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Harmonic function for which the second dilatation is α-spiral

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Abstract

Let $f = h + \overline{g}$ be a harmonic function in the unit disc $\mathbb{D}$. We will give some properties of f under the condition the second dilatation is α-spiral.

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1 Introduction

A planar harmonic mapping in the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected, the mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. As usual, we call h the analytic part of f and g the co-analytic part of f. An elegant and complete account of the theory of planar harmonic mapping is given in Duren's monograph [1].

Lewy [2] proved in 1936 that the harmonic function f is locally univalent in a simply connected domain $\mathbb{D}_1$ if and only if its Jacobian

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$$

is different from zero in $\mathbb{D}_1$. In view of this result, locally univalent harmonic mappings in the unit disc are either sense-reversing if

$$|g'(z)| > |h'(z)|$$

in $\mathbb{D}_1$ or sense-preserving if

$$|g'(z)| < |h'(z)|$$

in $\mathbb{D}_1$. Throughout this paper, we will restrict ourselves to the study of sense-preserving harmonic mappings. However, since f is sense-preserving if and only if $\overline{f}$ is sense-reserving, all the results obtained in this article regarding sense-preserving harmonic mappings can be adapted to sense-reversing ones. Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc and the second-complex dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ in $\mathbb{D}$; therefore, we can take $h(z) = z + a_2 z^2 + \cdots$,

$g(z) = b_1 z + b_2 z^2 + \cdots$. Thus, the class of all harmonic mappings being sense-preserving in the unit disc can be defined by

$$S_H = \{f = h(z) + \overline{g(z)} | h(z) = z + a_2 z^2 + \cdots,$$
$$g(z) = b_1 z + b_2 z^2 + \cdots, f \text{ sense-preserving}\}.$$

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by P the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ which are regular in $\mathbb{D}$ such that

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$ for all $z \in \mathbb{D}$.

Next, let S^* denote the family of functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ which are regular in $\mathbb{D}$ such that

$$z\frac{s'(z)}{s(z)} = p(z) \tag{1.2}$$

for some $p(z) \in P$ for all $z \in \mathbb{D}$.

Let $s_1(z) = z + \alpha_2 z^2 + \alpha_3 z_3 + \cdots$ and $s_2(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Now, we consider the following class of harmonic mappings in the plane:

$$S^*_{\mathrm{HPST}}(\alpha) = \left\{f = h(z) + \overline{g(z)} | f \in S_H, h(z) \in S^*, \right.$$
$$\left. \mathrm{Re}\big(e^{i\alpha} w(z)\big) = \mathrm{Re}\left(e^{i\alpha}\frac{g'(z)}{h'(z)}\right) > 0, |\alpha| < \frac{\pi}{2}\right\}. \tag{1.3}$$

In the present paper, we will investigate the class $S^*_{\mathrm{HPST}}(\alpha)$.

We will need the following lemma and theorem in the sequel.

Theorem 1.1 ([3, 4]) *Let $h(z)$ be an element of S^*, then*

$$\frac{r}{(1+r)^2} \le |h(z)| \le \frac{r}{(1-r)^2},$$

for all $|z| = r < 1$.

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^3}.$$

These inequalities are sharp because the extremal function is $h(z) = \frac{z}{(1-z)^2}$.

Lemma 1.2 ([2, 5]) *Let $h(z)$ and $g(z)$ be regular in $\mathbb{D}$, $h(z)$ map $|z| < 1$ onto a many-sheeted starlike region,* $\mathrm{Re}(e^{i\alpha}\frac{g'(z)}{h'(z)}) > 0$, $|\alpha| < \frac{\pi}{2}$ *for $|z| < 1$. $h(0) = g(0) = 0$. Then* $\mathrm{Re}(e^{i\alpha}\frac{g(z)}{h(z)}) > 0$ *for $|z| < 1$.*

2 Main results

Lemma 2.1 *Let* $f = h(z) + \overline{g(z)}$ *be an element of* $S^*_{\mathrm{HPST}}(\alpha)$ *then*

$$\frac{|b_1| - r}{1 - |b_1|r} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + r}{1 + |b_1|r} \tag{2.1}$$

for all $|z| = r < 1$. *This inequality is sharp because the extremal function is*

$$e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{z + b}{1 + \overline{b}z},$$

where $b = e^{i\alpha}b_1$.

