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FERMATEAN FUZZY MATRICES

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Abstract. In this paper, we introduce the concept of Fermatean fuzzy matrices, which are direct extensions of an intuitionistic fuzzy matrices. Then we define some algebraic operations, such as max-min, min-max, complement, algebraic sum, algebraic product, scalar multiplication (nA) and exponentiation (A^n). We also investigate their algebraic properties of these operations. Furthermore, we define two operators, namely the necessity and possibility to convert FFM into a ordinary FM and then dicuss some of their basic properties. Finally, we define a new operation(@) on Fermatean fuzzy matrices and discuss distributive laws in the case where the operations of $\oplus_F, \otimes_F, \wedge_F$ and $\vee_F$ are combined each other.

Keywords: Intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, Fermatean fuzzy matrix, Algebraic sum, Algebraic product.

AMS Subject Classification: 03E72, 15B15, 15B99.

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. [3] and simultaneously Im et al. [2] to generalize the concept of Thomason's [9] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ and $0 \leq \mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$. Since the IFS was proposed, it has received a lot of attention in many fields, such as pattern recognition, medical diagnosis, and so on. But if the sum of the membership degree and the nonmembership degree is greater than 1, the IFM is no longer applicable. Khan and Pal [4] defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product etc. and proved equality between IFMs. After the introduction of IFM theory, many researchers attempted the important role in IFM theory [1, 5, 6, 11].
Yager [10] introduced the concept of a Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. Zhang and Xu [12] studied various binary operations over PFS and also proposed a decision making algorithm based on PFS. Using the theory of PFS, in [7] we defined the Pythagorean fuzzy matrix (PFM) and its algebraic operations. Each element in an PFM is expressed by an ordered pair $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ and $0 \leq \mu^2_{a_{ij}} + \nu^2_{a_{ij}} \leq 1$. Also,they constructed nA and A^n of a Pythagorean fuzzy matrix A

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and using these operations.Further, we defined the Hamacher operations on Pythagorean fuzzy matrices and proved that the set of all PFMs forms a commutative monoid [8]. Since the PFM was brought up, it has been widely applied in different fields, such as investment decision making, service quality of domestic airline, collaborative-based recommender systems, and so on. Although the PFM generalizes the IFM, it cannot describe the following decision information. A panel of experts were invited to give their opinions about the feasibility of an investment plan, and they were divided into two independent groups to make a decision. One group considered the degree of the feasibility of the investment plan as 0.9, while the other group considered the nonmembership degree as 0.6. It was clearly seen that $0.9 + 0.6 > 1$, $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, and thus it could not be described by IFM and PFM. To describe such evaluation information, in this paper we have proposed Fermatean fuzzy matrix (FFM)and its algebraic operations. Each element in an PFM is expressed by an ordered pair $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ and $0 \le \mu^3_{a_{ij}} + \nu^3_{a_{ij}} \le 1$. Also, we can get $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \le 1$, which is good enough to apply the FFM to control it.
The part of this paper is as follows. In *Preliminaries* section, we give some basic definitions of IFMs and PFMs. In *Fermatean fuzzy matrices* section, we introduce Fermatean fuzzy matrices and its basic operations, then examples are given. In *IFM operations on Fermatean fuzzy matrices* section, we defined an IFM operations on Fermatean fuzzy matrices and prove their desirable properties. In *Necessity and possibility on Fermatean fuzzy matrices* section, we define necessity and possibility on Fermatean fuzzy matrices and proved some algebraic properties of these operations. In *New operation(@) on Fermatean fuzzy matrices* section, we define a new operation(@) on Fermatean fuzzy matrices and investigated their algebraic properties. we write the *Conclusion* of the paper in the last section.

2. Preliminaries

In this section, some basic concepts related to the intuitionistic fuzzy matrix (IFM) and Pythagorean fuzzy matrix (PFM) have been given.

Definition 2.1. [3] *An intuitionistic fuzzy matrix (IFM) is a pair* $A = \left[\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle \right]$ *of a non negative real numbers* $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ *satisfying* $0 \le \mu_{a_{ij}} + \nu_{a_{ij}} \le 1$ *for all* i, j.

Definition 2.2. [7] *A Pythagorean fuzzy matrix (PFM) is a pair* $A = \left[\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle \right]$ *of non negative real numbers* $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ *satisfying the condition* $0 \le \mu^2_{a_{ij}} + \nu^2_{a_{ij}} \le 1$, *for all* i, j. *Where* $\mu_{a_{ij}} \in [0, 1]$ *is called the degree of membership and* $\nu_{a_{ij}} \in [0, 1]$ *is called the degree of non-membership.*

3. Fermatean fuzzy matrices

In this section, we briefly introduce the Fermatean fuzzy matrices (FFM) and define, then examples are given.

Definition 3.1. *A Fermatean fuzzy matrix (FFM) is a pair* $A = \left[\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle \right]$ *of non negative real numbers* $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0, 1]$ *satisfying the condition* $0 \le \mu^3_{a_{ij}} + \nu^3_{a_{ij}} \le 1$, *for all* i, j. *Where* $\mu_{a_{ij}} \in [0, 1]$ *is called the degree of membership and* $\nu_{a_{ij}} \in [0, 1]$ *is called the degree of non-membership.*

For understanding the FFM better, we give an instance to illuminate the understandability of the FFM: We can definitely get $0.9 + 0.6 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy matrices. Also, we can get $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \leq 1$, which is good enough to apply the FFM to control it.

