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SOME NEW EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS INVOLVING ψ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. This paper concerns the boundary value problem for a fractional differential equation involving a generalized Caputo fractional derivative in $b-$metric spaces. The used fractional operator is given by the kernel $k(t,s) = \psi(t) - \psi(s)$ and the derivative operator $\frac{1}{\psi'(t)}\frac{d}{dt}$. Some existence results are obtained based on fixed point theorem of α-$\phi-$Graghty contraction type mapping. In the end, we provide some illustrative examples to justify the acquired results.

Keywords: $\psi-$Caputo fractional derivative, Boundary value problem (BVP), α–$\phi-$Geraghty contractive type mapping, Fixed point (FP).

AMS Subject Classification: 83-02, 99A00.

1. introduction

Fractional calculus has been studied extensively due to its practical applications in science and engineering. A comprehensive study about fractional differential equation and its applications is provided in [25]. Recently numerous interesting results concerning the existence, uniqueness and stability of the solution or the positive solution of some fractional differential equations are given applying some FP results. However most of these problems have been handled with respect to the standard derivatives of Riemann–Liouville (RL), Caputo and Hadamard [1, 12, 15, 16, 20, 21, 24, 23].

Almeida et al. in 2017 introduced a generalization of Caputo by some interesting properties [13, 14]. Some articles that present studies about the theory and analysis of $\phi-$fractional differential equations can be found in [2, 3, 4, 27, 28] and references therein.

In [5, 6, 7, 8, 9, 10, 11], Afshari and coauthors introduced the notion of generalized α–$\phi-$Geraghty multivalued mappings and their applications in complete $b-$metric spaces ($b - MS$s).

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Following the work of Aydi and Almeida, in this paper, by utilizing FP results of $\alpha-$ $\phi-$Geraghty contraction type mappings, we present new results on the fractional BVPs involving a $\psi-$Caputo fractional derivative operator in complete $(b-MS\text{s})$. We denote $\mathcal{J}=[0,1]$ and "if and only if" with "iff".

Definition 1.1. [22] *Let $\gamma>0$ and ψ be an increasing function, having a continuous derivative ψ' on (a,b). The left-sided $\psi-RL$ fractional integral of a function ζ with respect to ψ expressed as*

$$I_{a^+}^{\gamma;\psi}\zeta(\varrho)=\frac{1}{\Gamma(\alpha)}\int_a^{\varrho}\psi'(\varsigma)\left[\psi(\varrho)-\psi(\varsigma)\right]^{\alpha-1}\zeta(\varsigma)d\varsigma,\ \ \varrho>a.$$

provided that $I_{a^+}^{\alpha;\psi}$ is exists. Note that when $\psi(\varrho)=\varrho$, we obtain the known classical RL fractional integral.

Definition 1.2. [22, 25] *Let $\gamma>0$, n be the smallest integer greater than or equal to γ and $\zeta\in L^p[a,b]$, $p\geq 1$ let $\psi\in C^n[a,b]$ an increasing function such that $\psi'(\varrho)\neq 0$, for all $\varrho\in[a,b]$. The left-sided $\psi-RL$ fractional derivative of ζ of order α is given by*

$$D_{a^+}^{\gamma;\psi}h(\varrho)=\left(\frac{1}{\psi'(\varrho)}\frac{d}{d\varrho}\right)^n I_{a^+}^{n-\gamma,\psi}\zeta(\varrho).$$

Definition 1.3. [13, 14] *Let $n-1<\gamma<n$, $\zeta\in C^n[a,b]$, and let $\psi\in C^n[a,b]$ an increasing function such that $\psi'(\varrho)\neq 0$, for all $\varrho\in[a,b]$. The left-sided $\psi-$Caputo fractional derivative of ζ of order α is given by*

$${}^C D_{a^+}^{\gamma;\psi}\zeta(\varrho)=I_{a^+}^{n-\gamma,\psi}\ D^{n,\psi}\zeta(\varrho),$$

where $D^{n,\psi}:=\left(\frac{1}{\psi'(\varrho)}\frac{d}{d\varrho}\right)^n$, and $n=[\gamma]+1$.

We consider BVP:

$$\begin{cases} -{}^C D_0^{\gamma,\psi}\zeta(\varrho)=f(\varrho,\zeta(\varrho)),\ \varrho\in\mathcal{J}, \\ \zeta'(0)=0,\ \varpi^C D_0^{\gamma-1,\psi}\zeta(1)+\zeta(\eta)=0, \end{cases}\tag{1}$$

where $1<\gamma<2$, $\psi\in C^2(\mathcal{J})$, $\psi'(\varrho)>0$, ${}^C D_0^{\theta,\psi}$ is the ψ-Caputo fractional derivative of order θ, $\theta\in\{\gamma,\gamma-1\}$, $0\leq\eta\leq 1$, $\varpi>0$ and $f:\mathcal{J}\times\mathbb{R}^+\to\mathbb{R}$ is a continuous.

