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CENTRAL DERIVATION OF SOME CLASSES OF LEIBNIZ ALGEBRAS

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Abstract. In this study, we deal with central derivation of finite low dimensional Leibniz algebras. We provide some properties of the central derivation algebras. Description of the central derivation algebras, with their dimensions, for complex Leibniz algebras of dimensions two, three and four are given and summarized in tabular form. The result is then used to determine which centroid is decomposable or indecomposable.

Keywords: Central Derivation, Leibniz algebra, Lie Algebra, centroid.

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1. Introduction

In 1965, Bloh [1] introduced Leibniz algebras as a generalization of Lie algebras and in 1993, Loday [3] studied the properties of Leibniz algebras. Rakhimov and Al-Hossain [5] studied the derivations of some Classes of finite dimensional Leibniz Algebra. Hassan and AbdGhafur [6] studied the centroids and quasi-centroids of some Classes of finite dimensional Leibniz Algebra. Biyogman et. al. [7] did some studies on central derivation of nilpotent lie superalgebra. Narayan et. al. [8] studied lie central derivation and found the dimension of central derivation equal lie-stem Leibniz algebras. In this paper, we study the central derivation algebras of low dimensional Leibniz algebras and determine which classifications are indecomposable or decomposable by using the central derivation. The outline of this paper will cover the following: section 1 provides overview of this article. Section 2 gives preliminaries and some previous results used throughout the article. In section 3, an algorithm for finding central derivations, from previous literatures, of finite dimensional Leibniz algebras. This algorithm is used to compute the central derivations of low dimensional Leibniz algebras.

2. Preliminaries

This section contains definitions used and some earlier results used throughout the paper.

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Definition 2.1. *A Lie algebra L over a field $\mathbb{K}$ is an algebra satisfying the following conditions:*

$$[x,x] \;=\; 0, \forall\ x \in L \tag{1}$$

$$\Big[[x,y],z\Big] + \Big[[y,z],x\Big] + \Big[[z,x],y\Big] \;=\; 0,\ \ \forall\ \ x,y,z \in L. \tag{2}$$

Definition 2.2. *A Leibniz algebra L is a vector space over a field $\mathbb{F}$ equipped with a bilinear map*

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the Leibniz identity:

$$\Big[x,[y,z]\Big] = \Big[[x,y],z\Big] - \Big[[x,z],y\Big], \quad \textit{for all} \quad x,y,z \in L.$$

Definition 2.3. *A derivation of Leibniz algebra L is a linear transformation $d : L \to L$ satisfying*

$$d[x,y] = [d(x),y] + [x,d(y)], \quad \textit{for all} \quad x,y \in Z.$$

for all $x,y \in L$.
The set of all derivations of a Leibniz algebra L is denoted by $Der(L)$.
For a Liebniz algebra L we define

$$L = L^1,\ L^{k+1} = [L^k, L],\ k \geqslant 1\,.$$

Clearly,

$$L^1 \supseteq L^2 \supseteq \cdots$$

Definition 2.4. *Let L and L_1 be two Liebniz algebras over a field $\mathbb{K}$. A mapping $\mu : L \longrightarrow L_1$ is a homomorphism if satisfying*

$$\mu[x_1,x_2] = [\mu(x_1),\mu(x_2)], \quad \textit{for all} \quad x_1,x_2 \in L.$$

for all $x,y \in L$. We can say μ is an isomorphism if it is bijective and μ is an endomorphism when we have a linear map $\mu : L \longrightarrow L$. The set of all endomorphisms of L is denoted by $End(L)$.

Definition 2.5. *Let L be a Leibniz algebra over a field $\mathbb{K}$. The set satisfying*

$$\Gamma(L) = \Big\{\varphi \in End(L) |\ \varphi[a,b] = [a,\varphi(b)] = [\varphi(a),b], \quad \textit{for all} \quad a,b \in L\Big\}$$

for all $x,y \in L$, is called the centroids of L.

Definition 2.6. *Let L_1 be a nonempty subset of Leibniz algebra L. The subset*

$$C_L(L_1) = \{x \in L \mid [x,L_1] = [L_1,x] = 0\}$$

for all $x,y \in L$, is said to be the centralizer of L_1 in L. Obviously, $C_L(L) = C(L)$ is the center of L, and we called an Ideal I of Leibniz algebra L if $IL \subseteq I$ and $LI \subseteq I$.