Theorem 3.1. *The FFMs is larger than the set of PFMs and IFMs.*

Proof. Any intuitionistic fuzzy matrix $(\mu_{a_{ij}}, \nu_{a_{ij}})$ that is an IFM is also a PFM and a FFM.
For any two fuzzy matrices $A, B \in [0,1]$, we get $\mu^3_{a_{ij}} \leq \mu^2_{a_{ij}} \leq \mu_{a_{ij}}$ and $\nu^3_{a_{ij}} \leq \nu^2_{a_{ij}} \leq \nu_{a_{ij}}$.
Thus $\mu_{a_{ij}} + \nu_{a_{ij}} \leq 1 \Rightarrow \mu^2_{a_{ij}} + \nu^2_{a_{ij}} \leq 1 \Rightarrow \mu^3_{a_{ij}} + \nu^3_{a_{ij}} \leq 1$.
Consider a point $(0.9, 0.6)$, we see that $(0.9)^3 + (0.6)^3 \leq 1$, thus this is an FFM.

The difference between IFM, PFM and FFM as shown in Figure.1

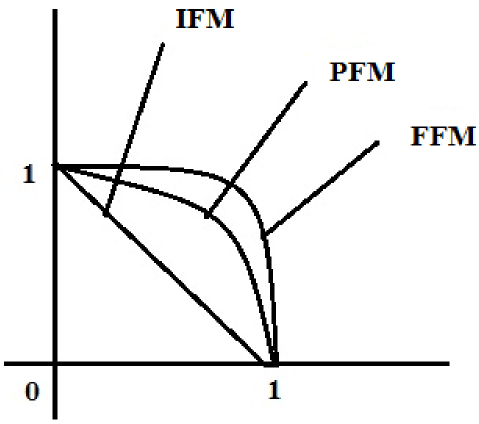

Figure 1. Comparison of space of IFMs, PFMs and FFMs.

Since $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 \geq 1$ and $0.9 + 0.6 \geq 1$, therefore $(0.9, 0.6)$ is neither a PFM nor an IFM. □

Example

[1] $\mathbf{A} = \begin{bmatrix} (0.5, 0.5) & (0.2, 0.4) \\ (0.3, 0.4) & (0.4, 0.4) \end{bmatrix}$ is an IFM.

[2] $\mathbf{A} = \begin{bmatrix} (0.7, 0.6) & (0.2, 0.4) \\ (0.3, 0.4) & (0.4, 0.4) \end{bmatrix}$ is not an IFM, but it A is a PFM.

[3] $\mathbf{A} = \begin{bmatrix} (0.9, 0.6) & (0.2, 0.4) \\ (0.3, 0.4) & (0.4, 0.4) \end{bmatrix}$ is not an IFM and PFM, but it A is a FFM.

This development can be evidently recognized in Figure 1. Here we notice that IFMs are all points beneath the line $\mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$, the PFMs are all points with $\mu^2_{a_{ij}} + \nu^2_{a_{ij}} \leq 1$, and the FFMs are all points with $\mu^3_{a_{ij}} + \nu^3_{a_{ij}} \leq 1$. We see then that the FFMs enable for the presentation of a bigger body of nonstandard membership function then IFMs and PFMs.
Let $F_{m \times n}$ denote the set of all the Fermatean fuzzy matrices.

4. IFM operations on Fermatean fuzzy matrices

In this section we propose the definition of Fermatean fuzzy matrices (FFM) and some operations on FFM. Also, we prove some algebraic properties, such as commutativity, associativity, identity, distributivity and De Morgan's laws over complement.

Definition 4.1. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$ *and* $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be two Fermatean fuzzy matrices of the same size. Then*
(*i*) $A \vee_F B = \left[\left\langle \max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\} \min\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right\rangle\right]$
(*ii*) $A \wedge_F B = \left[\left\langle \min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\} \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right\rangle\right]$
(*iii*) $A^C = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$.

Definition 4.2. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$ *and* $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be two Fermatean fuzzy matrices of the same size, then*
(*i*) $A \oplus_F B = \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle\right]$
(*ii*) $A \otimes_F B = \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}} \right\rangle\right]$
(*iii*) $nA = \left[\left\langle \sqrt[3]{1-(1-\mu^3_{a_{ij}})^n}, (\nu_{a_{ij}})^n \right\rangle\right]$
(*iv*) $A^n = \left[\left\langle \mu^n_{a_{ij}}, \sqrt[3]{1-(1-\mu^3_{a_{ij}})^n} \right\rangle\right]$
where $+, -$ *and* $.$ *are ordinary addition, subtraction and multiplication respectively.*

Theorem 4.1. *For* $A, B \in F_{m\times n}$, *then*
(*i*) $A \oplus_F B = B \oplus_F A$
(*ii*) $A \otimes_F B = B \otimes_F A$
(*iii*) $n(A \oplus_F B) = nA \oplus_F nB, n > 0$
(*iv*) $(n_1 + n_2)A = n_1A \oplus_F n_2A, n_1, n_2 > 0$
(*v*) $(A \oplus_F B)^n = A^n \otimes_F B^n, n > 0$
(*vi*) $A^{n_1} \otimes_F A^{n_2} = A^{(n_1+n_2)}, n_1, n_2 > 0$.

