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SOME FIXED POINT RESULTS FOR β-ADMISSIBLE MULTI-VALUED F-CONTRACTIONS

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ABSTRACT. In the present paper, we prove some fixed point results for β- admissible multi-valued F- contractions on metric spaces. This type of contraction is a generalization of some multi-valued contractions including Nedler's and Berinde's. Finally, we obtain a fixed point result for β- generalized Suzuki type multivalued F- contraction.

Keywords: Fixed point, Multi-valued F-contraction, Complete metric space.

AMS Subject Classification: 47H10, 47H04.

1. Introduction

Fixed point theory for multivalued operators was first studied by Nadler in [7] in 1969, who extended Banach's fixed point theorem [4] for set-valued functions. Many fixed point theorems have been proved by various authors as generalizations to Nadler's theorem (see[1, 6]).
Recently, D. Wardowski [11] introduced the concept of F-contraction for single-valued mappings and proved a fixed point theorem which generalizes some well-known results in the literature. The method was extended by Sgroi and Vetro [8] to the multivalued F-contractions in metric spaces by using Hausdorff metric.
In this paper, by considering the recent technique of Wardowski [11] and M. A. Miandaragh et al [6] we present a new generalized F-contraction, and improve the main result in [1, 2, 8] and [11].

2. Preliminaries

Let (X, d) be a metric space. We denote by 2^X the family of all nonempty subsets of X and by $CB(X)$ the family of all nonempty closed and bounded subsets of X. For $A \in 2^X$ and $x \in X$, $D(x, A) = \inf\{d(x, a) : a \in A\}$. For every $A, B \in CB(X)$, let

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}.$$

Such a function H is called generalized Hausdorff metric induced by d.

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Definition 2.1. [6] *Let X be a set, $T : X \to 2^X$ a multivalued mapping and $\beta : 2^X \times 2^X \to [0, \infty)$ a mapping. We say that T is β-admissible whenever $\beta(A, B) \geq 1$ implies $\beta(Tx, Ty) \geq 1$ for all $x \in A$ and $y \in B$, where A and B are subsets of X.*
We say that T is β-convergent whenever for each convergent sequence $\{x_n\}$ with $x_n \to x$, there exists a natural number N such that $\beta(Tx_n, Tx) \geq 1$ for all $n \geq N$.

Definition 2.2. [3] *Let $F : (0, \infty) \to R$ and $\theta : (0, \infty) \to (0, \infty)$ be two mappings. Let Δ be the set of all pairs (θ, F) satisfying the following:*
$\delta 1$) $\theta(t_n) \nrightarrow 0$ for each strictly decreasing sequence $\{t_n\}$;
$\delta 2$) F is a strictly increasing function;
$\delta 3$) For each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$
$\delta 4$) If $t_n \downarrow 0$ and $\theta(t_n) \leq F(t_n) - F(t_{n+1})$ for each $n \in \mathbb{N}$, then we have $\sum_{n=1}^{\infty} t_n < \infty$

Example 2.1. [3] *Let $F(t) = ln(t)$ and $\theta(t) = -ln(\alpha(t))$ for each $t \in (0, \infty)$, where $\alpha : (0, \infty) \to (0, 1)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$, for all $t \in [0, \infty)$. Then $(\theta, F) \in \Delta$.*

Definition 2.3. *Let $\mathcal{R}$ denote the class of all continuous functions $g : [0, \infty)^5 \to [0, \infty)$ with the following properties:*
1) $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) = 1$
2) g is a homogenous function, that is,

$$g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5),$$

for all $\alpha \geq 0$ and $(x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5$
3) If $x_i < y_i$ for $i = 1, ..., 4$, then $g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0)$ and $g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4)$.

Definition 2.4. [5] *Let X be a metric space. A subset $C \subseteq X$ is said to be approximative if the set*

$$P_C(x) = \{y \in C : d(x, y) = D(C, x)\}, \qquad \forall x \in X,$$

is nonempty.
A mapping $T : X \to 2^X$ is said to be approximative multivalued mapping , AV for short, if Tx is approximative for each $x \in X$.

3. Fixed Point Theory

Now, we are ready to state and prove our main results.

