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FUZZY ESSENTIAL SUBMODULES WITH RESPECT TO AN ARBITRARY FUZZY SUBMODULE

S. NIMBHORKAR[1], J. KHUBCHANDANI[2], §

Abstract. In this paper, we extend the concepts of a fuzzy essential submodule and a fuzzy complement submodule to the concepts of fuzzy essential submodule with respect to an arbitrary fuzzy submodule and fuzzy complement submodule with respect to an arbitrary fuzzy submodule respectively. We give some characterizations and properties of such fuzzy submodules.

Keywords: Fuzzy submodule, Fuzzy essential submodule, Fuzzy complement submodule.

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1. Introduction

The well known concept of a fuzzy set was introduced by Zadeh [11] in 1965. Since then this concept has been applied to many areas in Mathematics. Rosenfeld [7] applied this concept to groups. Naegoita and Ralescu [5] applied this concept to modules and defined fuzzy submodules. Pan [6] defined a fuzzy finitely generated module and a fuzzy quotient module. The concept of an essential fuzzy submodule and various properties of such modules was studied by Kalita [3]. Safaeeyan and Saboori [9] studied essential submodules with respect to an arbitrary submodule. Al-Dhaheri and Al-Bahrami [1] have considered essential submodules with respect to an arbitrary submodule.

In this paper, we have fuzzify the concepts of an essential submodule with respect to an arbitrary submodule and a complement submodule with respect to arbitrary submodule and obtain various properties related to these concepts.

2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity and M an unitary R-module with the zero element θ.

[1] Formerly of Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India.
e-mail: sknimbhorkar@gmail.com; ORCID: https://orcid.org/0000-0001-5432-8819.

[2] Department of Mathematics, Dr.Vitthalrao Vikhe Patil College of Engineering, Ahmednagar, 414001, India.
e-mail: khubchandani_jyoti@yahoo.com; ORCID: https://orcid.org/0000-0003-3155-0817.

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Definition 2.1. [4] *A fuzzy subset of an R-module M, is a mapping μ from M to* $[0, 1]$.

Definition 2.2. [4] *If $N \subseteq M$ and $\alpha \in [0,1]^M$, then α_N is defined as,*

$$\alpha_N(x) = \begin{cases} \alpha, & if\ x \in N, \\ 0, & otherwise. \end{cases}$$

If $N = \{x\}$, then α_x is often called a fuzzy point and is denoted by χ_α.
If $\alpha = 1$, then 1_N is called the characteristic function of N and is denoted by χ_N.

The following properties are well-known.
If $\mu, \sigma \in [0,1]^M$, then

(i) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$;
(ii) $\mu \cup \sigma = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;
(iii) $\mu \cap \sigma = min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$;
(iv) $(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$.

Definition 2.3. [4] *Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Let $\mu \in [0,1]^X$ and $\sigma \in [0,1]^Y$. Then the image $f(\mu) \in [0,1]^Y$ and the inverse image $f^{-1}(\sigma) \in [0,1]^X$ are defined as follows:*
For every $y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x \in X, f(x) = y\}, & if f^{-1}(y) \neq \emptyset, \\ 0, & otherwise. \end{cases}$$

and $f^{-1}(\sigma)(x) = \sigma(f(x))$ for all $x \in X$.

Definition 2.4. [4] *Let M be an R-module. A fuzzy subset μ of M is said to be a fuzzy submodule, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:*

(i) $\mu(\theta) = 1$,
(ii) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
(iii) $\mu(rx) \geq \mu(x)$.

The set of all fuzzy submodules of M is denoted by $F(M)$.
The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by
$\mu^* = \{x \in M \mid \mu(x) > 0\}$.

Definition 2.5. [4, Theorem 4.2.1] *Let $\mu, \sigma \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as $\left(\frac{\nu}{\mu}\right)([x]) = \vee\{\nu(z) \mid z \in [x]\}$, $\forall x \in \nu^*$, where $[x]$ denotes the coset $x + \mu^*$.*

Definition 2.6. [10] *A fuzzy submodule σ of M is called a fuzzy direct sum of two fuzzy submodules μ and ν if $\sigma = \mu + \nu$ and $\mu \cap \nu = \chi_\theta$.*

Definition 2.7. [10] *An element $\mu \in F(M)$ is called a fuzzy direct summand of M if there exists $\gamma \in F(M)$ such that χ_M is a fuzzy direct sum of μ, ν.*

Definition 2.8. [3] *A fuzzy submodule μ of M is called an essential fuzzy submodule of M, denoted by $\mu \trianglelefteq M$ if for every non zero fuzzy submodule σ of M, $\mu \cap \sigma \neq \chi_\theta$.*

Example 2.1. *Let $R = \mathbb{Z}_8$ under addition and multiplication modulo 8 and $M = \mathbb{Z}_8 \bigoplus \mathbb{Z}_8$. Then M is an R-module. Define $\sigma : M \to [0,1]$ as :*

$$\sigma(x) = \begin{cases} 1, & if\ x = \{(0,0)\}, \\ 1\backslash 3, & if\ x \in \{(0,4), (4,0), (4,4)\}, \\ 0, & otherwise. \end{cases}$$

Then σ is an essential fuzzy submodule of M.

We recall some known results which we shall use in this paper.