Proof. (*i*)
$$\begin{aligned} A \oplus_F B &= \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle\right] \\ &= \left[\left\langle \sqrt[3]{\mu^3_{b_{ij}} + \mu^3_{a_{ij}} - \mu^3_{b_{ij}}\mu^3_{a_{ij}}}, \nu_{b_{ij}}\nu_{a_{ij}} \right\rangle\right] \\ &= B \oplus_F A. \end{aligned}$$

(*ii*)
$$\begin{aligned} A \otimes_F B &= \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}} \right\rangle\right] \\ &= \left[\left\langle \mu_{b_{ij}}\mu_{a_{ij}}, \sqrt[3]{\nu^3_{b_{ij}} + \nu^3_{a_{ij}} - \nu^3_{b_{ij}}\nu^3_{a_{ij}}} \right\rangle\right] \\ &= B \otimes_F A. \end{aligned}$$

(*iii*)
$$\begin{aligned} n(A \oplus_F B) &= n\left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle\right] \\ &= \left[\left\langle \sqrt[3]{1-\left[1-(\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}})\right]^n}, (\nu_{a_{ij}}\nu_{b_{ij}})^n \right\rangle\right] \\ &= \left[\left\langle \sqrt[3]{1-(1-\mu^3_{a_{ij}})^n(1-\mu^3_{b_{ij}})^n}, (\nu_{a_{ij}}\nu_{b_{ij}})^n \right\rangle\right] \end{aligned}$$

$$nA \oplus_F nB = \left[\left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, (\nu_{a_{ij}})^n \right\rangle \oplus_F \left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, (\nu_{a_{ij}})^n \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n(1-\mu_{a_{ij}}^3)^n}, (\nu_{a_{ij}}\nu_{b_{ij}})^n \right\rangle \right]$$
$$= n(A \oplus_F B).$$

(iv)
$$(n_1+n_2)A = \left[\left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_1+n_2}}, (\nu_{a_{ij}})^{n_1+n_2} \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_1}(1-\mu_{a_{ij}}^3)^{n_2}}, (\nu_{a_{ij}}\nu_{b_{ij}})^{n_1+n_2} \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_1}}, (\nu_{a_{ij}})^{n_1} \right\rangle \oplus_F \left\langle \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_2}}, (\nu_{a_{ij}})^{n_2} \right\rangle \right]$$
$$= n_1 A \oplus_F n_2 A.$$

(v)
$$(A \otimes_F B)^n = \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu_{a_{ij}}^3 + \nu_{b_{ij}}^3 - \nu_{a_{ij}}^3\nu_{b_{ij}}^3} \right\rangle \right]^n$$
$$= \left[\left\langle \left(\mu_{a_{ij}}\mu_{b_{ij}}\right)^n, \sqrt[3]{1-\left(1-\nu_{a_{ij}}^3 - \nu_{b_{ij}}^3 + \nu_{a_{ij}}^3\nu_{b_{ij}}^3\right)^n} \right\rangle \right]^n$$
$$= \left[\left\langle \left(\mu_{a_{ij}}\right)^n \left(\mu_{b_{ij}}\right)^n, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n(1-\mu_{a_{ij}}^3)^n} \right\rangle \right]$$
$$= \left[\left\langle \mu_{a_{ij}}^n, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n} \right\rangle \otimes_F \left\langle \mu_{b_{ij}}^n, \sqrt[3]{1-(1-\mu_{b_{ij}}^3)^n} \right\rangle \right]$$
$$= A^n \otimes_F B^n.$$

(vi)
$$A^{n_1} \otimes_F A^{n_2} = \left[\left\langle \mu_{a_{ij}}^{n_1}, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_1}} \right\rangle \otimes_F \left\langle \mu_{a_{ij}}^{n_2}, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_2}} \right\rangle \right]$$
$$= \left[\left\langle \mu_{a_{ij}}^{n_1+n_2}, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^{n_1+n_2}} \right\rangle \right]$$
$$= A^{(n_1+n_2)}.$$
□

Theorem 4.2. *For $A, B \in F_{m\times n}$, then*
(i) $A \wedge_F B = B \wedge_F A$
(ii) $A \vee_F B = B \vee_F A$
(iii) $A \wedge_F (B \wedge_F C) = (A \wedge_F B) \wedge_F C$
(iv) $A \vee_F (B \vee_F C) = (A \vee_F B) \vee_F C$
(v) $n(A \wedge_F B) = nA \wedge_F nB$
(vi) $n(A \vee_F B) = nA \vee_F nB$
(vii) $(A \wedge_F B)^n = A^n \wedge_F B^n$
(viii) $(A \vee_F B)^n = A^n \vee_F B^n$.