We consider:

$$-{}^C\ D_0^{\gamma}\zeta(\varrho)=\zeta(\varrho),\ a<\varrho<b.\tag{2}$$
$$\zeta'(0)=0,\ \varpi^C D_a^{\gamma-1}\zeta(b)+\zeta(\delta)=0,$$

where $1<\gamma<2$, $\varpi>0$, $a\leq\delta\leq b$, and $\zeta\in C([a,b])$.

Lemma 1.1. [18] *$\zeta\in C([a,b])$ is a solution to (2) if and only if*

$$\zeta(\varrho)=\int_a^b\mathcal{G}(\varrho,\varsigma)\zeta(\varsigma)d\varsigma,\ a\leq\varrho\leq b,$$

where $\mathcal{G}$ given by:

$$\mathcal{G}(\varrho,\varsigma)=\varpi+H_\delta(\varsigma)-H_\varrho(\varsigma),\tag{3}$$

and for $\varrho\in[a,b]$, $H_\varrho:[a,b]\to\mathbb{R}$ defined as:

$$H_\delta(\varsigma)=\begin{cases}\frac{(\varrho-\varsigma)^{\alpha-1}}{\Gamma(\gamma)}, & a\leq\varsigma\leq\varrho\leq b,\\ 0, & a\leq\varrho\leq\varsigma\leq b.\end{cases}$$

Lemma 1.2. [18] *The function $\mathcal{G}$ satisfies the following:*
(i) $\mathcal{G}$ is continuous on $[a,b]\times[a,b]$.
(ii) We have

$$\max\{\mathcal{G}(\varrho,\varsigma): a\le\varrho,\varsigma\le b\}=\varpi+\frac{(\delta-a)^{\alpha-1}}{\Gamma(\gamma)}$$

and

$$\min\{\mathcal{G}(\varrho,\varsigma): a\le\varrho,\varsigma\le b\}=\varpi-\frac{(b-\delta)^{\alpha-1}}{\Gamma(\gamma)}.$$

Lemma 1.3. [17] *$\zeta\in C^2(\mathcal{J})$ is a solution to (1) if and only if $\omega\in C^2([a,b])$ is a solution to:*

$$-{}^{C}D_a^{\gamma}\omega(\varsigma)=f(\psi^{-1}(\varsigma),\omega(\varsigma)),\ a<\varsigma<b. \tag{4}$$
$$\omega'(a)=0,\ \varpi{}^{C}D_a^{\gamma-1}\omega(b)+\omega(\delta)=0,$$

where $a=\psi(0)$, $b=\psi(1)$, $\delta=\psi(\eta)$, and $\omega=\zeta(\psi^{-1}(\varsigma))$, $\psi(0)\le\varsigma\le\psi(1)$.

Definition 1.4. [19] *Let $M\neq\emptyset$ and $s\ge 1$. A mapping $d\colon M\times M\to\mathbb{R}_0^+$ is said to be a b-metric if*

(bM_1) $d(\varrho,\varsigma)=0$ iff $\varsigma=\varrho$;
(bM_2) $d(\varrho,\varsigma)=d(\varsigma,\varrho)$;
(bM_3) $d(\varrho,z)\le s[d(\varrho,\varsigma)+d(\varsigma,z)]$.

Let Φ be set of all increasing and continuous functions $\phi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the property: $\phi(c\varrho)\le c\phi(\varrho)\le c\varrho$ for $c>1$ and $\phi(0)=0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\mu:\mathbb{R}^+\to[0,\frac{1}{s^2})$ for some $s\ge 1$.

Definition 1.5. [8] *Let (M,d) be a $b-MS$ (with constant $s\ge 1$) and $\Lambda: M\to M$, we say that Λ is a generalized α–$\phi-$Geraghty contraction whenever there exist $\alpha: M\times M\to\mathbb{R}^+$ such that*

$$\alpha(\varrho,\varsigma)\phi(s^3 d(\Lambda\varrho,\Lambda\varsigma))\le\mu(\phi(d(\varrho,\varsigma)))\phi(d(\varrho,\varsigma)),$$

for $\varrho,\varsigma\in M$, where $\mu\in\mathcal{F}$ and $\phi\in\Phi$.