Theorem 2.1. [3, Theorem 3.2.5] *A submodule A of M is essential in M if and only if χ_A is an essential fuzzy submodule of M.*

Lemma 2.1. [8, Lemma 5.3.1] *Let μ, $\nu \in F(M)$ be such that $\mu \subseteq \nu$. Then $\left(\dfrac{\nu}{\mu}\right)^* = \dfrac{\nu^*}{\mu^*}$.*

Theorem 2.2. [3, Theorem 3.2.15] *Let μ be a nonzero fuzzy submodule of M. Then $\mu \trianglelefteq M$ if and only if $\mu^* \trianglelefteq M$.*

Theorem 2.3. [3, Theorem 3.2.16] *Let μ and σ be two nonzero fuzzy submodule of an R-module M. Then $\mu \trianglelefteq \sigma$ if and only if $\mu^* \trianglelefteq \sigma^*$.*

Theorem 2.4. [3, Theorem 3.2.19] *If $f : M \to K$ is a module homomorphism and μ is an essential fuzzy submodule of K, then $f^{-1}(\mu)$ is an essential fuzzy submodule of M.*

Definition 2.9. [3] *Let μ be fuzzy submodule of an R-module M. A relative complement for μ in M is any fuzzy submodule σ of M which is maximal with respect to the property $\mu \cap \sigma = \chi_\theta$.*

Proposition 2.1. [3, Proposition 3.1.19] *Let $\{\mu_\alpha | \alpha \in \Omega\} \subseteq F(M)$. Then $\sum_{\alpha\in\Omega} \mu_\alpha^* = (\sum_{\alpha\in\Omega} \mu_\alpha)^*$.*

Proposition 2.2. [3, Proposition 3.1.20] *Let $\{\mu_\alpha | \alpha \in \Omega\} \subseteq F(M)$. Then $(\bigcap_{\alpha\in\Omega} \mu_\alpha)^* \subseteq \bigcap_{\alpha\in\Omega} \mu_\alpha^*$.*

Proposition 2.3. [9, Proposition 2.2] *For each $m, n \in Z$, $mZ \trianglelefteq_{nZ} mZ + nZ$.*

Theorem 2.5. [9, Proposition 2.13] *Let $T_1 \leqslant K_1 \leqslant M_1 \leqslant M$ and $T_2 \leqslant K_2 \leqslant M_2 \leqslant M$ such that $M_1 \cap M_2 = T_1 \cap T_2$. Then, $K_1 + K_2 \trianglelefteq_{T_1+T_2} M_1 + M_2$ if and only if $K_1 \trianglelefteq_{T_1} M_1$ and $K_2 \trianglelefteq_{T_2} M_2$.*

Lemma 2.2. [9, Lemma 2.15] *Let M and N be right R-modules, T and K be submodules of N and $f \in Hom_R(M, N)$. If $K \trianglelefteq_T N$, then $f^{-1}(K) \trianglelefteq_{f^{-1}(T)} M$.*

Corollary 2.1. [9, Corollary 2.6] *Let $T_1 \leqslant K_1 \leqslant M_1 \leqslant M$ and $T_2 \leqslant K_2 \leqslant M_2 \leqslant M$ such that $M_1 \cap M_2 = T_1 \cap T_2$. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq_{kerf} M$. Moreover, if f is an epimorphism, then $K \trianglelefteq N$ if and only if $f^{-1}(K) \trianglelefteq_{kerf} M$.*

3. Fuzzy essential submodules with respect to an arbitrary fuzzy submodule

In this section we introduce the concept of a fuzzy essential submodules with respect to an arbitrary fuzzy submodule. We obtain some properties of these modules.

Definition 3.1. *Let σ be a proper fuzzy submodule of an R-module M. A fuzzy submodule μ of M is called σ- essential denoted by $\mu \trianglelefteq_\sigma M$, provided that $\mu \nsubseteq \sigma$ and for any fuzzy submodule δ of M, $\mu \cap \delta \subseteq \sigma$ implies that $\delta \subseteq \sigma$.*

Example 3.1. *Consider the module $M = \mathbb{Z}_{16}$ under addition and multiplication modulo 16 over the ring $\mathbb{Z}$.*
Define μ and σ from M to $[0, 1]$ as:

$$\mu(x) = \begin{cases} 1, \ \ if\ x = \{0\}, \\ 0.7, if\ x = \{2, 4, 6, 8, 10, 12, 14\}, \\ 0, otherwise. \end{cases}$$

$$\sigma(x) = \begin{cases} 1, & if\ x = \{0\}, \\ 0.5, if\ x = \{4, 8, 12\}, \\ 0, otherwise. \end{cases}$$

Then μ is a σ-essential submodule of M.

Remark 3.1. *μ is an essential fuzzy essential submodule of M if and only if $\mu \trianglelefteq_{\chi_\theta} M$.*

Theorem 3.1. *Let $\mu, \sigma \in F(M)$. Then $\mu \trianglelefteq_\sigma M$ if and only if $\mu^* \trianglelefteq_{\sigma^*} M$.*

Proof. Assume, that $\mu \trianglelefteq_\sigma M$. Let A be a submodule of M such that $\mu^* \cap A \subseteq \sigma^*$. This implies $(\mu \cap \chi_A)^* \subseteq \sigma^*$ and so $\mu \cap \chi_A \subseteq \sigma$. But, $\mu \trianglelefteq_\sigma M$ and so $\chi_A \subseteq \sigma$ gives $A \subseteq \sigma^*$. Hence, $\mu^* \trianglelefteq_{\sigma^*} M$.