Theorem 3.1. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(g(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx))), \quad (1)$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$, where $g \in \mathcal{R}$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Let $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Since T is AV, we can choose a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) = D(x_n, Tx_n)$ for all $n \geq 0$. Since T

is β-admissible and $\beta(A, Tx_0) \geq 1$, it is easy to see that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all $n \geq 1$. Since F is a strictly increasing, we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(D(x_n, Tx_n)) \leq F(H(Tx_{n-1}, Tx_n)) \\ &\leq F(\beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n)), \end{aligned} \tag{2}$$

for all $n \in \mathbb{N}$. From (1) and (2), we have

$$\begin{aligned} &\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \\ &\leq \theta(d(x_{n-1}, x_n)) + F(\beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n)) \\ &\leq F(g(d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1}))) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n))) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)), \end{aligned} \tag{3}$$

for each $n \in \mathbb{N}$. Since F is strictly increasing, we get

$$d(x_n, x_{n+1}) < g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0),$$

for each $n \in \mathbb{N}$. Now we claim that $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, otherwise if there exist $n \in \mathbb{N}$ such that $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$, then by the fact that $g \in \mathcal{R}$ we have

$$\begin{aligned} d(x_n, x_{n+1}) &< g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), 2d(x_n, x_{n+1}), 0) \\ &= d(x_n, x_{n+1})g(1, 1, 1, 2, 0) = d(x_n, x_{n+1}), \text{ (by using Definition 2.3)} \end{aligned}$$

which is a contradiction. Therefore $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence, then by using (3) we have

$$\begin{aligned} &\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), 2d(x_{n-1}, x_n), 0))) \\ &= F(d(x_{n-1}, x_n)g(1, 1, 1, 2, 0)) = F(d(x_{n-1}, x_n)), \text{ (by using Definition 2.3)} \end{aligned}$$

for each $n \in \mathbb{N}$. Thus,

$$\theta(d(x_{n-1}, x_n)) \leq F(d(x_{n-1}, x_n)) - F(d(x_n, x_{n+1})). \tag{4}$$

Let $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$, for some $r \geq 0$. We show that $r = 0$. On contrary, suppose that $r > 0$. From (4) we obtain

$$\sum_{i=1}^{n-1} \theta(d(x_{i-1}, x_i)) \leq F(d(x_1, x_2)) - F(d(x_n, x_{n+1})), \tag{5}$$

for each $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is strictly decreasing, then from ($\delta 1$) we have $\theta(d(x_n, x_{n+1})) \nrightarrow 0$. Thus, $\sum_{i=1}^{\infty} \theta(d(x_i, x_{i+1})) = +\infty$, and from (5), $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Then by ($\delta 3$) we have $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Which is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

From (4), (6) and ($\delta 4$) we have $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete $x_n \to x \in X$. Now, we prove that x is a fixed point of T.
If there exits a strictly increasing sequence $\{n_k\}$ such that $x_{n_k} \in Tx$ for all $k \in \mathbb{N}$. Since $x_{n_k} \to x$, we get $D(x, Tx) = 0$. Since Tx is closed we get $x \in Tx$ and the proof is complete.

So, we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tx$, for each $n > n_0$. This implies that $Tx_n \neq Tx$, for each $n > n_0$. Since T is β-convergent we obtain

$$\begin{aligned} F(D(x_{n+1}, Tx)) &\leq \theta(d(x_n, x)) + F(H(Tx_n, Tx)) \\ &\leq \theta(d(x_n, x)) + F(\beta(Tx_n, Tx)H(Tx_n, Tx)) \\ &\leq F(g(d(x_n, x), D(x_n, Tx_n), D(x, Tx), D(x_n, Tx), D(x, Tx_n))) \\ &\leq F(g(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), D(x, x_{n+1}))), \end{aligned} \tag{7}$$

for all $n \geq n_0$. Since F is strictly increasing, inequality (7) reduces to

$$D(x_{n+1}, Tx) < g(d(x_n, x), d(x_n, x_{n+1}), D(x, Tx), D(x_n, Tx), D(x, x_{n+1})),$$

for all $n \geq n_0$. Now if $x \in Tx$, then proof is complete, otherwise, letting $n \to \infty$ in the previous inequality, we obtain

$$\begin{aligned} D(x, Tx) &\leq g(0, 0, D(x, Tx), D(x, Tx), 0) \\ &= D(x, Tx)g(0, 0, 1, 1, 0) \\ &< D(x, Tx)g(1, 1, 1, 2, 0) \\ &= D(x, Tx). \end{aligned}$$

which is a contradiction. Hence $x \in Tx$ and proof is complete. □

Corollary 3.1. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(d(x, y))$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Define $g(x_1, x_2, x_3, x_4, x_5) = x_1$. Then $g \in \mathcal{R}$, by using Theorem 3.1, T has a fixed point in X. □