Proof. *(i)*
$$(A \wedge_F B) = \left(\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right)$$
$$= \left(\min\left\{\mu_{b_{ij}}, \mu_{a_{ij}}\right\}, \max\left\{\nu_{b_{ij}}, \nu_{a_{ij}}\right\}\right)$$
$$= B \wedge_F A.$$

(iii)
$$A \wedge_F (B \wedge_F C) = \left(\mu_{a_{ij}}, \nu_{b_{ij}}\right) \wedge_F \left(\min\left\{\mu_{b_{ij}}, \mu_{c_{ij}}\right\}, \max\left\{\nu_{b_{ij}}, \nu_{c_{ij}}\right\}\right)$$
$$= \left(\min\left\{\mu_{a_{ij}}, \min\left\{\mu_{b_{ij}}, \mu_{c_{ij}}\right\}\right\}, \max\left\{\nu_{a_{ij}}, \max\left\{\nu_{b_{ij}}, \nu_{c_{ij}}\right\}\right\}\right)$$
$$= \left(\min\left\{\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \mu_{c_{ij}}\right\}, \max\left\{\max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}, \nu_{c_{ij}}\right\}\right)$$
$$= \left(\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right) \wedge_F \left(\mu_{c_{ij}}, \nu_{c_{ij}}\right)$$
$$= (A \wedge_F B) \wedge_F C$$

(vi)
$$n(A \vee_F B) = nA \vee_F nB$$
$$= n\left(\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right)$$

$$= \left[\left\langle \sqrt[3]{1-(1-\max\left\{\mu_{a_{ij}}^3, \mu_{b_{ij}}^3\right\})^n}, \min\left\{(\nu_{a_{ij}})^n, (\nu_{b_{ij}})^n\right\} \right\rangle \right]$$

$$nA \vee_F nB = \left[\left\langle \left(\sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, \nu_{a_{ij}}{}^n \right) \vee \left(\sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, (\nu_{a_{ij}})^n \right) \right\rangle \right]$$
$$= \left[\left\langle \max\left\{ \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, \sqrt[3]{1-(1-\mu_{b_{ij}}^3)^n} \right\}, \min\left\{(\nu_{a_{ij}})^n, (\nu_{b_{ij}})^n\right\} \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{1-(1-\max\left\{\mu_{a_{ij}}^3, \mu_{b_{ij}}^3\right\})^n}, \min\left\{(\nu_{a_{ij}})^n, (\nu_{b_{ij}})^n\right\} \right\rangle \right]$$
$$= n(A \vee_F B).$$

□

Theorem 4.3. *For $A, B \in F_{m\times n}$, then*
(i) $(A \wedge_F B)^C = A^C \vee_F B^C$
(ii) $(A \vee_F B)^C = A^C \wedge_F B^C$
(iii) $(A \oplus_F B)^C = A^C \otimes_F B^C$
(iv) $(A \otimes_F B)^C = A^C \oplus_F B^C$
(v) $(A^C)^n = (nA)^C$
(vi) $n(A^C) = (A^n)^C$.

Proof. (i)
$$(A \wedge_F B)^C = \left[\left\langle \left(\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right)^C \right\rangle \right]$$
$$= \left[\left\langle \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}, \min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\} \right\rangle \right]$$
$$= \left(\nu_{a_{ij}}, \mu_{a_{ij}}\right) \vee_F \left(\nu_{b_{ij}}, \mu_{b_{ij}}\right)$$
$$= A^C \vee_F B^C$$

(iii)
$$(A \oplus_F B)^C = \left[\left\langle \left(\sqrt[3]{\mu_{a_{ij}}^3 + \mu_{b_{ij}}^3 - \mu_{a_{ij}}^3\mu_{b_{ij}}^3}, \nu_{a_{ij}}\nu_{b_{ij}} \right)^C \right\rangle \right]$$
$$= \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu_{a_{ij}}^3 + \nu_{b_{ij}}^3 - \nu_{a_{ij}}^3\nu_{b_{ij}}^3} \right\rangle \right]$$
$$= (\nu_{a_{ij}}\mu_{b_{ij}}) \otimes (\nu_{b_{ij}}\mu_{b_{ij}})$$
$$= A^C \otimes_F B^C.$$

(v)
$$(A^C)^n = (\nu_{a_{ij}}, \mu_{a_{ij}})^n$$
$$= \left[\left\langle \nu_{a_{ij}}^n, \sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n} \right\rangle \right]$$
$$= \left[\left\langle \left(\sqrt[3]{1-(1-\mu_{a_{ij}}^3)^n}, \nu_{a_{ij}}^n \right)^C \right\rangle \right]$$
$$= (nA)^C.$$

□

Theorem 4.4. *For $A, B, C \in F_{m\times n}$, then*
(i) $(A \vee_F B) \wedge_F C = (A \wedge_F C) \vee_F (B \wedge_F C)$
(ii) $(A \wedge_F B) \vee_F C = (A \vee_F C) \wedge_F (B \vee_F C)$
(iii) $(A \vee_F B) \oplus_F C = (A \oplus_F C) \vee_F (B \oplus_F C)$
(iv) $(A \wedge_F B) \oplus_F C = (A \oplus_F C) \wedge_F (B \oplus_F C)$
(v) $(A \vee_F B) \otimes_F C = (A \otimes_F C) \vee_F (B \otimes_F C)$
(vi) $(A \wedge_F B) \otimes_F C = (A \otimes_F C) \wedge_F (B \otimes_F C)$.