Proof Since

$$\begin{aligned}
&w(z) = \frac{g'(z)}{h'(z)} = \frac{(b_1 z + b_2 z^2 + \cdots)'}{(z + a_2 z^2 + \cdots)'} = \frac{b_1 + 2b_2 z + \cdots}{1 + 2a_2 z + \cdots},\\
&W(z) = e^{i\alpha}w(z) = e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{e^{i\alpha}b_1 + 2e^{i\alpha}b_2 z + \cdots}{1 + 2a_2 z + \cdots} \quad\Rightarrow\quad W(0) = e^{i\alpha}b_1 = b,\\
&\left|W(z)\right| = \left|e^{i\alpha}w(z)\right| = \left|e^{i\alpha}\right|\left|w(z)\right| = \left|w(z)\right| < 1,
\end{aligned}$$

then the function

$$\phi(z) = \frac{W(z) - W(0)}{1 - \overline{W(0)}W(z)} = \frac{W(0) - b}{1 - \overline{b}W(0)} = \frac{b - b}{1 - b^2} = 0$$

satisfies the condition of the Schwarz lemma. Using the definition of subordination, we have

$$W(z) = e^{i\alpha}w(z) = e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{b + \phi(z)}{1 + \overline{b}\phi(z)} \quad\Leftrightarrow\quad e^{i\alpha}\frac{g'(z)}{h'(z)} \prec \frac{b + z}{1 + \overline{b}z}.$$

On the other hand, the transformation $(\frac{b+z}{1+\overline{b}z})$ maps $|z| < 1$ onto the disc with the center

$$C(r) = \left(\frac{\alpha_1(1 - r^2)}{1 - |b_1|^2 r^2}, \frac{\alpha_2(1 - r^2)}{1 - |b_1|^2 r^2}\right), \quad b = \alpha_1 + i\alpha_2$$

and the radius

$$\rho(r) = \frac{(1 - |b_1|^2)r}{1 - |b_1|^2 r^2}.$$

Therefore, we can write

$$\left|e^{i\alpha}\frac{g'(z)}{h'(z)} - \frac{b_1(1 - r^2)}{1 - |b_1|^2 r^2}\right| \leq \frac{(1 - |b_1|^2)r}{1 - |b_1|^2 r^2} \tag{2.2}$$

which gives (2.1). □

Corollary 2.2 *Let $f \in S^*_{\mathrm{HPST}}(\alpha)$, then*

$$\frac{r(|b_1|-r)}{(1+r)^2(1-|b_1|r)} \le |g(z)| \le \frac{r(|b_1|+r)}{(1-r)^2(1+|b_1|r)}, \tag{2.3}$$

$$\frac{(1-r)(|b_1|-r)}{(1+r)^3(1-|b_1|r)} \le |g'(z)| \le \frac{(1+r)(|b_1|+r)}{(1-r)^3(1+|b_1|r)} \tag{2.4}$$

for all $|z| = r < 1$.

Proof Using Lemma 1.2 and Lemma 2.1, then we can write

$$|h(z)|\frac{|b_1|-r}{1-|b_1|r} \le |g(z)| \le |h(z)|\frac{|b_1|+r}{1+|b_1|r}, \tag{2.5}$$

$$|h'(z)|\frac{|b_1|-r}{1-|b_1|r} \le |g'(z)| \le |h'(z)|\frac{|b_1|+r}{1+|b_1|r}. \tag{2.6}$$

If we use Theorem 1.1 in the inequalities (2.5) and (2.6), we get (2.3) and (2.4). □

Corollary 2.3 *Let $f = h(z) + \overline{g(z)}$ be an element of $S^*_{HPTS}(\alpha)$, then*

$$\frac{(1-|b_1|^2)(1-r)^3}{(1+r)^5(1+|b_1|r)^2} \le J_{f(z)} \le \frac{(1-|b_1|^2)(1+r)^3}{(1-r)^5(1+|b_1|r)^2} \tag{2.7}$$

for all $|z| = r < 1$.

Proof Since

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2\bigl(1 - |w(z)|^2\bigr), \tag{2.8}$$

using Lemma 2.1 and Theorem 1.1 in the equality (2.8) and after simple calculations, we get (2.7). □

Corollary 2.4 *If $f = h(z) + \overline{g(z)}$ is an element of $S^*_{HPTS}(\alpha)$, then*

$$\begin{aligned}
&\frac{1}{(1+a)^3(-1+r)^2}\bigl(2(1+a)\bigl(a(-2+r)-r\bigr)r + (-1+a)^2(-1+r)^2\log(1-r) \\
&\quad - (-1+a)^2(-1+r)^2\log(1+ar)\bigr) \le |f(z)| \le \frac{1}{(1+a)^2(-1+r^2)^2} \\
&\quad \bigl(-2r\bigl(-1+r+4ar+r^2+r^3-a^2\bigl(-1+r(-3+r+r^2)\bigr)\bigr) \\
&\quad + (-1+a)^2\bigl(-1+r^2\bigr)^2\log(1-r) - (-1+a)^2\bigl(-1+r^2\bigr)^2\log(1+ar)\bigr),
\end{aligned} \tag{2.9}$$

where $a = |b_1|$ for all $|z| = r < 1$.