Conversely, assume that $\mu^* \trianglelefteq_{\sigma^*} M$. Let δ be an fuzzy submodule of M such that $\mu \cap \delta \subseteq \sigma$. This implies, $(\mu \cap \delta)^* \subseteq \sigma^*$ and so $\mu^* \cap \delta^* \subseteq \sigma^*$. But, $\mu^* \trianglelefteq_{\sigma^*} M$ and so $\delta^* \subseteq \sigma^*$ consequently, $\delta \subseteq \sigma$. Thus, $\mu \trianglelefteq_\sigma M$. □

Theorem 3.2. *Let $\alpha, \beta \in F(M)$. If $\alpha \trianglelefteq_\sigma M$ and $\beta \trianglelefteq_\sigma M$. Then $\alpha \cap \beta \trianglelefteq_\sigma M$.*

Proof. Let γ be a fuzzy submodule of the module M such that $(\alpha \cap \beta) \cap \gamma \subseteq \sigma$. Then $\alpha \cap (\beta \cap \gamma) \subseteq \sigma$. As $\alpha \trianglelefteq_\sigma M$ gives $\beta \cap \gamma \subseteq \sigma$. Since, $\beta \trianglelefteq_\sigma M$ we conclude that $\gamma \subseteq \sigma$. Thus, $\alpha \cap \beta \trianglelefteq_\sigma M$. □

Definition 3.2. *Let $\mu, \delta \in F(M)$ be such that $\mu \subseteq \delta$. Then μ is called σ-essential of δ, denoted by $\mu \trianglelefteq_\sigma \delta$, provided that $\mu \nsubseteq \sigma$ and for any fuzzy submodule γ of M satisfying $\gamma \subseteq \delta$ and $\mu \cap \gamma \subseteq \sigma$ implies $\gamma \subseteq \sigma$.*

Theorem 3.3. *Let $\mu, \delta \in F(M)$. Then $\mu \trianglelefteq_\sigma \delta$ if and only if $\mu^* \trianglelefteq_{\sigma^*} \delta^*$.*

Proof. Assume, that $\mu \trianglelefteq_\sigma \delta$. Let A be an submodule of δ^* such that $\mu^* \cap A \subseteq \sigma^*$. (I)
Define a fuzzy submodule γ as follows:

$$\gamma(A) = \begin{cases} \sigma(x), & if\ x \in A, \\ 0, & if\ x \notin A. \end{cases}$$

Clearly, $\gamma^* = A$. Hence, (I) becomes, $\mu^* \cap \gamma^* \subseteq \sigma^*$. This implies $(\mu \cap \gamma)^* \subseteq \sigma^*$. Thus $\mu \cap \gamma \subseteq \sigma$. Since, $\mu \trianglelefteq_\sigma \delta$ and so $\gamma \subseteq \sigma$ implies $\gamma^* \subseteq \sigma^*$. Thus, $\mu^* \trianglelefteq_{\sigma^*} \delta^*$.

Conversely, assume that $\mu^* \trianglelefteq_{\sigma^*} \delta^*$. Let $\nu \subseteq \delta$ be such that $\mu \cap \nu \subseteq \sigma$. Hence, $(\mu \cap \nu)^* \subseteq \sigma^*$ and so $\mu^* \cap \nu^* \subseteq \sigma^*$. As, $\mu^* \trianglelefteq_{\sigma^*} \delta^*$ we get $\nu^* \subseteq \sigma^*$ Thus, $\nu \subseteq \sigma$ and so, $\mu \trianglelefteq_\sigma \delta$. □

Theorem 3.4. *Let $\mu, \gamma, \delta \in F(M)$ such that $\mu \subseteq \gamma \subseteq \delta$. Then $\mu \trianglelefteq_\sigma \delta$ if and only if $\mu \trianglelefteq_\sigma \gamma$ and $\gamma \trianglelefteq_\sigma \delta$.*

Proof. Let $\mu \trianglelefteq_\sigma \delta$ and $\xi \subseteq \gamma$ be such that $\mu \cap \xi \subseteq \sigma$. (I)
As $\xi \subseteq \gamma$ and $\gamma \subseteq \delta$ gives $\xi \subseteq \delta$. Thus, from (I) and using $\mu \trianglelefteq_\sigma \delta$ we get $\xi \subseteq \sigma$. This implies $\mu \trianglelefteq_\sigma \gamma$.
Again, let $\nu \subseteq \delta$ be such that $\nu \cap \gamma \subseteq \sigma$. (II)
As $\mu \subseteq \gamma$ implies $\mu \cap \nu \subseteq \gamma \cap \nu$ and so (II) becomes $\mu \cap \nu \subseteq \sigma$. But, given $\mu \trianglelefteq_\sigma \delta$ implies $\nu \subseteq \sigma$ gives $\gamma \trianglelefteq_\sigma \delta$ and thus, $\mu \trianglelefteq_\sigma \gamma$ and $\gamma \trianglelefteq_\sigma \delta$.

Conversely, assume that $\mu \trianglelefteq_\sigma \gamma$ and $\gamma \trianglelefteq_\sigma \delta$. Let $\xi \subseteq \delta$ be such that $\mu \cap \xi \subseteq \sigma$. Then $\mu \cap (\gamma \cap \xi) \subseteq \sigma$. As, $\gamma \cap \xi \subseteq \gamma$ and $\mu \trianglelefteq_\sigma \gamma$ we get $\gamma \cap \xi \subseteq \sigma$. Also, $\gamma \trianglelefteq_\sigma \delta$ implies $\xi \subseteq \sigma$. Thus, $\mu \trianglelefteq_\sigma \delta$. □

We prove the following characterization.