Corollary 3.2. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CL(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\begin{aligned} \theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(ad(x, y) &+ b[D(x, Tx) + D(y, Ty)] \\ &+ c[D(x, Ty) + D(y, Tx)]), \end{aligned}$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$, and $a, b, c > 0$ with $a + 2b + 2c = 1$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Define $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + b(x_2 + x_3) + c(x_4 + x_5)$. Then $g \in \mathcal{R}$, by using Theorem 3.1, T has a fixed point in X. □

Corollary 3.3. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(ad(x, y) + bD(x, Tx) + cD(y, Ty)),$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$, and $a, b, c > 0$ with $a + b + c = 1$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Define $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + cx_3$. Then $g \in \mathcal{R}$, by using Theorem 3.1, T has a fixed point in X. □

Corollary 3.4. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(\max\{d(x, y), D(x, Tx), D(y, Ty)\}),$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$, Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Define $g(x_1, x_2, x_3, x_4, x_5) = \max\{x_1, x_2, x_3\}$. Then $g \in \mathcal{R}$, by using Theorem 3.1, T has a fixed point in X. □

Corollary 3.5. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that there exists $(\theta, F) \in \Delta$ such that*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(d(x, y) + LD(y, Tx)),$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$, and $L \geq 0$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Define $g(x_1, x_2, x_3, x_4, x_5) = x_1 + Lx_3$. Then $g \in \mathcal{R}$, by using Theorem 3.1, T has a fixed point in X. □

In below we explain a generalization of Theorem 3.2 of [10].

Corollary 3.6. *Let (X, d) be a complete metric space, $\beta : 2^X \times 2^X \to [0, \infty)$ be a mapping and $T : X \to CB(X)$ a β-admissible, β-convergent and satisfying AV. Assume that,*

$$\beta(Tx, Ty)H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, with $\beta(Tx, Ty)H(Tx, Ty) \neq 0$ where $\alpha : (0, \infty) \to (0, 1)$ is a function such that $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, \infty)$. Suppose that there exist $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Let $F(x) = ln(x)$ and $\theta(x) = -ln(\alpha(x))$ for each $x \in (0, \infty)$, and $g(x_1, x_2, x_3, x_4, x_5) = x_1$ then $(\theta, F) \in \Delta$ and $g \in \mathcal{R}$. Hence by using Theorem 3.1, T has a fixed point in X. □

Example 3.1. *Let $X = \{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2^n}, ...\} \cup \{0, 1\}$, $d(x, y) = |x - y|$, for all $x, y \in X$. Let $T : X \to CB(X)$ defined by*

$$Tx = \begin{cases} \{\frac{1}{2^n}\} & \text{if } x = \frac{1}{2^n},\ n = 1, 2, 3, ..., \\ \{0\} & \text{if } x = 0 \\ \{1, \frac{1}{2}\} & \text{if } x = 1. \end{cases} \tag{8}$$

Put $x = 1$, $y = \frac{1}{2}$. Then, we have

$$H(T1, T\frac{1}{2}) = \frac{1}{2} = d(1, \frac{1}{2}) + LD(\frac{1}{2}, \{1, \frac{1}{2}\}).$$

Then for all $F \in \mathcal{F}$ and $\tau > 0$, we have

$$\tau + F(T1, T\frac{1}{2}) > F(d(1, \frac{1}{2}) + LD(\frac{1}{2}, \{1, \frac{1}{2}\})).$$