Definition 2.7. *Let L be Leibniz algebra. We say that L is indecomposable if it can not be written as a direct sum of its ideals. Otherwise the L is called decomposable.*

Definition 2.8. *Let L be Leibniz algebra and $\varphi \in End(L)$. Then φ is called a central derivation, if $\varphi(L) \subseteq C(L)$ and $\varphi(L^2) = 0$.*

Let the set of all central derivation of a Leibniz algebra L be denoted by $CL(L)$. The $CL(L)$ is an associative algebra with respect the composition operation $\circ$ and it is a Lie algebra with respect to the bracket $[a_1,a_2] = a_1 \circ a_2 - a_2 \circ a_1$.

Theorem 2.1. *Let $g : L_1 \longrightarrow L_2$ be an isomorphism of Leibniz algebras $(L_1, \star)$ and $(L_2, *)$ over a field $\mathbb{K}$. The mapping $\phi : End(L_1) \longrightarrow End(L_2)$ defined by $\phi(c) = g \circ c \circ g^{-1}$ is an isomorphism of $CL(L_1)$ and $CL(L_2)$, that is*

$$\phi(CL(L_1)) = CL(L_2).$$

Proof. Due to the isomorphism relation, we have $x * y = g(g^{-1}(x) \star g^{-1}(y))$. Assume $c \in End(L_1)$ such that;

$$d\big(g^{-1}(x_1) \star g^{-1}(x_2)\big) = c(g^{-1}(x_1)) \star g^{-1}(x_2) + g^{-1}(x_1) \star c(g^{-1}(x_2)).$$

Applying the mapping g on this equation we have

$$g \circ c \circ g^{-1}(x_1 * x_2) = \big(g \circ c \circ g^{-1}(x_1)\big) * x_2 + x_1 * \big(g \circ c \circ g^{-1}(x_2)\big),$$

that is $g \circ c \circ g^{-1} \in CL(L_2)$. □

Proposition 2.1. *Let L be Leibniz algebra. Then*

i) $CL(L) \subseteq Der(L)$.;
iii) $[\Gamma(L), \Gamma(L)] \subseteq CL(L)$;
iv) $(\varphi \circ d)[x, y] = [(\varphi \circ d)(x), y] + [x, \varphi \circ d(y)]$;
v) $[d, \varphi](x, y) = [[d, \varphi](x), y] + [x, [d, \varphi](y)]]$.

Proof. We prove i) and the others can be obtained by definitions.

Let L be Leibniz algebra, $\varphi_1 \in CL(L)$ and for all $x, y \in L$. We need to show that

$$\varphi_1[x, y] \quad = \quad [\varphi_1(x), y] = [x, \varphi_1(y)] = 0.$$

Thus,

$$\varphi_1[x, y] = [\varphi_1(x), y] = 0 \tag{3}$$

$$\varphi_1[x, y] = [x, \varphi_1(y)] = 0. \tag{4}$$

If we add (3) with (4), we get

$$\varphi_1[x, y] + \varphi_1[x, y] \quad = \quad [\varphi_1(x), y] + [x, \varphi_1(y)].$$

Therefore, $\varphi_1[x, y] = [\varphi_1(x), y] + [x, \varphi_1(y)] \subseteq Der(L)$ since $\varphi_1[x, y] = 0$. □

Theorem 2.2. *Let Z be Leibniz algebra. Then for any $\varphi \in \Gamma(L)$ and $d \in Der(L)$ one has the following.*

a) $Der(L) \cap \Gamma(L) = CL(L)$;
b) $d \circ \varphi$ *is contained in* $\Gamma(L)$ *if and only if* $\varphi \circ d$ *is a central derivation of* L;
c) $d \circ \varphi$ *is a derivation of* L *if and only if* $[d, \varphi]$ *is a central derivation of* L.