Definition 1.6. [26] *Let $M\neq\emptyset$, $\Lambda: M\to M$ and $\alpha: M\times M\to[0,\infty)$. Λ is $\alpha-$admissible if for $\varrho,\varsigma\in M$, we have*

$$\alpha(\varrho,\varsigma)\ge 1\Longrightarrow\alpha(\Lambda\varrho,\Lambda\varsigma)\ge 1. \tag{5}$$

Theorem 1.1. [8] *Take (M,d) a complete $b-MS$ and $\Lambda: M\to M$ is a generalized α–$\phi-$Geraghty contraction, also*
(i) Λ is $\alpha-$admissible;
(ii) $\exists\ \varrho_0\in M$; $\alpha(\varrho_0,\Lambda\varrho_0)\ge 1$;
(iii) If $\{\varrho_n\}\subseteq M$ with $\varrho_n\to\varrho$ and $\alpha(\varrho_n,\varrho_{n+1})\ge 1$, then $\alpha(\varrho_n,\varrho)\ge 1$.
Then Λ has a FP.

Let $M=C([a,b],\mathbb{R})$ $(0<a<b<\infty)$ and let $d: M\times M\to[a,\infty)$ be given by

$$d(\zeta,\vartheta)=\|(\zeta-\vartheta)^2\|_\infty=\sup_{\varrho\in\mathcal{J}}(\zeta(\varrho)-\vartheta(\varrho))^2.$$

Then, (M,d) is a complete $b-MS$ with $s=2$.

Theorem 1.2. *Suppose that there exist functions $\tau:\mathbb{R}^2\to\mathbb{R}$ and $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ such that*
(i)

$$\begin{aligned}&|f(\psi^{-1}(\varsigma),w(\varsigma))-f(\psi^{-1}(\varsigma),z(\varsigma))|\\ \leq\ & \frac{1}{2\sqrt{2}}\frac{\Gamma(\gamma)}{\varpi\Gamma(\gamma)+(\delta-a)^{\alpha-1}}\sqrt{\phi(\|(\zeta-\vartheta)^2\|_\infty)\mu(\phi(\|(\zeta-\vartheta)^2\|_\infty))},\end{aligned}$$

where $\phi\in\Phi$, $\mu\in\mathcal{F}$ and $w(\varrho)=\zeta(\psi^{-1}(\varrho))$ and $z(\varrho)=\vartheta(\psi^{-1}(\varrho))$.
(ii) there exists $\zeta_0\in C(\mathcal{J})$ such that $\tau(\zeta_0(\varrho),\int_0^1\mathcal{G}(\varrho,\varsigma)f(\psi^{-1}(\varsigma),w_0(\varsigma))d\varsigma)\geq 0$, for $\varrho\in\mathcal{J}$ where $w_0(\varrho)=\zeta_0(\psi^{-1}(\varrho))$;
(iii) for $\varrho\in\mathcal{J}$ and $\zeta,\vartheta\in C(\mathcal{J})$, $\tau(\zeta(\varrho),\vartheta(\varrho))\geq 0$ implies

$$\tau\left(\int_0^1\mathcal{G}(\varrho,\varsigma)f(\psi^{-1}(\varsigma),w(\varsigma))d\varsigma,\int_0^1\mathcal{G}(\varrho,\varsigma)f(\psi^{-1}(\varsigma),z(\varsigma))d\varsigma\right)\geq 0,$$

where $w(\varrho)=\zeta(\psi^{-1}(\varrho))$ and $z(\varrho)=\vartheta(\psi^{-1}(\varrho))$;
(iv) if $\{\zeta_n\}$ is a sequence in $C(\mathcal{J})$ such that $\zeta_n\to\zeta$ in $C(\mathcal{J})$ and $\tau(\zeta_n,\zeta_{n+1})\geq 0$, then we have $\tau(\zeta_n,\zeta)\geq 0$.
Then, the problem (1) has at minimum one solution.

Proof. By Lemmas 1.3 and 1.1, $\zeta\in C^2(\mathcal{J})$ is a solution of (1) if and only if it's a solution of $\zeta(\varrho)=\int_0^1\mathcal{G}(\varrho,\varsigma)f(\psi^{-1}(\varsigma),w(\varsigma))d\varsigma$, where $w(\varrho)=\zeta(\psi^{-1}(\varrho))$ for $\varrho\in\mathcal{J}$. Define, $O: C^2(\mathcal{J})\to C^2(\mathcal{J})$ by $O\zeta(\varrho)=\int_0^1\mathcal{G}(\varrho,\varsigma)f(\psi)f(\psi^{-1}(\varsigma),w(\varsigma))d\varsigma$. Now, we show a FP of the operator O. Let $\zeta,\vartheta\in C^2(\mathcal{J})$ be such that $\tau(\zeta(\varrho),\vartheta(\varrho))\geq 0$. Using (i), we get

