex u_n which dominates the central vertex and u_1. Also the vertex u_3 dominates the central vertex and u_2. Since both $G[D]$ and $G[V-D]$ are connected. Hence D is $\gamma^{cc}_{pf}-$set of W_n with order $n-2$.
2. If $n=3$ then $D^{-1}=V-D$. If $n=4$ then D consist of the two vertices of the cycle which are in $V-D$. Hence D^{-1} is $\gamma^{-cc}_{pf}-$set of W_n with order two. When $n=5$ then if the central vertex belongs to D^{-1} then it is dominates the three vertices of D which is contradiction, also if the central vertex don't belongs to D^{-1} then there exist one vertex in D don't dominated by D^{-1}, hence there is no inverse pitchfork dominating set. If $n \geq 6$ then $\gamma^{cc}_{pf}(W_n) > \frac{n+1}{2}$ and there is no D^{-1} according to Note (1.1).
Proof 3. and 4. follow from (1) and (2). □

Theorem 3.5. *Let $\overline{P}_n$ be a complement path graph, then:*
1. $\overline{P}_n$ has a doubly connected pitchfork domination if and only if $n \geq 5$ such that

$$\gamma^{cc}_{pf}(\overline{P}_n) = \begin{cases} 2, & \textit{if } n = 5, 6 \\ 3, & \textit{if } n = 7 \\ n-2, & \textit{if } n \geq 8. \end{cases}$$

2. $\overline{P}_n$ has an inverse doubly connected pitchfork domination if and only if $n = 5, 6, 7$ such that

$$\gamma^{-cc}_{pf}(\overline{P}_n) = \begin{cases} 2, & \textit{if } n = 5 \\ \lceil \frac{n}{3} \rceil + 1, & \textit{if } n = 6, 7 \end{cases}$$

3. $\gamma^{cc}_{pf}(\overline{P}_n) + \gamma^{-cc}_{pf}(\overline{P}_n) = n$ if and only if $n = 7$.

Proof. Since $\overline{P}_2$ is null graph, $\overline{P}_3$ has an isolated vertex and since every chosen of a pitchfork dominating set D in $\overline{P}_4$ give $G[D]$ or $G[V-D]$ disconnected. Hence $\overline{P}_2$, $\overline{P}_3$ and $\overline{P}_4$ has no doubly connected pitchfork domination.
1. It is clear for $n = 5, 6, 7$. If $n \geq 8$, then the best choice "but not the only one" for connected pitchfork dominating set is $D = \{v_2, v_3, \cdots, v_{n-1}\}$. Hence D is a $\gamma^{cc}_{pf}-$set with order $n-2$.
2. It is clear for $n = 5, 6, 7$. If $n \geq 8$, then $\overline{P}_n$ has no inverse connected pitchfork domination according to Note (1.1) since $\gamma^{cc}_{pf}(\overline{P}_n) > \frac{n}{2}$.
3. Since $\gamma^{cc}_{pf}(\overline{P}_7) = 3$ and $\gamma^{-cc}_{pf}(\overline{P}_7) = 4$. □

Theorem 3.6. *Let $\overline{C}_n$ be a complement cycle graph, then:*
1. $\overline{C}_n$ has a doubly connected pitchfork domination if and only if $n \geq 6$ such that

$$\gamma^{cc}_{pf}(\overline{C}_n) = \begin{cases} 2, & \textit{if } n = 6 \\ n-4, & \textit{if } n = 7, 8 \\ 6, & \textit{if } n = 9 \\ n-2, & \textit{if } n \geq 10. \end{cases}$$

2. $\overline{C}_n$ has an inverse doubly connected pitchfork domination if and only if $n = 6, 7, 8$ such that

$$\gamma^{-cc}_{pf}(\overline{C}_n) = \begin{cases} 2, & \textit{if } n = 6 \\ n-4, & \textit{if } n = 7, 8 \end{cases}$$

3. $\gamma^{cc}_{pf}(\overline{C}_n) + \gamma^{-cc}_{pf}(\overline{C}_n) = n$ if and only if $n = 8$.

Proof. Since $\overline{C}_3$ is null graph, $\overline{C}_4$ is disconnected graph and every chosen of a pitchfork dominating set D in $\overline{C}_5$ give $G[V-D]$ disconnected then $\overline{C}_3$, $\overline{C}_4$ and $\overline{C}_5$ has no doubly connected pitchfork domination.

1. There are four cases as follows: If $n = 6$ let $D = \{v_k,\, v_{k+3}\}$ for any integer $1 \leq k \leq 6$. If $n = 7$ let us label the vertices of $\overline{C}_7$ as $\{v_i;\ i = 1,\, 2,\, \cdots,\, 7\}$ and Let $D = \{v_i;\ i\, is\, odd,\ \ i \neq 7\}$. If $n = 8$ let $D = \{v_i;\ i\, is\, odd\}$. If $n = 9$ let D consists of the firstly two vertices from every three consecutive vertices. If $n \geq 10$ let D consists of all the vertices except two vertices, but we must avoid choose $V - D = \{v_i, v_{i+2}\}$ since v_{i+1} don't dominates any vertex. In all above cases, D is a pitchfork dominating set, since $G[D]$ so that $G[V - D]$ are connected graphs, then D is a $\gamma_{pf}^{cc}-$set of $\overline{C}_n$.
2. According to the $\gamma_{pf}^{cc}-$set D in proof 1, let us choose D^{-1} as follows: If $n = 6$ then let $D^{-1} = \{v_{k+1},\, v_{k+4}\}$ for any integer $1 \leq k \leq 6$. If $n = 7,\, 8$ then let $D^{-1} = \{v_i;\ i\, is\, even\}$. D^{-1} is an inverse pitchfork dominating set, since $G[D^{-1}]$ so that $G[V - D^{-1}]$ are connected graphs, then D^{-1} is a $\gamma_{pf}^{-cc}-$set of $\overline{C}_n$. If $n \geq 9$ then $\overline{C}_n$ has no inverse doubly connected pitchfork domination by Note (1.1) since $\gamma_{pf}^{cc}(\overline{C}_n) > \frac{n}{2}$.
3. It is clear since $\gamma_{pf}^{cc}(\overline{C}_8) = 4 = \gamma_{pf}^{-cc}(\overline{C}_8)$. □

Notice that $\overline{K}_n$, $\overline{W}_n$ and $\overline{K}_{n,m}$ have no doubly connected pitchfork domination, since $\overline{K}_n$ is null graph, $\overline{W}_n$ has an isolated vertex and $\overline{K}_{n,m}$ is disconnected graph contains two disjoint complete components.

4. Conclusions

The goal of this paper is to introduce new parameters of domination, doubly connected pitchfork domination and it's inverse. Some bounds and properties of these new types of dominations are studied and applied on some known graphs.

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Mohammed Abdali Abdlhusein is an assistant professor at Thiqar University (Iraq), College of Education for Pure Sciences in the Department of Mathematics. He is a Ph.D. student in the College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad. His main subjects are q-series and domination in graph theory.

Manal Naji Al-Harere has been an assistant professor, Ph.D. in Mathematics at Ain Shams University, Egypt since 2014. She is a member of the Faculty of Applied Sciences Department at University of Technology, Iraq. Her main interests are Labeling and Domination in Graph Theory.