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Some properties of starlike harmonic mappings

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Abstract

A fundamental result of this paper shows that the transformation

$$F = \frac{az(h(\frac{z+a}{1+\overline{a}z}) + \overline{g(\frac{z+a}{1+\overline{a}z})})}{(h(a) + \overline{g(a)})(z+a)(1+\overline{a}z)}$$

defines a function in $S^0_{HS^*}$ whenever $f = h(z) + \overline{g(z)}$ is $S^0_{HS^*}$, and we will give an application of this fundamental result.

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1 Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$; denote by $\mathcal{P}$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

regular in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Next, let $s_1(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ and $s_2(z) = z + d_2 z^2 + d_3 z^3 + \cdots$ be regular functions in $\mathbb{D}$, if there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinated to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Moreover, univalent harmonic functions are generalizations of univalent regular functions; the point of departure is the canonical representation

$$f = h(z) + \overline{g(z)}, \quad g(0) = 0 \tag{1.2}$$

of a harmonic function f in the unit disc $\mathbb{D}$ as the sum of a regular function $h(z)$ and the conjugate of a regular function $g(z)$. With the convention that $g(0) = 0$, the representation is unique. The power series expansions of $h(z)$ and $g(z)$ are denoted by

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.3}$$

If f is a sense-preserving harmonic mapping of $\mathbb{D}$ onto some other region, then, by Lewy theorem, its Jacobian is strictly positive, *i.e.*,

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0. \tag{1.4}$$

Equivalently [1], the inequality $|g'(z)| < |h'(z)|$ holds for all $z \in \mathbb{D}$. This shows, in particular, that $h'(z) \neq 0$, so there is no loss of generality in supposing that $h(0) = 0$ and $h'(0) = 1$. The class of all sense-preserving harmonic mappings of the disc with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by S_H. Thus S_H contains the standard class S of regular univalent functions. Although the regular part $h(z)$ of a function $f \in S_H$ is locally univalent, it will become apparent that it need not be univalent. The class of functions $f \in S_H$ with $g'(0) = 0$ will be denoted by S_H^0. At the same time, we note that S_H is a normal family and S_H^0 is a compact normal family [2].

Finally, let $f = h(z) + \overline{g(z)}$ be an element S_H (or S_H^0). If f satisfies the condition

$$\frac{\partial}{\partial\theta}\left(\operatorname{Arg} f\left(re^{i\theta}\right)\right) = \operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) > 0 \tag{1.5}$$

then f is called harmonic starlike function. The class of such functions is denoted by S_{HS^*} (or $S_{HS^*}^0$). Also, let $f = h(z) + \overline{g(z)}$ be an element S_H (or S_H^0). If f satisfies the condition

$$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\left(\operatorname{Arg} f\left(re^{i\theta}\right)\right)\right) = \operatorname{Re}\left(\frac{z(zh'(z))' - \overline{z(zg'(z))'}}{zh'(z) + \overline{zg'(z)}}\right) > 0, \tag{1.6}$$

then f is called a convex harmonic function. The class of convex harmonic functions is denoted by S_{HC} (or S_{HC}^0).

For the aim of this paper, we will need the following lemma and theorem.

Lemma 1.1 ([2, p.51]) *If $f = h(z) + \overline{g(z)} \in S_{HC}$, then there exist angles α and β such that*

$$\operatorname{Re}\left[\left(e^{i\alpha}h'(z) + e^{-i\alpha}g'(z)\right)\left(e^{i\beta} - e^{-i\beta}z^2\right)\right] > 0 \tag{1.7}$$

for all $z \in \mathbb{D}$.

Theorem 1.2 ([2, p.108]) *If $f = h(z) + \overline{g(z)} \in S_H$ is a starlike function and if $H(z)$ and $G(z)$ are the regular functions defined by $zH'(z) = h(z)$, $zG'(z) = -g(z)$, $H(0) = G(0) = 0$, then $F = H(z) + \overline{G(z)}$ is a convex function.*

2 Main results

Lemma 2.1 *Let $f = h(z) + \overline{g(z)}$ be an element of S_{HC}^0, then*

$$\frac{G(\alpha,\beta,-r)}{(1+r^2)^2} \le \left|h'(z) + e^{-2i\alpha}g'(z)\right| \le \frac{G(\alpha,\beta,r)}{(1-r^2)^2}, \tag{2.1}$$

where

$$G(\alpha,\beta,r) = 2\cos(\alpha+\beta)r + \sqrt{1+\left[2\cos(\alpha+\beta)\right]r^2 + r^4},$$
$$\cos(\alpha+\beta) > 0.$$