$$\begin{aligned}&|O\zeta(\varrho)-O\vartheta(\varrho)|^2=\left|\int_0^1\mathcal{G}(\varrho,\varsigma)(f(\psi^{-1}(\varsigma),w(\varsigma))-f(\psi^{-1}(\varsigma),z(\varsigma)))d\varsigma\right|^2\\ &\leq\left[\int_0^1\mathcal{G}(\varrho,\varsigma)|f(\psi^{-1}(\varsigma),w(\varsigma))-f(\psi^{-1}(\varsigma),z(\varsigma))|d\varsigma\right]^2\\ &\leq\left[\int_0^1\mathcal{G}(\varrho,\varsigma)\frac{1}{2\sqrt{2}}\frac{\Gamma(\gamma)}{\varpi\Gamma(\gamma)+(\delta-a)^{\alpha-1}}\sqrt{\phi(\|(\zeta-\vartheta)^2\|_\infty)\mu(\phi(\|(\zeta-\vartheta)^2\|_\infty))}d\varsigma\right]^2\\ &=\frac{1}{8}\phi(\|(\zeta-\vartheta)^2\|_\infty)\mu(\phi(\|(\zeta-\vartheta)^2\|_\infty)).\end{aligned}$$

Therefore for $\zeta,\vartheta\in C^2(\mathcal{J})$ with $\tau(\zeta(\varrho),\vartheta(\varrho))\geq 0$, we have

$$\|(Ou-O\vartheta)^2\|_\infty\leq\frac{1}{8}\phi(\|\zeta-\vartheta\|_\infty^2)\mu(\phi(\|\zeta-\vartheta\|_\infty^2)).$$

Put, $\alpha: C^2(\mathcal{J})\times C^2(\mathcal{J})\to\mathbb{R}^+$ by

$$\alpha(\zeta,\vartheta)=\begin{cases}1 & \tau(\zeta(\varrho),\vartheta(\varrho))\geq 0,\quad \varrho\in\mathcal{J},\\ 0 & else.\end{cases}$$

This implies that for $\zeta,\vartheta\in C^2(\mathcal{J})$ with $\tau(\zeta(\varrho),\vartheta(\varrho))\geq 0$,

$$\alpha(\zeta,\vartheta)8d(O\zeta,O\vartheta)\leq 8d(O\zeta,O\vartheta)\leq\phi(\mu(d(\zeta,\vartheta)))\phi(d(\zeta,\vartheta)),\quad \mu\in\mathcal{F}.$$

From (iii),

$$\alpha(\zeta,\vartheta)\geq 1\Rightarrow\tau(\zeta(\varrho),\vartheta(\varrho))\geq 0\Rightarrow\tau(O(\zeta),O(\vartheta))\geq 0\Rightarrow\alpha(O(\zeta),O(\vartheta))\geq 1,$$

for $\zeta,\vartheta\in C^2(\mathcal{J})$. Thus, O is $\alpha-$admissible. By (ii), $\exists\ \zeta_0\in C^2(\mathcal{J})$; $\alpha(\zeta_0,O\zeta_0)\geq 1$. By Theorem 1.1, we realize w^* with $\zeta^*=O\zeta^*$. □

Example 1.1. *Let us consider the fractional BVP:*

$$-^C D_{0+}^{\frac{3}{2},e^{\varrho}}\zeta(\varrho) = f(\varrho,\zeta(\varrho)), \quad \varrho \in \mathcal{J}, \tag{6}$$
$$\zeta'(0) = 0,\ \varpi^C D_{0+}^{\frac{1}{2},e^{\varrho}}\zeta(1) + \zeta(\eta) = 0,\ 0 \le \eta \le 1, \varpi > 0.$$

By Lemma 1.3 $\zeta \in C^2(\mathcal{J})$ is a solution to (6) if and only if $w \in C^2([a,b])$ is a solution to the following problem

$$-^C D_{1+}^{\frac{3}{2}} w(\varsigma) = f(\varsigma, w(\varsigma)), \quad 1 \le \varsigma \le e, \tag{7}$$
$$w'(1) = 0,\ 2^C D_{1+}^{\frac{1}{2}} w(e) + w(e^{\eta}) = 0,\ 0 \le \eta \le 1.$$