Theorem 3.5. *Let $\sigma_1, \sigma_2, \gamma_1, \gamma_2 \in F(M)$ such that $\gamma_1 \cap \gamma_2 = \sigma_1 \cap \sigma_2$. Also, $\mu_1, \mu_2 \in F(M)$ be such that $\sigma_1 \subseteq \mu_1 \subseteq \gamma_1$ and $\sigma_2 \subseteq \mu_2 \subseteq \gamma_2$. Then $\mu_1 + \mu_2 \trianglelefteq_{\sigma_1+\sigma_2} \gamma_1 + \gamma_2$ if and only if $\mu_1 \trianglelefteq_{\sigma_1} \gamma_1$ and $\mu_2 \trianglelefteq_{\sigma_2} \gamma_2$.*

Proof. Assume that $\mu_1 + \mu_2 \trianglelefteq_{\sigma_1+\sigma_2} \gamma_1 + \gamma_2$.
$\Longrightarrow (\mu_1 + \mu_2)^* \trianglelefteq_{(\sigma_1+\sigma_2)^*} (\gamma_1 + \gamma_2)^*$ by Theorem 3.3.
$\Longrightarrow (\mu_1^* + \mu_2^*) \trianglelefteq_{(\sigma_1^*+\sigma_2^*)} (\gamma_1^* + \gamma_2^*)$ by Proposition 2.1.
$\Longrightarrow \mu_1^* \trianglelefteq_{\sigma_1^*} \gamma_2^*$ and $\mu_2^* \trianglelefteq_{\sigma_2^*} \gamma_2^*$ by Theorem 2.5.
$\Longrightarrow \mu_1 \trianglelefteq_{\sigma_1} \gamma_1$ and $\mu_2 \trianglelefteq_{\sigma_2} \gamma_2$ by Theorem 3.3.

Conversely, assume that $\mu_1 \trianglelefteq_{\sigma_1} \gamma_1$ and $\mu_2 \trianglelefteq_{\sigma_2} \gamma_2$.
$\Longrightarrow \mu_1^* \trianglelefteq_{\sigma_1^*} \gamma_2^*$ and $\mu_2^* \trianglelefteq_{\sigma_2^*} \gamma_2^*$ by Theorem 3.3.
$\Longrightarrow (\mu_1^* + \mu_2^*) \trianglelefteq_{(\sigma_1^*+\sigma_2^*)} (\gamma_1^* + \gamma_2^*)$ by Theorem 2.5.
$\Longrightarrow (\mu_1 + \mu_2)^* \trianglelefteq_{(\sigma_1+\sigma_2)^*} (\gamma_1 + \gamma_2)^*$ by Proposition 2.1.
$\Longrightarrow \mu_1 + \mu_2 \trianglelefteq_{\sigma_1+\sigma_2} \gamma_1 + \gamma_2$ by Theorem 3.3. □

We give relationships between inverse images.

Theorem 3.6. *Let $f : M \to N$ be a module epimorphism where M and N are R-modules. Let μ and σ be fuzzy subsets of N. If $\mu \trianglelefteq_{\sigma} N$, then $f^{-1}(\mu) \trianglelefteq_{f^{-1}(\sigma)} M$.*

Proof. From the proof of Theorem 2.4 we have, $f^{-1}(\mu^*) = (f^{-1}(\mu))^*$. (I)
As $\mu \trianglelefteq_{\sigma} N$ then by Theorem 3.1, $\mu^* \trianglelefteq_{\sigma^*} N$. Hence by Lemma 2.2, $f^{-1}(\mu^*) \trianglelefteq_{f^{-1}(\sigma^*)} M$. Thus, from (I), $(f^{-1}(\mu))^* \trianglelefteq_{(f^{-1}(\sigma))^*} M$. Hence, by Theorem 3.1 $f^{-1}(\mu) \trianglelefteq_{f^{-1}(\sigma)} M$. □

Theorem 3.7. *Let $f : M \to N$ be a module epimorphism where N is an R-module and μ be fuzzy subset of N. Then $f^{-1}(\mu) \trianglelefteq_{\chi_{kerf}} M$ if and only if $\mu \trianglelefteq N$.*

Proof. Assume, $f^{-1}(\mu) \trianglelefteq_{\chi_{kerf}} M$. From the proof of the Theorem 2.4, we have
$f^{-1}(\mu^*) = (f^{-1}(\mu))^*$. (I)
As $f^{-1}(\mu) \trianglelefteq_{\chi_{kerf}} M$ then by Theorem 3.1 $(f^{-1}(\mu))^* \trianglelefteq_{kerf} M$ and so by equation (I) $f^{-1}(\mu^*) \trianglelefteq_{\chi_{kerf}} M$ and by Corollary 2.1, $\mu^* \trianglelefteq N$. Hence, $\mu \trianglelefteq N$.

Conversely, assume $\mu \trianglelefteq N$ implies $\mu^* \trianglelefteq N$. Then by Corollary 2.1, $f^{-1}(\mu^*) \trianglelefteq_{kerf} M$ and by (I), $(f^{-1}(\mu))^* \trianglelefteq_{kerf} M$. Hence, by Theorem 3.1 $f^{-1}(\mu) \trianglelefteq_{\chi_{kerf}} M$. □

We prove some results related to fuzzy quotient submodules.