Proof. In the following, we shall prove (i),(iii),(v) and (ii),(iv),(vi) can be proved analogously.
(i)
$$(A \vee_F B) \wedge_F C = \left[\left\langle \min\left\{\max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \mu_{c_{ij}}\right\}, \max\left\{\min\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\} \nu_{c_{ij}}\right\} \right\rangle \right]$$
$$= \left[\left\langle \max\left\{\min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \min\left\{\mu_{a_{ij}}, \mu_{c_{ij}}\right\}\right\}, \min\left\{\max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}, \max\left\{\nu_{b_{ij}}, \nu_{c_{ij}}\right\}\right\} \right\rangle \right]$$

$= \left[\left\langle \left\{ \min\left\{\mu_{a_{ij}}, \mu_{c_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{c_{ij}}\right\}\right\} \bigvee \left\{ \min\left\{\nu_{b_{ij}}, \nu_{c_{ij}}\right\}, \max\left\{\nu_{b_{ij}}, \nu_{c_{ij}}\right\}\right\} \right\rangle \right]$
$= (A \wedge_F C) \vee_F (B \wedge_F C)$
Hence, $(A \vee_F B) \wedge_F C = (A \wedge_F C) \vee_F (B \wedge_F C)$

(iii) $(A \vee_F B) \oplus_F C = \left(\max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \min\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right) \oplus \left(\mu_{c_{ij}}, \nu_{c_{ij}}\right)$
$= \left[\left\langle \sqrt[3]{\max\left\{\mu^3_{a_{ij}}, \mu^3_{b_{ij}}\right\} + \mu^3_{c_{ij}} - \max\left\{\mu^3_{a_{ij}}, \mu^3_{b_{ij}}\right\}\mu^3_{c_{ij}}}, \min\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\nu_{c_{ij}} \right\rangle \right]$
$= \left[\left\langle \sqrt[3]{(1-\mu^3_{c_{ij}})\max\left\{\mu^3_{a_{ij}}, \mu^3_{b_{ij}}\right\} + \mu^3_{c_{ij}}}, \min\left\{\nu_{a_{ij}}\nu_{c_{ij}}, \nu_{b_{ij}}\nu_{c_{ij}}\right\} \right\rangle \right]$

$(A \oplus_F C) \vee_F (B \oplus_F C)$
$= \left[\left\langle \max\left\{ \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{c_{ij}} - \mu^3_{a_{ij}}\mu^3_{c_{ij}}}, \sqrt[3]{\mu^3_{b_{ij}} + \mu^3_{c_{ij}} - \mu^3_{b_{ij}}\mu^3_{c_{ij}}} \right\}, \min\left\{\nu_{a_{ij}}\nu_{c_{ij}}, \nu_{b_{ij}}\nu_{c_{ij}}\right\} \right\rangle \right]$
$= \left[\left\langle \max\left\{ \sqrt[3]{(1-\mu^3_{c_{ij}})\mu^3_{a_{ij}} + \mu^3_{c_{ij}}}, \sqrt[3]{(1-\mu^3_{c_{ij}})\mu^3_{b_{ij}} + \mu^3_{c_{ij}}} \right\}, \min\left\{\nu_{a_{ij}}\nu_{c_{ij}}, \nu_{b_{ij}}\nu_{c_{ij}}\right\} \right\rangle \right]$
$= \left[\left\langle \sqrt[3]{(1-\mu^3_{c_{ij}})\max\left\{\mu^3_{a_{ij}}, \mu^3_{b_{ij}}\right\} + \mu^3_{c_{ij}}}, \min\left\{\nu_{a_{ij}}\nu_{c_{ij}}, \nu_{b_{ij}}\nu_{c_{ij}}\right\} \right\rangle \right]$
$= (A \vee_F B) \oplus_F C$
Hence, $(A \vee_F B) \oplus_F C = (A \oplus_F C) \vee_F (B \oplus_F C)$.

(v) $(A \vee_F B) \otimes_F C$
$= \left[\left\langle \max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}\mu_{c_{ij}}, \sqrt[3]{\min\left\{\nu^3_{a_{ij}}, \nu^3_{b_{ij}}\right\} + \nu^3_{c_{ij}} - \min\left\{\nu^3_{a_{ij}}, \nu^3_{b_{ij}}\right\}\nu^3_{c_{ij}}} \right\rangle \right]$
$= \left[\left\langle \max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}\mu_{c_{ij}}, \sqrt[3]{(1-\nu^3_{c_{ij}})\min\left\{\nu^3_{a_{ij}}, \nu^3_{b_{ij}}\right\} + \nu^3_{c_{ij}}} \right\rangle \right]$

$(A \otimes_F C) \vee_F (B \otimes_F C)$
$= \left[\left\langle \max\left\{\mu_{a_{ij}}\mu_{c_{ij}}, \mu_{b_{ij}}\mu_{c_{ij}}\right\}, \min\left\{ \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{c_{ij}} - \nu^3_{a_{ij}}\nu^3_{c_{ij}}}, \sqrt[3]{\nu^3_{b_{ij}} + \nu^3_{c_{ij}} - \nu^3_{b_{ij}}\nu^3_{c_{ij}}} \right\} \right\rangle \right]$
$= \left[\left\langle \max\left\{\mu_{a_{ij}}\mu_{c_{ij}}, \mu_{b_{ij}}\mu_{c_{ij}}\right\}, \min\left\{ \sqrt[3]{(1-\nu^3_{c_{ij}})\nu^3_{a_{ij}} + \nu^3_{c_{ij}}}, \sqrt[3]{(1-\nu^3_{c_{ij}})\nu^3_{b_{ij}} + \nu^3_{c_{ij}}} \right\} \right\rangle \right]$
$= \left[\left\langle \max\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}\mu_{c_{ij}}, \sqrt[3]{(1-\nu^3_{c_{ij}})\min\left\{\nu^3_{a_{ij}}, \nu^3_{b_{ij}}\right\} + \nu^3_{c_{ij}}} \right\rangle \right]$
$= (A \vee_F B) \otimes_F C$.
Hence, $(A \vee_F B) \otimes_F C = (A \otimes_F C) \vee_F (B \otimes_F C)$. □