Proof Using Theorem 1.2, we write

$$p(z) = \left(e^{i\alpha}h'(z) + e^{-i\alpha}g'(z)\right)\left(e^{i\beta} - e^{-i\beta}z^2\right), \quad \operatorname{Re} p(z) > 0,$$
$$p(0) = \left(e^{i\alpha}h'(0) + e^{-i\alpha}g'(0)\right)\left(e^{-i\beta} - e^{i\beta}0^2\right) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

On the other hand, since

$$p(z) = \left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right] + p_1 z + p_2 z^2 + \cdots$$

is regular and satisfies the condition $\operatorname{Re} p(z) > 0$, with $\cos(\alpha+\beta) > 0$, the function

$$p_1(z) = \frac{1}{\cos(\alpha+\beta)}\left[p(z) - i\sin(\alpha+\beta)\right] \tag{2.2}$$

is an element of $\mathcal{P}$ [4]. Therefore, we have

$$\left|p_1(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}. \tag{2.3}$$

After simple calculations from (2.3), we get (2.1). □

Corollary 2.2 *Let $f = h(z) + \overline{g(z)}$ be an element of S^0_{HC}, then*

$$\frac{G(\alpha,\beta,-r)}{(1+r^2)^2(1-r)} \le \left|h'(z)\right| \le \frac{G(\alpha,\beta,r)}{(1-r)^3(1+r)^2}, \tag{2.4}$$
$$\frac{|w(z)|G(\alpha,\beta,-r)}{(1+r^2)^2(1-r)} \le \left|g'(z)\right| \le \frac{rG(\alpha,\beta,r)}{(1-r)^3(1+r)^2}. \tag{2.5}$$

Proof Since $f \in S^0_{HC}$, then $g'(z) = h'(z)w(z)$ and the second dilatation $w(z)$ satisfies the condition of Schwarz lemma, then the inequality (2.1) can be written in the form

$$\frac{G(\alpha,\beta,-r)}{|1+e^{-2i\alpha}w(z)|(1+r^2)^2(1-r)} \le \left|h'(z)\right| \le \frac{G(\alpha,\beta,r)}{|1+e^{-2i\alpha}w(z)|(1-r^2)^2} \tag{2.6}$$

which is given in (2.4) and (2.5). □

Corollary 2.3 *Let $f = h(z) + g(z)$ be an element of S^0_{CH}, then*

$$\frac{rG(\alpha,\beta,-r)}{(1+r^2)^2(1-r)} \le \left|h(z)\right| \le \frac{rG(\alpha,\beta,r)}{(1-r)^3(1+r)^2}, \tag{2.7}$$
$$\frac{|w(z)|rG(\alpha,\beta,-r)}{(1+r^2)^2(1-r)} \le \left|g(z)\right| \le \frac{r^2G(\alpha,\beta,r)}{(1-r)^3(1+r)^2}. \tag{2.8}$$

Proof Using Theorem 1.2 and Corollary 2.2, we obtain (2.7) and (2.8). □

Theorem 2.4 *If $f = h(z) + \overline{g(z)}$ is in $S^0_{HS^*}$ and a is in $\mathbb{D}$, then*

$$F = \frac{az(h(\frac{z+a}{1+\overline{a}z}) + \overline{g(\frac{z+a}{1+\overline{a}z})})}{(h(a) + \overline{g(a)})(z+a)(1+\overline{a}z)} \tag{2.9}$$

is likewise in $S^0_{HS^*}$.

Proof For ρ real, $0 < \rho < 1$, let

$$F_\rho = \frac{az(h(\rho(\frac{z+a}{1+\overline{a}z})) + \overline{g(\rho(\frac{z+a}{1+\overline{a}z}))})}{(h(\rho a) + \overline{g(\rho a)})(z+a)(1+\overline{a}z)} \tag{2.10}$$