Setting $\tau(\varrho,z) = \varrho z$, $\zeta_n(\varrho) = \dfrac{\varrho}{n^2+1}$, $\phi(\varrho) = \varrho$, $\mu(\varrho) = \frac{\varrho}{1+4\varrho}$ and also assuming that the following condition true:

$$|f(\varrho,\zeta(\varrho)) - f(\varrho,\vartheta(\varrho))| \le \frac{3\sqrt{\pi}}{32\sqrt{2}} \frac{(\varrho+3)(\psi(\varrho)-\psi(0))}{(\psi(1)-\psi(0))^3} \sqrt{\left\|(\zeta-\vartheta)^2\right\|_\infty \frac{\left\|(\zeta-\vartheta)^2\right\|_\infty}{1+4\left\|(\zeta-\vartheta)^2\right\|_\infty}},$$

we have;

$$|f(\varrho,\zeta(\varrho)) - f(\varrho,\vartheta(\varrho))| \le \frac{1}{2\sqrt{2}} \frac{\Gamma(\frac{3}{2})}{\varpi\Gamma(\frac{3}{2}) + (\delta-1)^{\frac{1}{2}}} \sqrt{\phi(\| (\zeta-\vartheta)^2 \|_\infty)\mu(\phi(\| (\zeta-\vartheta)^2 \|_\infty))}.$$

If $\varsigma_0(\varrho) = \varrho$, then

$$\tau(\varsigma_0(\varrho), \int_0^1 \mathcal{G}(\varrho,\varsigma) f(\psi^{-1}(\varsigma), w(\varsigma))) d\varsigma \ge 0.$$

for $\varrho \in \mathcal{J}$, also,
$\tau(\varsigma(\varrho), z(\varrho)) = \varsigma(\varrho) z(\varrho) \ge 0$ implies that

$$\tau(\int_0^1 \mathcal{G}(\varrho,\varsigma) f(\psi^{-1}(\varsigma), w(\varsigma)) d\varsigma, \int_0^1 \mathcal{G}(\varrho,\varsigma) f(\psi^{-1}(\varsigma), w(\varsigma))) d\varsigma \ge 0;$$

by (1.2), (6) has at minimum one solution.

Now, we discuss the fractional differential equation of the form

$$\begin{cases} -^C D_{0+}^{\gamma,\psi}\zeta(\varrho) = f(t,\zeta(\varsigma)), \quad \varrho \in \mathcal{J}, \\ \zeta'(0) = 0, \quad \zeta(0) + \psi\zeta(1) = \int_0^1 \mathcal{G}(\varsigma,\zeta(\varsigma)) d\varsigma. \end{cases} \tag{8}$$

where $1 < \gamma < 2$, $0 < \psi < 1$, $D_{0+}^{\gamma,\psi}$ is the generalized fractional derivative of order γ in the sense of Caputo introduced by Almeida in [13], and $f, g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$.

Lemma 1.4. *Let $1 < \gamma < 2$, $0 < \psi < 1$, and $\zeta, r : \mathcal{J} \to \mathbb{R}$ are continuous functions. Then the function $\zeta(\varrho) \in C(\mathcal{J})$ is a solution of the following problem*

$$\begin{cases} -^C D_{0+}^{\gamma,\psi}\zeta(\varrho) = \zeta(\varrho), \quad \varrho \in \mathcal{J}, \\ \zeta_\psi^{[1]}(0) = 0, \quad \zeta(0) + \psi\zeta(1) = \int_0^1 r(\varsigma) d\varsigma. \end{cases} \tag{9}$$

if and only if $\zeta \in C(\mathcal{J})$ is a solution of

$$\zeta(\varrho) = \frac{1}{\Gamma(\gamma)} \int_0^1 \psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)\zeta(\varsigma) d\varsigma + \frac{\mathrm{H}_\psi(\varrho)}{\mathrm{H}_\psi(1)} \frac{1}{\psi} \int_0^1 r(\varsigma) d\varsigma. \tag{10}$$

where

$$\mathcal{G}_2(\varrho,\varsigma)=\begin{cases} \mathrm{R}^{\gamma}_{\psi}(\varrho,\varsigma)+\frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\mathrm{R}^{\gamma}_{\psi}(1,\varsigma), & 0\le\varsigma\le\varrho\le 1,\\ \frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\mathrm{R}^{\gamma}_{\psi}(1,\varsigma), & 0\le\varrho\le\varsigma\le 1.\end{cases} \tag{11}$$

Here $\mathrm{H}_{\psi}(\varrho):=[\psi(\varrho)-\psi(0)]$, $\mathrm{R}^{\gamma}_{\psi}(\varrho,\varsigma):=[\psi(\varrho)-\psi(\varsigma)]^{\gamma-1}$, *and* $\mathcal{G}_2(\varrho,\varsigma)$ *is called Green function of BVP (9).*