Proof. a) If we assume $\varphi \in Der(L) \cap \Gamma(L)$ then we can say $\varphi \in \Gamma(L)$ and $\varphi \in Der(L)$. Let $x, y \in L$. Then from derivation: $\varphi[x, y] = [\varphi(x), y] + [x, \varphi(y)]$ and from centroid: $\varphi[x, y] = [\varphi(x), y] = [x, \varphi(y)]$, so we get $\varphi[x, y] = [\varphi(x), y] = [x, \varphi(y)] = 0$. Therefore $\varphi(L^2) = 0$ and $\varphi(L) \subseteq Z(L)$. Then $\Gamma(L) \cap Der(L) \subseteq CL(L)$.
To show the inverse inclusion, let $\varphi \in C(L)$. Then $0 = \varphi[x, y] = [\varphi(x), y] = [x, \varphi(y)]$. Thus $\varphi \in \Gamma(L) \cap Der(L)$. This implies $Der(L) \cap \Gamma(L) = C(L)$.

b). Assume for any $\varphi \in \Gamma(L)$, $d \in Der(L)$, $\forall x, y \in L$.
If $d \circ \varphi$ is contained in $\Gamma(L)$, then

$$(d \circ \varphi)[x, y] = [(d \circ \varphi)(x), y] = [x, (d \circ \varphi)(y)]. \tag{5}$$

Also,

$$(d \circ \varphi)[x, y] = d \circ [\varphi(x), y] = d \circ [x, \varphi(y)],$$

then,

$$(d \circ \varphi)[x, y] = d \circ [\varphi(x), y] = [(d \circ \varphi)(x), y] + [x, (\varphi \circ d)(y)] \tag{6}$$

$$(d \circ \varphi)[x, y] = d \circ [x, \varphi(y)] = [(\varphi \circ d)(x), y] + [x, (d \circ \varphi)(y)] \tag{7}$$

By equation (5) and equation (6), we get $[x, (\varphi \circ d)(y)] = 0$ and by equation (5) and equation (7), we get $[(\varphi \circ d)(x), y] = 0$. By (v) in Proposition 2.1, we get $(\varphi \circ d)[x, y] = 0$. Then $\varphi \circ d$ is central derivation of L.
To show the inverse inclusion, let $\varphi \circ d$ be a central derivation of L. Then $0 = (\varphi \circ d)[x, y] = [x, (\varphi \circ d)(y)] = [(\varphi \circ d)(x), y]$. Subtract equation (7) from equation (6) to obtain $[(d \circ \varphi)(x), y] = [x, (d \circ \varphi)(y)]$. Then we get $(d \circ \varphi)[x, y] = [(d \circ \varphi)(x), y] = [x, (d \circ \varphi)(y)]$ by equation (6).

c). Assume $d \circ \varphi \in Der(L)$ and $\forall x, y \in L$. If $d \circ \varphi \in Der(L)$. Then

$$(d \circ \varphi)[x, y] = [(d \circ \varphi)(x), y] + [x, (d \circ \varphi)(y)]. \tag{8}$$

By equation (6) and (8), we get $[x, (\varphi \circ d)(y)] = [x, (d \circ \varphi)(y)]$. On Simplification, it becomes $[x, [d, \varphi](y)] = 0$. By (iv) in Proposition 2.1, $[d, \varphi]$ is a central derivation of L. To show the inverse inclusion, if $[d, \varphi]$ is a central derivation of L, then $\left[x, [d, \varphi](y)\right] = 0$ and we get $[x, (d \circ \varphi)(y)] = [x, (\varphi \circ d)(y)]$. After substitution into equation (6), then it easy to prove $d \circ \varphi \in Der(L)$. □

Theorem 2.3. *If $L = L_1 \bigoplus L_2$, where L_1 and L_2 are ideals of L. Then,* $\Gamma(L) = \Gamma(L_1) \bigoplus \Gamma(L2) \bigoplus \mathcal{C}_1 \bigoplus \mathcal{C}_2$ *where $\mathcal{C}_i$*

$$\mathcal{C}_i = \left\{\phi \in \operatorname{End}(A_i, A_j) \,|\phi(A_i) \subseteq \mathcal{C}(A_j), \phi\left(A_i^2\right) = 0\right\}, 1 \leq i \neq j \leq 2$$