Theorem 3.8. *Let $\mu, \sigma \in F(M)$. If $\mu \trianglelefteq_{\sigma} M$. Then $\left(\dfrac{\mu+\sigma}{\sigma}\right) \trianglelefteq \left(\dfrac{\chi_M}{\sigma}\right)$.*

Proof. Let δ be a fuzzy submodule such that $\sigma \subseteq \delta$ and $\left(\dfrac{\delta}{\sigma}\right)$ be a fuzzy submodule of $\left(\dfrac{\chi_M}{\sigma}\right)$ such that $\left(\dfrac{\delta}{\sigma}\right) \cap \left(\dfrac{\mu+\sigma}{\sigma}\right) = \chi_{\sigma^*}$. Then $\left(\dfrac{\delta}{\sigma}\right)^* \cap \left(\dfrac{\mu+\sigma}{\sigma}\right)^* = \{\sigma^*\}$ and so $\left(\dfrac{\delta^*}{\sigma^*}\right) \cap \left(\dfrac{\mu^*+\sigma^*}{\sigma^*}\right) = \{\sigma^*\}$. (I)
Let $x \in \mu^* \cap \delta^*$. Then $x + \sigma^* \in (\mu^* \cap \delta^*) + \sigma^* = (\delta^* \cap \mu^*) + \sigma^*$.
Using $\sigma \subseteq \delta$ we get $\sigma^* \subseteq \delta^*$ and by modular law, $x + \sigma^* \in \delta^* \cap (\sigma^* + \mu^*)$.
Thus, $x + \sigma^* \in \left(\dfrac{\delta^*}{\sigma^*}\right) \cap \left(\dfrac{\sigma^*+\mu^*}{\sigma^*}\right) = \{\sigma^*\}$ by (I).
Hence $x + \sigma^* \in \sigma^*$ so $x \in \sigma^*$.
Thus, $\mu^* \cap \delta^* \subseteq \sigma^*$ and so $(\mu \cap \delta)^* \subseteq \sigma^*$. Thus $\mu \cap \delta \subseteq \sigma$.
But given that, $\mu \trianglelefteq_{\sigma} M$ implies $\delta \subseteq \sigma$ and hence, $\left(\dfrac{\mu+\sigma}{\sigma}\right) \trianglelefteq \left(\dfrac{\chi_M}{\sigma}\right)$. □

Theorem 3.9. *Let $\mu, \sigma \in F(M)$ be such that $\sigma \subseteq \mu$. Then $\mu \trianglelefteq_\sigma M$ if and only if $\left(\frac{\mu}{\sigma}\right) \trianglelefteq \left(\frac{\chi_M}{\sigma}\right)$.*

Proof. Assume, that $\mu, \sigma \in F(M)$ and $\sigma \subseteq \mu$. Let $\left(\frac{\alpha}{\sigma}\right)$ be a fuzzy submodule of $\left(\frac{\chi_M}{\sigma}\right)$ such that $\left(\frac{\alpha}{\sigma}\right) \cap \left(\frac{\mu}{\sigma}\right) = \chi_{\{\sigma^*\}}$. Then, $\left(\frac{\alpha}{\sigma}\right)^* \cap \left(\frac{\mu}{\sigma}\right)^* = \{\sigma^*\}$. (I)

Let $x \in \alpha^* \cap \mu^*$. Then $x + \sigma^* \in \left(\frac{\alpha^*}{\sigma^*}\right) \cap \left(\frac{\mu^*}{\sigma^*}\right)$ implies $x + \sigma^* \in \left(\frac{\alpha}{\sigma}\right)^* \cap \left(\frac{\mu}{\sigma}\right)^* = \{\sigma^*\}$.

Hence from (I) we get $x + \sigma^* = \sigma^*$ and so $x \in \sigma^*$. This gives, $\alpha^* \cap \mu^* \subseteq \sigma^*$. (II)

But it is given that $\mu \trianglelefteq_\sigma M$ and by Theorem 3.1 $\mu^* \trianglelefteq_{\sigma^*} M$. So from (II), $\alpha^* \subseteq \sigma^*$. Hence, $\alpha = \sigma$. Therefore, $\left(\frac{\alpha}{\sigma}\right) = \chi_{\{\sigma^*\}}$ and hence, $\left(\frac{\mu}{\sigma}\right) \trianglelefteq \left(\frac{\chi_M}{\sigma}\right)$.

Conversely, assume that $\left(\frac{\mu}{\sigma}\right) \trianglelefteq \left(\frac{\chi_M}{\sigma}\right)$ implies $\left(\frac{\mu}{\sigma}\right)^* \trianglelefteq \left(\frac{\chi_M}{\sigma}\right)^*$. (III)

Let δ be an fuzzy submodule of M such that $\mu \cap \delta \subseteq \sigma$.
This implies $(\mu \cap \delta)^* \subseteq \sigma^*$
$\Rightarrow \mu^* \cap \delta^* \subseteq \sigma^*$
$\Rightarrow \left(\frac{\mu^*}{\sigma^*}\right) \cap \left(\frac{\delta^*}{\sigma^*}\right) \subseteq \{\sigma^*\}$
$\Rightarrow \left(\frac{\mu}{\sigma}\right)^* \cap \left(\frac{\delta}{\sigma}\right)^* \subseteq \{\sigma^*\}$.
Hence from equation (III), $\left(\frac{\delta}{\sigma}\right)^* \subseteq \{\sigma^*\}$.
Now assume that $x \notin \sigma^*$. Then $\left(\frac{\delta}{\sigma}\right)(x + \sigma^*) = 0$. This implies $\vee\{\delta(x + y) | y \in \sigma^*\} = 0$ and so $\delta(x + \theta) = \delta(x) = 0$. Thus $x \notin \delta^*$.
Hence, $\delta^* \subseteq \sigma^*$. Therefore, $\delta \subseteq \sigma$. Hence, $\mu \trianglelefteq_\sigma M$. □

We give an example to illustrate the concept of a fuzzy essential submodule with respect to arbitrary submodule.