then we have

$$\begin{aligned} &\frac{zF_{\rho z} - \overline{z}F_{\rho\overline{z}}}{F_\rho} \\ &\quad = 1 - \frac{z}{z+a} + \frac{\overline{a}z}{1+\overline{a}z} + \frac{(1-|a|)z}{(1+\overline{a}z)(z+a)} \cdot \frac{(\rho(\frac{z+a}{1+\overline{a}z}))h'(\rho(\frac{z+a}{1+\overline{a}z}))}{h(\rho(\frac{z+a}{1+\overline{a}z})) + \overline{g(\rho(\frac{z+a}{1+\overline{a}z}))}} \\ &\qquad - \frac{(1-|a|^2)\overline{z}}{(1+\overline{a}z)(z+a)} \cdot \frac{\overline{\rho(\frac{z+a}{1+\overline{a}z})g'(\rho(\frac{z+a}{1+\overline{a}z}))}}{h(\rho(\frac{z+a}{1+\overline{a}z})) + \overline{g(\rho(\frac{z+a}{1+\overline{a}z}))}}. \end{aligned} \tag{2.11}$$

Letting $z = e^{i\theta}$ and $w = \rho(\frac{z+a}{1+\overline{a}z})$ in (2.11) and after the straightforward calculations, we obtain

$$\operatorname{Re}\left(\frac{zF_z - \overline{z}F_{\overline{z}}}{F}\right) = \frac{1-|a|^2}{|a+e^{i\theta}|^2} \operatorname{Re}\left(\frac{wh'(w) - \overline{w\rho'(w)}}{h(w) + \overline{\rho(w)}}\right) > 0, \tag{2.12}$$

and we conclude that

$$F_\rho = \frac{az(h(\rho(\frac{z+a}{1+\overline{a}z})) + \overline{g(\rho(\frac{z+a}{1+\overline{a}z}))})}{(h(\rho a) + \overline{g(\rho a)})(z+a)(1+\overline{a}z)}$$

is in $S^0_{HS^*}$ for every admissible ρ. From the compactness of $S^0_{HS^*}$ [2] and (2.11), we infer that $F = \lim_{\rho \to 1} F_\rho$ is in $S^0_{HS^*}$. We also note that this theorem is a generalization of the theorem of Libera and Ziegler [3]. □

Corollary 2.5 *Let $f = h(z) + \overline{g(z)}$ be an element of $S^0_{HS^*}$, then*

$$\frac{\frac{(1-k)|u|}{1-k|u|^2} G(\alpha, \beta, -\frac{(1-k)u}{1-k|u|^2})}{(1 + \frac{(1-k)^2|u|^2}{1-k|u|^2})^2 (1 - \frac{(1-k)|u|}{1-k|u|^2})} \le \left|\frac{h(u)}{h(ku) + \overline{g(ku)}}\right| \le \frac{\frac{(1-k)|u|}{1-k|u|^2} G(\alpha, \beta, \frac{(1-k)u}{1-k|u|^2})}{(1 - \frac{(1-k)|u|}{1-k|u|^2})^3 (1 + \frac{(1-k)|u|}{1-k|u|^2})^2}, \tag{2.13}$$

$$\begin{aligned} &\frac{|w(\frac{(1-k)|u|}{1-k|u|^2})| \frac{(1-k)|u|}{1-k|u|^2} G(\alpha, \beta, \frac{(1-k)|u|}{1-k|u|^2})}{(1 + \frac{(1-k)^2|u|^2}{1-k|u|^2})^2 (1 - \frac{(1-k)|u|}{1-k|u|^2})} \\ &\quad \le \left|\frac{g(u)}{g(ku) + \overline{g(ku)}}\right| \le \frac{\frac{(1-k)|u|}{1-k|u|^2} G(\alpha, \beta, \frac{(1-k)u}{1-k|u|^2})}{(1 - \frac{(1-k)|u|}{1-k|u|^2})^3 (1 + \frac{(1-k)|u|}{1-k|u|^2})^2}. \end{aligned} \tag{2.14}$$

Proof Using Theorem 2.4, we have

$$\begin{cases} F = \dfrac{a.z.h(\frac{z+a}{1+\overline{a}z})}{(h(a)+\overline{g(a)})(z+a)(1+\overline{a}z)} + \dfrac{a.z.\overline{g(\frac{z+a}{1+\overline{a}z})}}{(h(a)+\overline{g(a)})(z+a)(1+\overline{a}z)} \\ = H(z)+\overline{G(z)}. \end{cases} \tag{2.15}$$

If we apply Corollary 2.3 to $H(z)$ and $G(z)$ by taking

$$u = \frac{z+a}{1+\overline{a}z} \quad \Leftrightarrow \quad z = \frac{u-a}{1+\overline{a}u}$$

$a = ku$, $-1 < k < 1$ and after straightforward calculations, we get (2.13) and (2.14). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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