Theorem 4.5. *For any FFM A, then*
(i) $(A \oplus_F O) = (O \oplus_F A) = A$,
(ii) $(A \otimes_F J) = (J \otimes_F A) = A$.

Proof. (i) $A \oplus_F O = \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \oplus_F \langle 0, 1\rangle$
$= \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + 0 - \mu^3_{a_{ij}}.0}, \nu_{a_{ij}}.1 \right\rangle \right]$
$= \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \right]$
$= A$.

Similarly, we can prove $O \oplus_F A = A$.

(ii) $A \otimes_F J = \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \otimes_F \langle 1, 0\rangle$
$= \left[\left\langle \mu_{a_{ij}}.1, \sqrt[3]{{\nu_{a_{ij}}}^3 + 0 - {\nu_{a_{ij}}}^3.0} \right\rangle \right]$

$$= \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \right]$$
$$= A$$

Similarly, we can prove $J \otimes_F A = A$. $\square$

Theorem 4.6. *For any FFM A, then*
(i) $(A \oplus_F J) = (J \oplus_F A) = J$,
(ii) $(A \otimes_F O) = (O \otimes_F A) = O$.

Proof. (i) $(A \oplus_F J) = \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \oplus_F \langle 1, 0\rangle$
$$= \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + 1 - \mu^3_{a_{ij}}.1}, a_{ij}.0 \right\rangle \right]$$
$$= \langle 1, 0\rangle = J$$

Similarly, we can prove $J \oplus_F A = J$.

(ii) $(A \otimes_F O) = \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \otimes_F \langle 0, 1\rangle$
$$= \left[\left\langle \mu_{a_{ij}}.0, \sqrt[3]{\nu^3_{a_{ij}} + 1 - \nu^3_{a_{ij}}.1} \right\rangle \right]$$
$$= \langle 0, 1\rangle = O$$

Similarly, we can prove $O \otimes_F A = O$. $\square$

5. Necessity and Possibility operators on Fermatean fuzzy matrices

In this section, we define necessity and possibility operators for FFMs and proved their algebraic properties.

Definition 5.1. *For every FFM A, the necessity* ($\square$) *and possibility* ($\Diamond$) *operators are defined as follows,*
$$\square A = \left[\left\langle \mu_{a_{ij}}, \sqrt[3]{1 - \mu^3_{a_{ij}}} \right\rangle \right]$$
$$\Diamond A = \left[\left\langle \sqrt[3]{1 - \nu^3_{a_{ij}}}, \nu_{a_{ij}} \right\rangle \right]$$

Theorem 5.1. *For* $A, B \in F_{m\times n}$, *then*
(i) $\square(A \oplus_F B) = \square A \oplus_F \square B$
(ii) $\Diamond(A \oplus_F B) = \Diamond A \oplus_F \Diamond B$.

Proof. (i) $\square(A \oplus_F B) = \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \sqrt[3]{1 - (\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}})} \right\rangle \right]$
$$\square A \oplus_F \square B = \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \sqrt[3]{(1 - \mu^3_{a_{ij}})}\sqrt[3]{(1 - \mu^3_{b_{ij}})} \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \sqrt[3]{1 - (\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}})} \right\rangle \right]$$
Hence, $\square(A \oplus_F B) = \square A \oplus_F \square B$.

(ii) $\Diamond(A \oplus_F B) = \left[\left\langle \sqrt[3]{1 - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle \right]$
$$\Diamond A \oplus_F \Diamond B = \left[\left\langle \sqrt[3]{(1 - \nu^3_{a_{ij}}) + (1 - \nu^3_{b_{ij}}) - (1 - \nu^3_{a_{ij}})(1 - \nu^3_{b_{ij}})}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle \right]$$
$$= \left[\left\langle \sqrt[3]{1 - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle \right]$$
Hence, $\Diamond(A \oplus_F B) = \Diamond A \oplus_F \Diamond B$. $\square$

Theorem 5.2. *For $A, B \in F_{m\times n}$, then*
(i) $\Box(A \otimes_F B) = \Box A \otimes_F \Box B$
(ii) $\Diamond(A \otimes_F B) = \Diamond A \otimes_F \Diamond B$.

Proof. (i) $\Box(A \otimes_F B) = \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{1-\mu^3_{a_{ij}}\mu^3_{b_{ij}}} \right\rangle \right]$

$$\Box A \otimes_F \Box B = \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{(1-\mu^3_{a_{ij}}) + (1-\mu^3_{b_{ij}}) - (1-\mu^3_{a_{ij}})(1-\mu^3_{b_{ij}})} \right\rangle \right]$$

$$= \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{1-\mu^3_{a_{ij}}\mu^3_{b_{ij}}} \right\rangle \right]$$

Hence, $\Box(A \otimes_F B) = \Box A \otimes_F \Box B$.