Example 3.2. *Let $M = R = \mathbb{Z}$. We define fuzzy subsets μ, σ and δ of $\mathbb{Z}$ as:*

$$\mu(x) = \begin{cases} 1, & if\ x = \{0\}, \\ 0.7, & if\ x \in \{\pm 3, \pm 6, \pm 9, ...\}, \\ 0, & if\ otherwise. \end{cases}$$

$$\sigma(x) = \begin{cases} 1, & if\ x = \{0\}, \\ 0.5, & if\ x = \{\pm 5, \pm 10, \pm 15, ...\}, \\ 0, & if\ otherwise. \end{cases}$$

$$\delta(x) = \begin{cases} 1, & if\ x = \{0\}, \\ 0.2, & if\ otherwise. \end{cases}$$

Then μ, σ and δ are fuzzy submodules of $\mathbb{Z}$. Also, $\mu^* = 3\mathbb{Z}$, $\sigma^* = 5\mathbb{Z}$ and $\delta^* = \mathbb{Z}$. Hence, by Proposition 2.3, $3\mathbb{Z} \trianglelefteq_{5\mathbb{Z}} (3\mathbb{Z} + 5\mathbb{Z})$, that is $\mu^* \trianglelefteq_{\sigma^*} (\mu^* + \sigma^*)$, where $\mu^* + \sigma^* = \delta^*$ and so $\mu^* \trianglelefteq_{\sigma^*} \delta^*$. Hence, by Theorem 3.3, $\mu \trianglelefteq_\sigma \delta$.

4. Fuzzy complement with respect to arbitrary fuzzy submodule

In this section we introduce the concept of a fuzzy complement with respect to an arbitrary fuzzy submodule and prove some results.

Definition 4.1. *Let μ be a fuzzy submodule and σ be a proper fuzzy submodule of an R-module M. A fuzzy submodule β of M is called σ-complement to μ, if β is maximal with respect to the property $\mu \cap \beta \subseteq \sigma$.*

Definition 4.2. *Let μ be a fuzzy submodule and σ be a proper submodule of a fuzzy submodule δ. Then a fuzzy submodule β of δ is called σ-complement to μ in δ if β is maximal with respect to $\mu \cap \beta \subseteq \sigma$.*

Remark 4.1. *Let M be an R-module. If $\sigma = \chi_\theta$, then fuzzy submodule β is σ-complement to μ if and only if β is complement for μ in M.*

Theorem 4.1. *Let $\mu, \delta \in F(M)$ and $\sigma = \mu \cap \delta$. If μ is σ-complement to δ. Then $\left(\dfrac{\delta + \mu}{\mu}\right) \trianglelefteq \left(\dfrac{\chi_M}{\mu}\right)$.*

Proof. Let ξ be a fuzzy submodule of M such that $\mu \subseteq \xi$ and $\left(\dfrac{\xi}{\mu}\right) \cap \left(\dfrac{\delta+\mu}{\mu}\right) = \chi_{\{\mu^*\}}$.
Then, $\left(\dfrac{\xi}{\mu}\right)^* \cap \left(\dfrac{\delta+\mu}{\mu}\right)^* = \{\mu^*\}$ implies $\left(\dfrac{\xi^*}{\mu^*}\right) \cap \left(\dfrac{\delta^*+\mu^*}{\mu^*}\right) = \{\mu^*\}$. (I)
Let $x \in \xi^* \cap \delta^*$. Therefore, $x + \mu^* \in (\xi^* \cap \delta^*) + \mu^* = \xi^* \cap (\delta^* + \mu^*)$ by using modular law. Hence $x + \mu^* \in \left(\dfrac{\xi^*}{\mu^*}\right) \cap \left(\dfrac{\delta^*+\mu^*}{\mu^*}\right) = \{\mu^*\}$ by (I). Thus $x + \mu^* \in \mu^*$ and so $x \in \mu^*$. Thus $\xi^* \cap \delta^* \subseteq \mu^*$. Consequently, $(\xi \cap \delta)^* \subseteq \mu^*$. Thus $\xi \cap \delta \subseteq \mu$. Hence $\xi \cap \delta \subseteq \mu \cap \delta = \sigma$. By assumption, $\mu = \xi$ and therefore, $\left(\dfrac{\xi}{\mu}\right)^* = \{\mu^*\}$. Hence, $\left(\dfrac{\delta+\mu}{\mu}) \trianglelefteq (\dfrac{\chi_M}{\mu}\right)$. $\square$

Theorem 4.2. *Let $\mu, \beta, \sigma \in F(M)$. If $\left(\dfrac{\mu}{\sigma}\right)$ is fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$. Then μ is σ-complement to β in M. The converse is true if $\sigma \subseteq \mu \cap \beta$.*

Proof. Let $\left(\dfrac{\mu}{\sigma}\right)$ be fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$.
Then, $\left(\dfrac{\mu}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.
$\Longrightarrow \left(\dfrac{\mu}{\sigma}\right)^* \cap \left(\dfrac{\beta}{\sigma}\right)^* = \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\mu^*}{\sigma^*}\right) \cap \left(\dfrac{\beta^*}{\sigma^*}\right) = \{\sigma^*\}$.
$\Longrightarrow \mu^* \cap \beta^* = \sigma^*$.
$\Longrightarrow \mu \cap \beta = \sigma$
$\Longrightarrow \mu$ is σ-complement to β in M.
Suppose that, $\mu \subseteq \alpha \subseteq \sigma$ is such that $\alpha \cap \beta \subseteq \sigma$.
$\Longrightarrow \alpha^* \cap \beta^* \subseteq \sigma^*$.
$\Longrightarrow \left(\dfrac{\alpha^*}{\sigma^*}\right) \cap \left(\dfrac{\beta^*}{\sigma^*}\right) \subseteq \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\alpha}{\sigma}\right)^* \cap \left(\dfrac{\beta}{\sigma}\right)^* \subseteq \{\sigma^*\}$.