Lemma 1.5. *For* $\gamma\in(1,2)$, $\mathcal{G}_2$ *satisfies the following:*

- **(i):** $\mathcal{G}_2(\varrho,\varsigma)$ *is continuous on* $\mathcal{J}\times\mathcal{J}$.
- **(ii):** $\mathcal{G}_2(\varrho,\varsigma)>0$, *for* $\varrho,\varsigma\in(0,1)$.
- **(iii):** *We have*

$$\max\{\mathcal{G}_2(\varrho,\varsigma):0\le\varrho,\varsigma\le 1\}=\mathcal{G}_2(1,\varsigma),$$

and

$$\min\{\mathcal{G}_2(\varrho,\varsigma):0\le\varrho,\varsigma\le 1\}=\frac{1}{2}\frac{\mathrm{H}_{\psi}(\varsigma)}{\mathrm{H}_{\psi}(1)}\mathcal{G}_2(1,\varsigma).$$

Theorem 1.3. *Suppose that there exist functions* $\tau:\mathbb{R}^2\to\mathbb{R}$ *and* $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ *such that*
(i) $\exists\ \mu\in\mathcal{F}$ *and* $\phi\in\Phi$ *such that*

$$|f(\varrho,\zeta(\varrho))-f(\varrho,\vartheta(\varrho))|\le\frac{\Gamma(\gamma)}{4\sqrt{2}}\sqrt{\phi(\|(\zeta-\vartheta)^2\|_{\infty})\mu(\phi(\|(\zeta-\vartheta)^2\|_{\infty}))}$$

and

$$|\mathcal{G}(\varrho,\zeta(\varrho))-\mathcal{G}(\varrho,\vartheta(\varrho))|\le\frac{\psi(\varrho)}{4\sqrt{2}}\sqrt{\phi(\|(\zeta-\vartheta)^2\|_{\infty})\mu(\phi(\|(\zeta-\vartheta)^2\|_{\infty})},$$

(ii) $\int_0^1\phi'(\varsigma)\mathcal{G}_2(1,\varsigma)d\varsigma<1$,
(iii) there exists $\zeta_0\in C(\mathcal{J})$ *with*

$$\tau\left(\zeta_0(\varrho),\int_0^1\psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)f(\varsigma,\zeta_0(\varsigma))d\varsigma+\frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\frac{1}{\psi}\int_0^1\mathcal{G}(\varsigma,\zeta_0(\varsigma))d\varsigma\right)\ge 0,$$

for $\varrho\in\mathcal{J}$,
(iv) for $\varrho\in\mathcal{J}$ *and* $\zeta,\vartheta\in C(\mathcal{J})$, $\tau(\zeta(\varrho),\vartheta(\varrho))\ge 0$ *implies*

$$\tau\left(\int_0^1\psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)f(\varsigma,\zeta(\varsigma))d\varsigma+\frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\frac{1}{\psi}\int_0^1\mathcal{G}(\varsigma,\zeta(\varsigma))d\varsigma\right.$$
$$\left.,\int_0^1\psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)f(\varsigma,\vartheta(\varsigma))d\varsigma+\frac{\mathrm{H}_{\phi}(\varrho)}{\mathrm{H}_{\phi}(1)}\frac{1}{\psi}\int_0^1\mathcal{G}(\varsigma,\vartheta(\varsigma))d\varsigma\right)\ge 0,$$

(v) if $\{\zeta_n\}\subseteq C(\mathcal{J})$ *with* $\zeta_n\to\zeta$ *in* $C(\mathcal{J})$ *and* $\tau(\zeta_n,\zeta_{n+1})\ge 0$, *then* $\tau(\zeta_n,\zeta)\ge 0$.
Forthwith, (8) has at minimum one solution.

Proof. By Lemma 1.4, $\zeta\in C(\mathcal{J})$ is a solution of (8) iff a solution of;

$$\zeta(\varrho)=\frac{1}{\Gamma(\gamma)}\int_0^1\psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)f(\varsigma,\zeta(\varsigma))d\varsigma+\frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\frac{1}{\psi}\int_0^1\mathcal{G}(\varsigma,\zeta(\varsigma))d\varsigma,\ \varrho\in\mathcal{J}.$$

Define the operator $O_1:C(\mathcal{J})\to C(\mathcal{J})$ by $O_1\zeta(\varrho)=\zeta(\varrho)$, i.e.,

$$O_1\zeta(\varrho)=\frac{1}{\Gamma(\gamma)}\int_0^1\psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)f(\varsigma,\zeta(\varsigma))d\varsigma+\frac{\mathrm{H}_{\psi}(\varrho)}{\mathrm{H}_{\psi}(1)}\frac{1}{\psi}\int_0^1\mathcal{G}(s,\zeta(s))ds,\ \varrho\in\mathcal{J}.$$