Proof Using Corollary 2.2 and Theorem 1.1, we obtain

$$\bigl(|h'(z)| - |g'(z)|\bigr) \ge \frac{(1-r^4)(1+|b_1|r) - (1+r^4)(|b_1|+r)}{(1-r)^3(1+r)^3(1+|b_1|r)},$$

and

$$\left(\left|h'(z)\right| + \left|g'(z)\right|\right) \leq \frac{(1+r)^2(1+|b_1|)}{(1-r)^3(1+|b_1|r)}.$$

Therefore, we have

$$\begin{aligned}&\left(\left|h'(z)\right| - \left|g'(z)\right|\right)|dz| \leq |df| \leq \left(\left|h'(z)\right| + \left|g'(z)\right|\right)|dz| \\ \Rightarrow\quad &\frac{(1-r)^4(1+|b_1|r) - (1+r)^4(|b_1|+r)}{(1-r)^3(1+r)^3(1+|b_1|r)}\,dr \leq |df| \leq \frac{(1+r)^2(1+|b_1|)}{(1-r)^3(1+|b_1|r)}\,dr. \end{aligned} \tag{2.10}$$

Integrating the last inequality (2.10), we get (2.9). □

Theorem 2.5 *Let $f = h(z) + \overline{g(z)}$ be an element of $S^*_{HPTS}(\alpha)$, then*

$$\sum_{k=1}^{n} |A_k|^2 \leq |t+1|^2 + \sum_{k=1}^{n} |B_k|^2 \tag{2.11}$$

where $A_k = (k+1)(\frac{b_{k+1}}{b_1} - a_{k+1})$; $B_k = (k+1)(\frac{b_{k+1}}{b_1} + ta_{k+1})$; a_k and b_k are the coefficients of the functions $h(z)$ and $g(z)$; $k = 1, 2, 3, \ldots, n$; $t = 2s - 1$; $s = e^{-i\alpha}\cos\alpha$.

Proof Since

$$g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots \quad \Rightarrow \quad g'(z) = b_1 + 2b_2 z + 3b_3 z^2 + \cdots.$$

We denote by $G(z) = \frac{1}{b_1} g(z)$

$$\begin{aligned} &G'(z) = \frac{1}{b_1} g'(z) = 1 + 2\frac{b_2}{b_1} z + 3\frac{b_3}{b_1} z^2 + \cdots, \qquad h(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \\ &h'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots, \end{aligned}$$

then we have

$$\begin{cases} \frac{1}{\cos\alpha}\left(e^{i\alpha}\frac{\frac{1}{b_1}g'(z)}{h'(z)} - i\sin\alpha\right) = p(z) \quad \Leftrightarrow \quad e^{i\alpha}\frac{\frac{1}{b_1}g'(z)}{h'(z)} = \cos\alpha\, p(z) + i\sin\alpha, \\ \quad \Leftrightarrow \quad \frac{\frac{1}{b_1}g'(z)}{h'(z)} = 1 + e^{-i\alpha}\cos\alpha(p(z) - 1). \end{cases} \tag{2.12}$$

Since $p(z)$ is in P, there is a function $\phi(z)$ satisfying the conditions of the Schwarz lemma such that

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} \quad \Leftrightarrow \quad p(z) - 1 = \frac{2\phi(z)}{1-\phi(z)}. \tag{2.13}$$

Using this equation in (2.12) and after the following calculations given above

$$\frac{\frac{1}{b_1}g'(z)}{h'(z)} = 1 + e^{-i\alpha}\cos\alpha\left(p(z) - 1\right) = 1 + s\left(\frac{2\phi(z)}{1-\phi(z)}\right) \quad \Rightarrow,$$

we get the following equality:

$$\frac{1}{b_1}g'(z) - h'(z) = \left(th'(z) + \frac{1}{b_1}g'(z)\right). \tag{2.14}$$

If $\phi(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$, we have

$$\sum_{k=1}^{n} A_k z^k + \sum_{k=n+1}^{\infty} D_k z^k = \left[(1+t) + \sum_{k=1}^{n} B_k z^k\right]\phi(z), \tag{2.15}$$

where

$$\sum_{k=n+1}^{\infty} D_k z^k = \sum_{k=n+1}^{\infty} A_k z^k - \left(c_1 B_n z^{n+1} + c_1 B_{n+1} z^{n+2} + \cdots\right).$$

Therefore, the equality (2.15) can be considered in the following form:

$$F(z) = G(z)\phi(z). \tag{2.16}$$

Using the Clunie method [6], then we can write

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|G\left(re^{i\theta}\right)\right|^2 d\theta,$$

which gives

$$\sum_{k=1}^{n} |A_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |D_k|^2 r^{2k} \le \left(|t+1|^2 + \sum_{k=1}^{n} |B_k|^2 r^{2k}\right). \tag{2.17}$$

Eventually, we will let $r \to 1^-$, then we have

$$\sum_{k=1}^{n} |A_k|^2 \le |t+1|^2 + \sum_{k=1}^{n} |B_k|^2.$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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