Therefore, Theorem 2.2 in [2] *which is the main result of* [2], *is not applicable to this example. Now, we define* $\beta : 2^X \times 2^X \to [0, \infty)$ *by*

$$\beta(A, B) = \begin{cases} 2 & \text{if } A, B \subseteq \{\frac{1}{2^n} : n \in \mathbb{N}\} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Clearly, if $\beta(Tx, Ty)H(Tx, Ty) > 0$ *for each* $x \neq y$*, and* $Tx, Ty \subseteq \{\frac{1}{2^n} : n \in \mathbb{N}\}$*. Then it is easy to see that*

$$\frac{\beta(Tx, Ty)H(Tx, Ty)}{d(x, y) + 7D(y, Tx)} \leq e^{-1}.$$

Then

$$1 + \ln(\beta(Tx, Ty)H(Tx, Ty)) \leq \ln(d(x, y) + 7D(y, Tx)).$$

Therefore by Corollary 3.5, T *has a fixed point in* X*. Note that* 0 *and* 1 *are fixed points of* T*.*

In 2008, Suzuki introduced a new type of mappings and a generalization of the Banach contraction principle in which the completeness can be also characterized by the existence of fixed points of these mappings [9]. We give our last result about fixed point of β-generalized Suzuki type (θ, F) multivalued contractions. Our result also extend main result of [1].

Theorem 3.2. *Let* (X, d) *be a complete metric space,* $\beta : 2^X \times 2^X \to [0, \infty)$ *be a mapping and* $T : X \to CB(X)$ *a* β*-admissible,* β*-convergent and satisfying AV. Assume that there exists* $(\theta, F) \in \Delta$ *such that* $\frac{1}{2}D(x, Tx) \leq d(x, y)$ *implies*

$$\theta(d(x, y)) + F(\beta(Tx, Ty)H(Tx, Ty)) \leq F(g(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx))), \tag{10}$$

for all $x, y \in X$*, with* $\beta(Tx, Ty)H(Tx, Ty) \neq 0$*, where* $g \in \mathcal{R}$*. Suppose that there exist* $A \subseteq X$ *and* $x_0 \in A$ *such that* $\beta(A, Tx_0) \geq 1$*. Then* T *has a fixed point.*

Proof. Let $A \subseteq X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Since T is AV, we can choose a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) = D(x_n, Tx_n)$ for all $n \geq 0$. Since T is β-admissible and $\beta(A, Tx_0) \geq 1$, it is easy to see that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all $n \geq 1$. Since $\frac{1}{2}D(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n)$ and

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(D(x_n, Tx_n)) \leq F(H(Tx_{n-1}, Tx_n)) \\ &\leq F(\beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n)), \end{aligned} \tag{11}$$

for all $n \in \mathbb{N}$. From (10) and (11), we have

$$\begin{aligned} &\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \\ &\leq \theta(d(x_{n-1}, x_n)) + F(\beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n)) \\ &\leq F(g(d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1}))) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n))) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0)), \end{aligned} \tag{12}$$

for each $n \in \mathbb{N}$. This implies that

$$d(x_n, x_{n+1}) < g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0),$$

for each $n \in \mathbb{N}$. Now we claim that $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, otherwise if there exist $n \in \mathbb{N}$ such that $d(x_n, x_{n-1}) \leq d(x_{n+1}, x_n)$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &< g(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), 0) \\ &= d(x_n, x_{n+1})g(1, 1, 1, 2, 0) = d(x_n, x_{n+1}), \text{ (by using Definition 2.3)} \end{aligned}$$

which is a contradiction. Therefore $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence, then by using (12) we have

$$\begin{aligned} &\theta(d(x_{n-1}, x_n)) + F(d(x_n, x_{n+1})) \\ &\leq F(g(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), 2d(x_{n-1}, x_n), 0))) \\ &= F(d(x_{n-1}, x_n)g(1, 1, 1, 2, 0)) = F(d(x_{n-1}, x_n)), \end{aligned} \tag{13}$$

for each $n \in \mathbb{N}$.
Thus

$$\theta(d(x_{n-1}, x_n)) \leq F(d(x_{n-1}, x_n)) - F(d(x_n, x_{n+1})).$$

Now, by a similar argument of Theorem 3.1 we deduce that $x_n \to x \in X$. We claim that either $\frac{1}{2}D(x_n, Tx_n) \leq d(x_n, x)$ or $\frac{1}{2}D(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x)$ for all $n \in \mathbb{N}$. If $\frac{1}{2}D(x_n, Tx_n) > d(x_n, x)$ and $\frac{1}{2}D(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, x)$ for some $n \geq 1$, then