We find a FP of O_1. Now, let $\zeta, \vartheta \in C(\mathcal{J})$ be such that $\tau(\zeta(\varrho), \vartheta(\varrho)) \geq 0$. By using (i), we get

$$
\begin{aligned}
|O_1\zeta(\varrho) - O_1\vartheta(\varrho)|^2 = & \left| \frac{1}{\Gamma(\gamma)} \int_0^1 \psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma)(f(\varsigma,\zeta(\varsigma)) - f(\varsigma,\vartheta(\varsigma)))d\varsigma \right. \\
& \left. + \frac{\mathrm{H}_\psi(\varrho)}{\mathrm{H}_\psi(1)} \frac{1}{\psi(\varsigma)} \int_0^1 (\mathcal{G}(\varsigma,\zeta(\varsigma)) - \mathcal{G}(\varsigma,\vartheta(\varsigma)))d\varsigma \right|^2 \\
\leq & \left[\frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(\varsigma)\mathcal{G}_2(\varrho,\varsigma) \left| f(\varsigma,\zeta(\varsigma)) - f(\varsigma,\vartheta(\varsigma)) \right| d\varsigma \right. \\
& \left. + \sup_{0\leq\varrho\leq 1} \frac{\mathrm{H}_\psi(\varrho)}{\mathrm{H}_\psi(1)} \frac{1}{\psi(\varsigma)} \int_0^1 |\mathcal{G}(\varsigma,\zeta(\varsigma)) - \mathcal{G}(\varsigma,\vartheta(\varsigma))| d\varsigma \right]^2 \\
\leq & \left[\frac{1}{\Gamma(\gamma)} \int_0^1 \psi'(\varsigma)\mathcal{G}_2(1,\varsigma) \frac{\Gamma(\gamma)}{4\sqrt{2}} \sqrt{\phi(\| (\zeta-\vartheta)^2 \|_\infty)\mu(\phi(\| (\zeta-\vartheta)^2 \|_\infty))} d\varsigma \right. \\
& \left. + \frac{1}{4\sqrt{2}} \sqrt{\phi(\| (\zeta-\vartheta)^2 \|_\infty)\mu(\phi(\| (\zeta-\vartheta)^2 \|_\infty))} \right]^2 \\
\leq & \left[\frac{1}{2\sqrt{2}} \sqrt{\phi(\| (\zeta-\vartheta)^2 \|_\infty)\mu(\phi(\| (\zeta-\vartheta)^2 \|_\infty))} \right]^2 \\
= & \frac{1}{8}\phi(\| (\zeta-\vartheta)^2 \|_\infty)\mu(\phi(\| (\zeta-\vartheta)^2 \|_\infty))
\end{aligned}
$$

Therefore

$$\| (O_1\zeta - O_1\vartheta)^2 \|_\infty \leq \frac{1}{8}\phi(\| \zeta-\vartheta \|_\infty^2)\mu(\phi(\| \zeta-\vartheta \|_\infty^2)).$$

Put, $\alpha : C(\mathcal{J}) \times C(\mathcal{J}) \to \mathbb{R}^+$ by

$$\alpha(\zeta,\vartheta) = \begin{cases} 1 & \tau(\zeta(\varrho),\vartheta(\varrho)) \geq 0, \quad \varrho \in \mathcal{J}, \\ 0 & else. \end{cases}$$

Implies that,

$$\alpha(\zeta,\vartheta)8d(O_1\zeta, O_1\vartheta) \leq 8d(O_1\zeta, O_1\vartheta) \leq \phi(\mu(d(\zeta,\vartheta)))\phi(d(\zeta,\vartheta)), \quad \mu \in \mathcal{F}.$$

From (iii),

$$
\begin{aligned}
\alpha(\zeta,\vartheta) \quad \geq \quad & 1 \Rightarrow \tau(\zeta(\varrho),\vartheta(\varrho)) \geq 0, \quad \forall \varrho \in \mathcal{J} \\
\Rightarrow \quad & \tau(O_1(\zeta), O_1(\vartheta)) \geq 0, \\
\Rightarrow \quad & \alpha(O_1(\zeta), O_1(\vartheta)) \geq 1.
\end{aligned}
$$

for $\zeta, \vartheta \in C(\mathcal{J})$. Thus, O_1 is $\alpha-$admissible. From (iii), $\exists\ \zeta_0 \in C(\mathcal{J})$; $\alpha(\zeta_0, O_1\zeta_0) \geq 1$. By (v) and 1.3, we realize ζ^* with $\zeta^* = F\zeta^*$, that is a solution of (8). □