$\Longrightarrow \left(\left(\dfrac{\alpha}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right)\right)^* \subseteq \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\alpha}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.
$\Longrightarrow \left(\dfrac{\alpha}{\sigma}\right)$ is fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$, a contradiction to the assumption.
Therefore, $\left(\dfrac{\mu}{\sigma}\right) = \left(\dfrac{\alpha}{\sigma}\right)$
$\Longrightarrow \left(\dfrac{\mu}{\sigma}\right)^* = \left(\dfrac{\alpha}{\sigma}\right)^*$.
$\Longrightarrow \dfrac{\mu^*}{\sigma^*} = \dfrac{\alpha^*}{\sigma^*}$
$\Longrightarrow \mu^* = \alpha^*$
$\Longrightarrow \mu = \alpha$. Hence, μ is σ-complement to β in M.

Conversely, μ is σ-complement to β in M and $\sigma \subseteq \mu \cap \beta$ then $\mu \cap \beta = \sigma$. Hence $(\mu \cap \beta)^* = (\sigma)^*$.
$\Longrightarrow \mu^* \cap \beta^* = \sigma^*$.
$\Longrightarrow \left(\dfrac{\mu^*}{\sigma^*}\right) \cap \left(\dfrac{\beta^*}{\sigma^*}\right) = \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\mu}{\sigma}\right)^* \cap \left(\dfrac{\beta}{\sigma}\right)^* = \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\mu}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.
$\Longrightarrow \left(\dfrac{\mu}{\sigma}\right)$ is fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$.
Suppose that $\left(\dfrac{\mu}{\sigma}\right) \subseteq \left(\dfrac{\alpha}{\sigma}\right) \subseteq \left(\dfrac{\chi_M}{\sigma}\right)$ such that $\left(\dfrac{\alpha}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.
$\Longrightarrow \left(\dfrac{\alpha}{\sigma}\right)^* \cap \left(\dfrac{\beta}{\sigma}\right)^* = \{\sigma^*\}$.
$\Longrightarrow \left(\dfrac{\alpha^*}{\sigma^*}\right) \cap \left(\dfrac{\beta^*}{\sigma^*}\right) = \{\sigma^*\}$.
$\Longrightarrow \alpha^* \cap \beta^* = \sigma^*$
$\Longrightarrow \alpha \cap \beta = \sigma$.
But, given μ is σ-complement to β in M, therefore, $\mu = \alpha$ and thus, $\left(\dfrac{\mu}{\sigma}\right) = \left(\dfrac{\alpha}{\sigma}\right)$. Thus, $\left(\dfrac{\mu}{\sigma}\right)$ is fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$. □

Theorem 4.3. *Let $\mu, \beta, \sigma \in F(M)$ such that $\left(\dfrac{\chi_M}{\sigma}\right) = \left(\dfrac{\mu}{\sigma}\right) \oplus \left(\dfrac{\beta}{\sigma}\right)$. Then μ is σ-complement to β in M.*

Proof. As $\left(\dfrac{\chi_M}{\sigma}\right) = \left(\dfrac{\mu}{\sigma}\right) \oplus \left(\dfrac{\beta}{\sigma}\right)$, we get $\left(\dfrac{\mu}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.

Let $\left(\dfrac{\mu}{\sigma}\right) \subseteq \left(\dfrac{\delta}{\sigma}\right) \subseteq \left(\dfrac{\chi_M}{\sigma}\right)$ be such that $\left(\dfrac{\delta}{\sigma}\right) \cap \left(\dfrac{\beta}{\sigma}\right) = \chi_{\{\sigma^*\}}$.
Since, $\left(\dfrac{\chi_M}{\sigma}\right) = \left(\dfrac{\mu}{\sigma}\right) + \left(\dfrac{\beta}{\sigma}\right)$ and $\left(\dfrac{\mu}{\sigma}\right) \subseteq \left(\dfrac{\delta}{\sigma}\right)$ we get $\left(\dfrac{\chi_M}{\sigma}\right) = \left(\dfrac{\delta}{\sigma}\right) \oplus \left(\dfrac{\beta}{\sigma}\right)$, so

$\left(\dfrac{\mu}{\sigma}\right) = \left(\dfrac{\delta}{\sigma}\right)$. Thus, $\left(\dfrac{\mu}{\sigma}\right)$ is fuzzy complement to $\left(\dfrac{\beta}{\sigma}\right)$ in $\left(\dfrac{\chi_M}{\sigma}\right)$. Hence by Theorem 4.2, μ is σ - complement to β in M. □

Theorem 4.4. *Let μ, β be fuzzy submodules of M. Then μ is complement to β in M if and only if $\mu \oplus \beta \trianglelefteq_\mu M$.*

Proof. Assume that μ is complement to β in M, then by Theorem 3.3.3 of [3], $\mu \oplus \beta \trianglelefteq M$. But by Theorem 3.3.10 of [3] , μ is closed in M. Hence, $\left(\dfrac{\mu \oplus \beta}{\mu}\right) \trianglelefteq \left(\dfrac{\chi_M}{\mu}\right)$ by Theorem 3.3.10 of [3]. Thus by Theorem 3.9 , $\mu \oplus \beta \trianglelefteq_\mu M$.