Proof. Let $\pi_i : L \to L_i$ be canonical projection, $i = 1, 2$. Then $\pi_1, \pi_2 \in \Gamma(L)$ and $\pi_1 + \pi_2 = \mathrm{id}_L$. For any $\phi \in \Gamma(L), \phi = (\pi_1 + \pi_2)\,\phi\,(\pi_1 + \pi_2) = \pi_1\phi\pi_1 + \pi_1\phi\pi_2 + \pi_2\phi\pi_1 + \pi_2\phi\pi_2$. Then $\pi_i\phi\pi_j \in \Gamma(L), i, j = 1, 2$, and $\pi_1\Gamma(L)\pi_1 + \pi_1\Gamma(L)\pi_2 + \pi_2\Gamma(L)\pi_1 + \pi_2\Gamma(L)\pi_2$ is direct sum. Now we define a mapping $\pi_1\Gamma(L)\pi_1 \to \Gamma(L_1)$ such that $\pi_1 f\pi_1 \mapsto \pi_1 f\pi_1|_{L_1}$, for any $f \in \Gamma(L)$. If $\pi_1 f\pi_1|_{L_1} = 0$, then by $\pi_1 f\pi_1|_{L_2} = 0$, we get that $\pi_1 f\pi_1 = 0$, and hence the above mapping is injection. For any $\phi \in \Gamma(L_1)$, we extend ϕ to L_2 such that $\phi|_{L_2} = 0$. Then the extended ϕ is in $\Gamma(L)$ and $\pi_1\phi\pi|_{L_1} = \phi$. Thus $\pi_1\Gamma(L)\pi_1 \cong \Gamma(L_1)$ as vector space. Similarly, we get that $\pi_2\Gamma(L)\pi_2 \cong \Gamma(L_2)$ Define a mapping $\pi_1\Gamma(L)\pi_2 \to \mathcal{C}_2$ such that $\pi_1 f\pi_2 \mapsto \pi_1 f\pi_2|_{L_2}$

For $x \in L_2, y \in L_1, \pi_1 f\pi_2(x)y = \pi_1 f\pi_2(xy) = 0,$ and $y\,(\pi_1 f\pi_2(x)) = \pi_1 f\pi_2(yx) =$ 0, then $\pi_1 f\pi_2(x) \in \mathcal{C}(L_1)$ For $x, y \in L_2, \pi_1 f\pi_2(xy) = 0$, then $\pi_1 f\pi_2(L_1) = 0$ and $\pi_1 f\pi_2|_{L_2} \in \mathcal{C}_2$ If $\pi_1 f\pi_2|_{L_2} = 0$ and $\pi_1 f\pi_2|_{L_1} = 0$, then $\pi_1 f\pi_2|_L = 0$, and the above mapping is injection.

For any $\phi \in \mathcal{C}_2$, we extend ϕ to L_1 denoted by $\bar{\phi}$ such that $\bar{\phi}\big|_{L_1} = 0$. Then $\bar{\phi} \in \Gamma(L)$ by the following three equations.

$$\begin{aligned}
&\bar{\phi}(xy) = \bar{\phi}\,(x_1y_1 + x_2y_2) = \phi\,(x_2y_2) = 0\\
&x\bar{\phi}(y) = (x_1 + x_2)\,\bar{\phi}\,(y_1 + y_2) = x_2\phi\,(y_2) = 0\\
&\bar{\phi}(x)y = \phi\,(x_2)\,(y_1 + y_2) = \phi\,(x_2)\,y = 0
\end{aligned}$$

So $\pi_1\bar{\phi}\pi_2\,(x_2) = \pi_1\bar{\phi}\,(x_2) = \pi\,(x_2)$. Then $\pi_1\bar{\phi}\pi_2 \mapsto \phi$, which says the mapping above is onto. Thus $\pi_1\Gamma(A)\pi_2 \cong \mathcal{C}_2$ as vector space. Similarly, we can get that $\pi_2\Gamma(L)\pi_1 \cong \mathcal{C}_1$. The theorem is proven. □

Corollary 2.1. *If $CL(L) = 0$, then $\Gamma(L)$ is decomposable.*

Proof. It is clear from theorm 2.3 and defintion 2.7. □

In the next section, we used an algorithm, described below, to find the central derivation of low-dimensional complex Leibniz algebras. Note that there is no one-dimensional Leibniz algebra except for abelian.