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(x_{n+1}, x) + d(x_n, x) \\ &< \frac{1}{2}D(x_{n+1}, Tx_{n+1}) + \frac{1}{2}D(x_n, Tx_n) \\ &\leq \frac{1}{2}d(x_{n+1}, x_{n+2}) + \frac{1}{2}D(x_n, x_{n+1}) \\ &\leq \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}D(x_n, x_{n+1}) \\ &= d(x_{n+1}, x_n), \end{aligned}$$

which is a contradiction. Thus either

$$\begin{aligned} &\theta(d(x_n, x)) + F(\beta(Tx_n, Tx)H(Tx_n, Tx)) \\ &\qquad \leq F(g(d(x_n, x), D(x, Tx), D(x_n, Tx_n), D(x_n, Tx), D(x, Tx_n))), \end{aligned}$$

or

$$\begin{aligned} &\theta(d(x_{n+1}, x)) + F(\beta(Tx_{n+1}, Tx)H(Tx_{n+1}, Tx)) \\ &\qquad \leq F(g(d(x_{n+1}, x), D(x, Tx), D(x_{n+1}, Tx_{n+1}), D(x_{n+1}, Tx), D(x, Tx_{n+1}))). \end{aligned}$$

Since T is β-convergent, in the first case we obtain

$$\begin{aligned} F(D(x_{n+1}, Tx)) &\leq F(H(Tx_n, Tx)) \\ &\leq F(\beta(Tx_n, Tx)H(Tx_n, Tx)) \\ &\leq \theta(d(x_n, x)) + F(\beta(Tx_n, Tx)H(Tx_n, Tx)) \\ &\leq F(g(d(x_n, x), D(x, Tx), D(x_n, Tx_n), D(x_n, Tx), D(x, Tx_n))) \\ &\leq F(g(d(x_n, x), D(x, Tx), D(x_n, x_{n+1}), D(x_n, Tx), D(x, x_{n+1}))). \end{aligned}$$

Thus we have

$$D(x_{n+1}, Tx) < g(d(x_n, x), D(x, Tx), D(x_n, x_{n+1}), D(x_n, Tx), D(x, x_{n+1})).$$

Now if $x \in Tx$, then proof is complete, otherwise, letting $n \to \infty$ in the previous inequality, we get

$$\begin{aligned} D(x,Tx) &< g(0,0,D(x,Tx),D(x,Tx),0) \\ &= D(x,Tx)g(0,0,1,1,0) \\ &< D(x,Tx)g(1,1,1,2,0) \\ &= D(x,Tx), \end{aligned}$$

which is a contradiction. Hence $x \in Tx$ and proof is complete.
Since T is β-convergent, in the second case we obtain by a similar argument that x is a fixed point and so the proof is complete. □

Example 3.2. *Let $X = [0,\infty)$ and $d(x,y) = |x-y|$, for all $x,y \in X$. We defined $T : X \to CB(X)$ by*

$$Tx = \begin{cases} [0, \frac{1}{4}e^{-r}x] & \text{if } x \in [0,1] \\ \{4x\} & \text{if } x \in (1,\infty) \end{cases}$$

for $r \geq 0$, and $\beta : 2^X \times 2^X \to [0,\infty)$ by

$$\beta(A,B) = \begin{cases} 2 & \text{if } A,B \subseteq [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then β satisfy conditions in Theorem (3.2). We will show that T satisfy the condition (10) for any $x,y \in [0,1]$ with $\frac{1}{2}D(x,Tx) \leq d(x,y)$. Let $x,y \in [0,1]$ and without loss of generality we suppose that $x \leq y$. Then we have $\frac{1}{2}D(x,Tx) = \frac{1}{2}(x - \frac{1}{4}e^{-r}x)$. Hence for $\frac{1}{2}D(x,Tx) \leq d(x,y)$, we must have $(\frac{3}{2} - \frac{1}{8}e^{-r})x \leq y$. Then it is easy to see that

$$\beta(Tx,Ty)H(Tx,Ty) = \frac{1}{2}e^{-r}d(x,y) \leq e^{-r}d(x,y).$$

Therefore

$$r + \ln(\beta(Tx,Ty)H(Tx,Ty)) \leq \ln(d(x,y))$$

Now, let $F(t) = \ln(t)$ and $g(x_1,x_2,x_3,x_4,x_5) = x_1$, then by Theorem (3.2), T has a fixed point in X. Note that, $0 \in T0$ is a fixed point of T.

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