Conversely, Let $\mu \oplus \beta \trianglelefteq_\mu M$ then $\mu \cap \beta = \chi_\theta$.
Now by Theorem 3.9, $\left(\dfrac{\mu \oplus \beta}{\mu}\right) \trianglelefteq \left(\dfrac{\chi_M}{\mu}\right)$.
Let α be a fuzzy submodule of M such that $\mu \subseteq \alpha$ and $\alpha \cap \beta = \chi_\theta$.
Now, $\left(\dfrac{\alpha}{\mu}\right) \subseteq \left(\dfrac{\chi_M}{\mu}\right)$ and by modular law,

$(\mu^* \oplus \beta^*) \cap \alpha^* = \mu^* \cap \alpha^* + \beta^* \cap \alpha^* = \mu^* + \beta^* \cap \alpha^* = \mu^* + \{\theta\} = \mu^*$. Hence
$\left(\dfrac{\mu^* \oplus \beta^*}{\mu^*}\right) \cap \left(\dfrac{\alpha^*}{\mu^*}\right) = \{\mu^*\}$ gives $\left(\dfrac{\mu \oplus \beta}{\mu}\right)^* \cap \left(\dfrac{\alpha}{\mu}\right)^* = \{\mu^*\}$ and thus,
$\left(\dfrac{\mu \oplus \beta}{\mu}\right) \cap \left(\dfrac{\alpha}{\mu}\right) = \chi_{\{\mu^*\}}$. But, $\left(\dfrac{\mu \oplus \beta}{\mu}\right) \trianglelefteq \left(\dfrac{\chi_M}{\mu}\right)$ gives $\left(\dfrac{\alpha}{\mu}\right) = \chi_{\{\mu^*\}}$ and so
$\left(\dfrac{\alpha}{\mu}\right)^* = \{\mu^*\}$.
Let $x \notin \mu^*$.
$\Longrightarrow \left(\dfrac{\alpha}{\mu}\right)(x + \mu^*) = 0$.
$\Longrightarrow \vee\{\alpha(x+y) | y \in \mu^*\} = 0$.
$\Longrightarrow \alpha(x+\theta) = \alpha(x) = 0$.
$\Longrightarrow x \notin \alpha^*$.
$\Longrightarrow \alpha^* \subseteq \mu^*$.
$\Longrightarrow \alpha = \mu$.
Hence, μ is complement to β in M. □

Theorem 4.5. *Let μ, α and β be fuzzy submodules of M. If $\mu \trianglelefteq M$ and β is fuzzy complement for α in M. Then $\mu + \alpha \trianglelefteq_\alpha M$.*

Proof. Let δ be an fuzzy submodule of M such that $\alpha \subseteq \delta$. Also, $\left(\dfrac{\delta}{\alpha}\right)$ be a fuzzy submodule of $\left(\dfrac{\chi_M}{\alpha}\right)$ such that $\left(\dfrac{\mu \oplus \alpha}{\alpha}\right) \cap \left(\dfrac{\delta}{\alpha}\right) = \chi_{\{\alpha^*\}}$
$\Longrightarrow \left(\dfrac{\mu \oplus \alpha}{\alpha}\right)^* \cap \left(\dfrac{\delta}{\alpha}\right)^* = \{\alpha^*\}$.
$\Longrightarrow \left(\dfrac{(\mu \oplus \alpha)^*}{\alpha^*}\right) \cap \left(\dfrac{\delta^*}{\alpha^*}\right) = \{\alpha^*\}$.
$\Longrightarrow (\mu + \alpha)^* \cap \delta^* = \alpha^*$.
$\Longrightarrow (\mu^* + \alpha^*) \cap \delta^* = \alpha^*$.
$\Longrightarrow \mu^* \cap \delta^* + \alpha^* \cap \delta^* = \alpha^*$.
$\Longrightarrow \mu^* \cap \delta^* + \alpha^* = \alpha^*$.
$\Longrightarrow \mu^* \cap \delta^* \subseteq \alpha^*$.
$\Longrightarrow \mu \cap \delta \subseteq \alpha$.

Hence, $(\mu \cap \delta) \cap \beta \subseteq \alpha \cap \beta = \chi_\theta$ because β is fuzzy complement for α in M. Implies $\mu \cap (\delta \cap \beta) = \chi_\theta$. But, $\mu \trianglelefteq M$ gives $\delta \cap \beta = \chi_\theta$. But given β is fuzzy complement for α in M and so $\alpha = \delta$, thus $\left(\dfrac{\mu \oplus \alpha}{\alpha}\right) \trianglelefteq \left(\dfrac{\chi_M}{\alpha}\right)$ and hence, by Theorem 3.9, $\mu + \alpha \trianglelefteq_\alpha M$. $\square$

5. Conclusion

In this paper fuzzy essential submodules with respect to an arbitrary fuzzy submodule and fuzzy complement submodules with respect to arbitrary fuzzy submodule have been studied. This concept of fuzzy essential submodules with respect to an arbitrary fuzzy submodule can be extended to fuzzy small-essential submodules with respect to an arbitrary fuzzy submodule.

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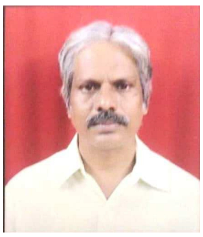
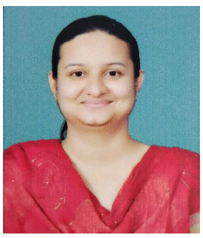


Shriram Khanderao Nimbhorkar is a retired professor of mathematics from Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India. His area of research is Algebra in particular, rings with involution, Lattice Theory and Graph Theory.



Jyoti Ashok Khubchandani completed her M.Sc. degree in mathematics from Savitribai Phule Pune University, Pune, India in 2009. She is working as an assistant professor of Mathematics in the Department of Applied Sciences, Dr. Vitthalrao Vikhe Patil College of Engineering, Ahmednagar, India. Her research area is Algebra and Fuzzy set theory.