3. An algorithm for finding central derivations

Firstly, let $\{e_1, e_2, ..., e_n\}$ be a basis of an n-dimensional Leibniz algebras L. Then

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^k e_k, \; i, j = 1, 2, ..., n.$$

The coefficients $\{\gamma_{ij}^k\}$ of the above linear combinations are called the structure constants. An element a of central derivation, $CL(L)$ being a linear transformation of the Leibniz algebras L is represented in a square matrix form $[a_{ij}]_{i,j=1,2,3,...,n}$, that is

$$\varphi(e_i) = \sum_{t=1}^{n} a_{ti} e_t, \; i = 1, 2, 3, ..., n.$$

According to the Theorem 2.2, the central derivation $CL(L)$ is equal intersection between centroid $\Gamma(L)$ and derivation $Der(L)$. From definition of centroid and central derivation we get an algorithm to find central derivation $CL(L)$ as given below:

$$\sum_{t=1}^{n} \gamma_{ij}^t a_{kt} = \sum_{t=1}^{n} a_{ti} \gamma_{tj}^k = \sum_{t=1}^{n} a_{tj} \gamma_{it}^k = 0, \;\; \forall \;\; i, j, k = 1, 2, 3, ..., n. \tag{9}$$

This approach can be applied to find the central derivations of complex Leibniz algebras in dimension 2, 3, and 4. Besides, we use the classification result from [5] and [4] together with the algorithm mentioned above to get the central derivation and the result is summarized in tabular form .

Theorem 3.1. *Any two-dimensional Leibniz algebra L is isomorphic to*
$L_1 : [e_1, e_1] \;\; = e_2.$
$L_2 : [e_1, e_2] \;\; = -[e_2, e_1] = e_2.$
$L_3 : [e_2, e_2] \;\; = [e_1, e_2] = e_1.$

Table 1. derivation of two-dimensional complex Zinbiel algebras

Isomorphism Class	Central Derivation	Dimension
L_1	$\begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}$	1
L_2	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0
L_3	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

Corollary 3.1. *Any two dimensional leibniz algebras, the centroid of L_1 is decomposable.*

Theorem 3.2. *An arbitrary non split three-dimensional complex Leibniz algebras is isomorphic to the following pairwise non isomorphic algebras:*

$L_1 : [e_3, e_2] = e_2, [e_2, e_2] = e_1, [e_1, e_3] = -2e_1, [e_2, e_3] = -e_2.$
$L_2 : [e_3, e_3] = e_1, [e_1, e_3] = \delta e_1, [e_2, e_3] = -e_2, [e_3, e_2] = e_2, \delta \in \mathbb{C}.$
$L_3 : [e_2, e_3] = e_1, [e_3, e_3] = \delta e_1, [e_2, e_2] = e_1, \delta \in \mathbb{C} \setminus \{0\}.$
$L_4 : [e_2, e_2] = e_1, [e_3, e_3] = e_2.$
$L_5 : [e_2, e_3] = e_1, [e_1, e_3] = e_2.$
$L_6 : [e_1, e_3] = e_2, [e_2, e_3] = \delta e_1 + e_2, \delta \in \mathbb{C}.$
$L_7 : [e_2, e_3] = e_2, [e_1, e_3] = e_1.$
$L_8 : [e_1, e_3] = e_2, [e_3, e_3] = e_1.$
$L_9 : [e_3, e_3] = e_1, [e_1, e_3] = e_1 + e_2.$
$L_{10} : [e_1, e_2] = e_1.$
$L_{11} : [e_2, e_3] = e_2, [e_1, e_3] = e_1 + e_2.$
$L_{12} : [e_1, e_2] = e_3, [e_2, e_3] = -e_2, [e_1, e_3] = 2e_1.$
$L_{13} : [e_1, e_3] = e_1, [e_2, e_3] = \delta e_3, \delta \in \mathbb{C} \setminus \{0\}.$

Corollary 3.2. *In any three dimensional Leibniz algebras, the centroid of $L_1, L_2, L_5, L_7, L_{11}, L_{12}$ and L_{13} are decomposable.*

Theorem 3.3. *The isomorphism class of four-dimensional complex nilpotent Leibniz algebras are given by the following*