(ii) It can be proved analogously. ☐

Theorem 5.3. *For $A, B \in F_{m\times n}$, then*
(i) $\left(\Box(A^C \oplus_F B^C)\right)^C = \Diamond A \otimes_F \Diamond B$
(ii) $\left(\Box(A^C \otimes_F B^C)\right)^C = \Diamond A \oplus_F \Diamond B$.

Proof. (i) $(A^C \oplus_F B^C) = \left[\left\langle \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \mu_{a_{ij}}\mu_{b_{ij}} \right\rangle \right]$

$$\Box(A^C \oplus_F B^C) = \left[\left\langle \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \sqrt[3]{1-(\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}})} \right\rangle \right]$$

$$\left(\Box(A^C \oplus_F B^C)\right)^C = \left[\left\langle \sqrt[3]{1-(\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}})}, \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}} \right\rangle \right]$$

$$= \Diamond A \otimes_F \Diamond B$$

(ii) It can be proved analogously. ☐

Theorem 5.4. *For $A, B \in F_{m\times n}$, then*
(i) $\left(\Diamond(A^C \oplus_F B^C)\right)^C = \Box A \otimes_F \Box B$
(ii) $\left(\Diamond(A^C \otimes_F B^C)\right)^C = \Box A \oplus_F \Box B$.

Proof. (i) $\Diamond(A^C \oplus_F B^C) = \left[\left\langle \sqrt[3]{1-\mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \mu_{a_{ij}}\mu_{b_{ij}} \right\rangle \right]$

$$\left(\Diamond(A^C \oplus_F B^C)\right)^C = \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{1-\mu^3_{a_{ij}}\mu^3_{b_{ij}}} \right\rangle \right]$$

$$= \Box A \otimes_F \Box B$$

(ii) It can be proved similarly. ☐

6. New operation (@) on Fermatean fuzzy matrices

In this section, we define the @ operation on FFMs and present their algebraic properties. We discuss the Distributivity law in the case the operation of Algebraic sum and Algebraic product, $\vee_F$ and $\wedge_F$ are combined each other.

Definition 6.1. *Let $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle \right], B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle \right]$ be any two FFMs,the new operation of FFM is defined by*

$$A@B = \left[\left\langle \sqrt[3]{\frac{\mu^3_{a_{ij}} + \mu^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\nu^3_{a_{ij}} + \nu^3_{b_{ij}}}{2}} \right\rangle \right].$$

Theorem 6.1. *For any FFM A, then $A@A = A$.*

Proof. $A@A = \left[\left\langle \sqrt[3]{\frac{\mu_{a_{ij}}^3+\mu_{a_{ij}}^3}{2}}, \sqrt[3]{\frac{\nu_{a_{ij}}^3+\nu_{a_{ij}}^3}{2}} \right\rangle\right]$

$$= \left[\left\langle \sqrt[3]{\frac{2\mu_{a_{ij}}^3}{2}}, \sqrt[3]{\frac{2\nu_{a_{ij}}^3}{2}} \right\rangle\right]$$

$$= \left[\left\langle \mu_{a_{ij}}^3, \nu_{a_{ij}}^3 \right\rangle\right]$$

$$= \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$$

$$= A.$$

□

Theorem 6.2. *For $A, B \in F_{m\times n}$, then*
(i) $(A \oplus_F B) \wedge_F (A \otimes_F B) = A \otimes_F B$,
(ii) $(A \oplus_F B) \vee_F (A \otimes_F B) = A \oplus_F B$,
(iii) $(A \oplus_F B) \wedge_F (A@B) = A@B$,
(iv) $(A \oplus_F B) \vee_F (A@B) = A \oplus_F B$,
(v) $(A \otimes_F B) \wedge_F (A@B) = A \otimes_F B$,
(vi) $(A \otimes_F B) \vee_F (A@B) = A@B$.

Proof. *(i)* $(A \oplus_P B) \wedge_F (A \otimes_F B)$

$$= \left[\left\langle \min\left\{ \sqrt[3]{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3-\mu_{a_{ij}}^3\mu_{b_{ij}}^3}, \mu_{a_{ij}}\mu_{b_{ij}} \right\}, \max\left\{ \nu_{a_{ij}}\nu_{b_{ij}}, \sqrt[3]{\nu_{a_{ij}}^3+\nu_{b_{ij}}^3-\nu_{a_{ij}}^3\nu_{b_{ij}}^3} \right\} \right\rangle\right]$$

$$= \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu_{a_{ij}}^3+\nu_{b_{ij}}^3-\nu_{a_{ij}}^3\nu_{b_{ij}}^3} \right\rangle\right]$$

$$= A \otimes_F B.$$

Hence, $(A \oplus_F B) \wedge_F (A \otimes_F B) = A \otimes_F B$.

(ii) $(A \oplus_F B) \vee_F (A \otimes_F B)$

$$= \left[\left\langle \max\left\{ \sqrt[3]{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3-\mu_{a_{ij}}^3\mu_{b_{ij}}^3}, \mu_{a_{ij}}\mu_{b_{ij}} \right\}, \min\left\{ \nu_{a_{ij}}\nu_{b_{ij}}, \sqrt[3]{\nu_{a_{ij}}^3+\nu_{b_{ij}}^3-\nu_{a_{ij}}^3\nu_{b_{ij}}^3} \right\} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3-\mu_{a_{ij}}^3\mu_{b_{ij}}^3}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle\right]$$

$$= A \oplus_F B.$$

Hence, $(A \oplus_F B) \vee_F (A \otimes_F B) = A \oplus_F B$.

(iii) $(A \oplus_F B) \wedge_F (A@B)$

$$= \left[\left\langle \min\left\{ \sqrt[3]{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3-\mu_{a_{ij}}^3\mu_{b_{ij}}^3}, \sqrt[3]{\frac{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3}{2}} \right\}, \max\left\{ \nu_{a_{ij}}\nu_{b_{ij}}, \sqrt[3]{\frac{\nu_{a_{ij}}^3+\nu_{b_{ij}}^3}{2}} \right\} \right\rangle\right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\mu_{a_{ij}}^3+\mu_{b_{ij}}^3}{2}}, \sqrt[3]{\frac{\nu_{a_{ij}}^3+\nu_{b_{ij}}^3}{2}} \right\rangle\right]$$

$$= A@B.$$

Hence, $(A \oplus_F B) \wedge_F (A@B) = A@B$.

(iv) $(A \oplus_F B) \vee_F (A@B)$

$$= \left[\left\langle \max\left\{ \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{a_{ij}}}, \sqrt[3]{\frac{\mu^3_{a_{ij}} + \mu^3_{b_{ij}}}{2}} \right\}, \min\left\{ \nu_{a_{ij}}\nu_{b_{ij}}, \sqrt[3]{\frac{\nu^3_{a_{ij}} + \nu^3_{b_{ij}}}{2}} \right\} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\mu^3_{a_{ij}} + \mu^3_{b_{ij}} - \mu^3_{a_{ij}}\mu^3_{b_{ij}}}, \nu_{a_{ij}}\nu_{b_{ij}} \right\rangle \right]$$

$= A \oplus_F B$.

Hence, $(A \oplus_F B) \vee_F (A@B) = A \oplus_F B$.

(v) $(A \otimes_F B) \wedge_F (A@B)$

$$= \left[\left\langle \min\left\{ \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\frac{\mu^3_{a_{ij}} + \mu^3_{b_{ij}}}{2}} \right\}, \max\left\{ \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \sqrt[3]{\frac{\nu^3_{a_{ij}} + \nu^3_{b_{ij}}}{2}} \right\} \right\rangle \right]$$

$$= \left[\left\langle \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}} \right\rangle \right]$$

$= A \otimes_F B$.

Hence, $(A \otimes_F B) \wedge_F (A@B) = A \otimes_F B$.

(vi) $(A \otimes_F B) \vee_F (A@B)$

$$= \left[\left\langle \max\left\{ \mu_{a_{ij}}\mu_{b_{ij}}, \sqrt[3]{\frac{\mu^3_{a_{ij}} + \mu^3_{b_{ij}}}{2}} \right\}, \min\left\{ \sqrt[3]{\nu^3_{a_{ij}} + \nu^3_{b_{ij}} - \nu^3_{a_{ij}}\nu^3_{b_{ij}}}, \sqrt[3]{\frac{\nu^3_{a_{ij}} + \nu^3_{b_{ij}}}{2}} \right\} \right\rangle \right]$$

$$= \left[\left\langle \sqrt[3]{\frac{\mu^3_{a_{ij}} + \mu^3_{b_{ij}}}{2}}, \sqrt[3]{\frac{\nu^3_{a_{ij}} + \nu^3_{b_{ij}}}{2}} \right\rangle \right]$$

$= A@B$

Hence, $(A \otimes_F B) \vee_F (A@B) = A@B$. □

Remark 6.1. *The Fermatean fuzzy matrix forms a commutative monoid, associativity, commutativity and identity under the Fermatean fuzzy matrix operation of algebraic sum and algebraic product. The distributive law also holds for* $\oplus_F, \otimes_F$ *and* $\wedge_F, \vee_F, @$ *are combined each other.*

7. Applications

The formation of Fermatean fuzzy matrices is commutative monoid structure, Fermatean fuzzy matrix and algebraic structure on this matrix, the results are applicable.

8. Conclusions

In this paper, we have introduced Fermatean fuzzy matrices and its basic algebraic operations. We also proved some algebraic properties of Fermatean fuzzy matrices, such as associativity, commutativity, identity, distributivity and De Morgan's laws over complement. Furthermore, we defined necessity and possibility operators on FFMs and investigated their algebraic properties. Finally, we have defined a new operation(@) on FFMs and discussed distributive laws in the case where the operations of $\oplus_F, \otimes_F, \wedge_F$ and $\vee_F$ are combined each other. This result can be applied further application of Fermatean fuzzy matric theory. For the development of Fermatean fuzzy commutative monoid structure and its algebraic property the results of this paper would be helpful. In the future, the application of the proposed aggregating operators of FFMs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

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I. Silambarasan for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.2.