$L_1 : [e_1, e_1] = e_2, [e_3, e_1] = e_4, [e_2, e_1] = e_3.$
$L_2 : [e_1, e_1] = e_3, [e_2, e_1] = e_3, [e_1, e_2] = e_4, [e_3, e_1] = e_4.$
$L_3 : [e_3, e_1] = e_4, [e_1, e_1] = e_3, [e_2, e_1] = e_3.$
$L_4 : [e_1, e_1] = e_3, [e_1, e_2] = \delta e_4, [e_2, e_2] = e_4, [e_2, e_1] = e_3,$
$e_4 = [e_3, e_1], \delta \in \{0, 1\}.$
$L_5 : [e_1, e_2] = e_4, [e_1, e_1] = e_3, [e_3, e_1] = e_4.$
$L_6 : [e_2, e_2] = e_4, [e_1, e_1] = e_3, [e_3, e_1] = e_4.$
$L_7 : [e_1, e_1] = e_4, [e_3, e_1] = e_4, [e_2, e_1] = e_3, [e_1, e_2] = -e_3,$
$-e_4 = [e_1, e_3].$
$L_8 : [e_1, e_1] = e_4, [e_3, e_1] = e_4, [e_2, e_1] = e_3, [e_1, e_2] = -e_3 + e_4,$
$-e_4 = [e_1, e_3].$
$L_9 : [e_1, e_1] = e_4, [e_2, e_2] = e_4, [e_2, e_1] = e_3, [e_1, e_2] = -e_3 + 2e_4,$
$-e_4 = [e_1, e_3], [e_3, e_1] = e_4.$
$L_{10} : [e_2, e_1] = e_3, [e_1, e_1] = e_4, [e_2, e_2] = e_4, [e_3, e_1] = e_4,$
$-e_3 = [e_1, e_2], [e_1, e_3] = -e_4.$
$L_{11} : [e_1, e_2] = e_3, [e_1, e_1] = e_4, [e_2, e_1] = -e_3, [e_2, e_2] = -2e_3 + e_4.$
$L_{12} : [e_2, e_1] = e_4, [e_1, e_2] = e_3, [e_2, e_2] = -e_3.$
$L_{13} : [e_1, e_2] = e_4, [e_1, e_1] = e_3, [e_2, e_1] = -\delta e_3, [e_2, e_2] = -e_4, \delta \in \mathbb{C}.$
$L_{14} : [e_3, e_3] = e_4, [e_1, e_2] = \delta e_4, [e_2, e_1] = -\delta e_4, [e_2, e_2] = e_4,$
$e_4 = [e_1, e_1], \delta \in \mathbb{C}.$
$L_{15} : [e_1, e_3] = e_4, [e_1, e_2] = e_4, [e_2, e_1] = -e_4, [e_2, e_2] = e_4.$
$L_{16} : [e_1, e_1] = e_4, [e_2, e_1] = -e_4, [e_1, e_2] = e_4, [e_3, e_3] = e_4.$
$L_{17} : [e_1, e_2] = e_3, [e_2, e_1] = e_4.$
$L_{18} : [e_2, e_2] = e_4, [e_1, e_2] = e_3, [e_2, e_1] = -e_3.$
$L_{19} : [e_2, e_1] = e_4, [e_2, e_2] = e_3.$
$L_{20} : [e_2, e_2] = e_3, [e_1, e_2] = e_4, [e_2, e_1] = \frac{1+\delta}{1-\delta} e_4, \delta \in \mathbb{C} \setminus \{1\}.$
$L_{21} : [e_2, e_1] = -e_4, [e_1, e_2] = e_4, [e_3, e_3] = e_4.$

TABLE 2. derivation of three-dimensional complex Leibniz algebras

Isomorphism Class	Central Derivation	Dimension
L_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
L_2	$\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\delta = 0)$	1
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\delta \neq 0)$	0
L_3	$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
L_4	$\begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
L_5	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
L_6	$\begin{pmatrix} a_{11} & 0 & a_{13} \\ -a_{11} & 0 & -a_{13} \\ 0 & 0 & 0 \end{pmatrix}, (\delta = 0)$	2
	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\delta \neq 0)$	0
L_7	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
L_8	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	1
L_9	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$	1
L_{10}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}$	2
L_{11}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
L_{12}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
L_{13}	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0

Corollary 3.3. *The centroid of four dimensional Leibniz algebras are indecomposable.*

Isomorphism Class	**Central Derivation**	**Dimension**
L_1	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix}$	1
L_2	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_3	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_4	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_5	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_6	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_7	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_8	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_9	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_{10}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	2
L_{11}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4

Leibniz algebra	**Central Derivation**	**Dim**
L_{12}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{13}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{14}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}$	3
L_{15}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}$	3
L_{16}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}$	3
L_{17}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{18}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{19}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{20}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$	4
L_{21}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}$	3

Corollary 3.4. *i) The dimensions of the central derivation of two-dimensional complex Leibniz algebras is one.*

ii) The dimensions of the central derivation of three-dimensional complex Leibniz algebras vary between one and two.

iii) The dimensions of the central derivation of four-dimensional complex Leibniz algebras vary between one and four.

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