Example 1.2. *Presume the ψ-Caputo fractional integral BVP:*

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2},\frac{e^\varrho}{3}} \xi(\varrho) = f(\varrho,\xi(\varrho)), \quad \varrho \in \varphi, \\ \xi'(0) = 0, \quad \xi(0) + \psi\xi(1) = \int_0^1 \mathcal{G}(\varsigma,\xi(\varsigma))d\varsigma, \end{cases} \tag{12}$$

where $\gamma = \frac{3}{2}$, $\psi(\varrho) = \frac{e^\varrho}{3}$, $0 < \psi < 1$. *Also f satisfies the following condition;*

$$|f(\varrho,\xi(\varrho)) - f(\varrho,\vartheta(\varrho))| \leq \frac{\Gamma(\frac{3}{2})}{48\sqrt{2}}(\varrho+3)\sqrt{\left\|(\xi-\vartheta)^2\right\|_\infty \frac{\left\|(\xi-\vartheta)^2\right\|_\infty}{1+4\left\|(\xi-\vartheta)^2\right\|_\infty}},$$

and

$$|\mathcal{G}(\varrho,\xi(\varrho))-\mathcal{G}(\varrho,\vartheta(\varrho))|\leq\frac{\varrho}{12\sqrt{2}}(\varrho+3)\sqrt{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}\frac{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}}{1+4\left\|(\xi-\vartheta)^{2}\right\|_{\infty}}},$$

Then;

$$\begin{aligned}|f(\varrho,\xi)-f(\varrho,\vartheta)| &\leq \frac{\Gamma(\frac{3}{2})}{8\sqrt{2}}(\varrho+3)\sqrt{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}\frac{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}}{\left\|1+4\left(\xi-\vartheta\right)^{2}\right\|_{\infty}}}\\ &\leq \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}}\sqrt{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}\frac{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}}{\left\|1+4\left(\xi-\vartheta\right)^{2}\right\|_{\infty}}}\\ &= \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}}\sqrt{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}\frac{\left\|(\xi-\vartheta)^{2}\right\|_{\infty}}{\left\|1+4\left(\xi-\vartheta\right)^{2}\right\|_{\infty}}}.\end{aligned}$$

We set $\phi(\varrho)=\varrho$, $\phi(0)=0$ *and* $\mu(t)=\frac{t}{1+4t}$. *Then*

$$|f(\varrho,\xi)-f(\varrho,\vartheta)|\leq\frac{\Gamma(\gamma)}{4\sqrt{2}}\sqrt{\phi(\|\ (\xi-\vartheta)^{2}\ \|_{\infty})\mu(\phi(\|\ (\xi-\vartheta)^{2}\ \|_{\infty}))},$$

Also,

$$|\mathcal{G}(t,\xi)-\mathcal{G}(t,\vartheta)|\leq\frac{\psi}{4\sqrt{2}}\sqrt{\phi(\|\ (\xi-\vartheta)^{2}\ \|_{\infty})\mu(\phi(\|\ (\xi-\vartheta)^{2}\ \|_{\infty}))},$$

Hence,

$$\int_0^1\psi'(\varsigma)\mathcal{G}_2(1,\varsigma)d\varsigma<1.$$

Case 1: *if* $0\leq\varsigma\leq\varrho\leq 1$,

$$\mathcal{G}_2(1,\varsigma)=2\mathrm{R}_{\psi}^{\gamma}(1,\varsigma)=2[\psi(1)-\psi(\varsigma)]^{\gamma-1}=2[e^{\frac{1}{3}}-e^{\frac{\varsigma}{3}}]^{\frac{1}{2}}.$$

Then,

$$\int_0^1\psi'(\varsigma)(\varsigma)\mathcal{G}_2(1,\varsigma)d\varsigma=\frac{2}{3}\int_0^1 e^{\frac{\varsigma}{3}}[e^{\frac{1}{3}}-e^{\frac{\varsigma}{3}}]^{\frac{1}{2}}d\varsigma=\frac{4}{3}\left(e^{\frac{1}{3}}-1\right)^{\frac{3}{2}}\approx 0.3<1.$$

Case 2: *if* $0\leq\varrho\leq\varsigma\leq 1$,

$$\mathcal{G}_2(1,\varsigma)=\mathrm{R}_{\psi}^{\gamma}(1,\varsigma)=[\psi(1)-\psi(\varsigma)]^{\gamma-1}=[e^{\frac{1}{3}}-e^{\frac{\varsigma}{3}}]^{\frac{1}{2}}.$$

Then,

$$\int_0^1\psi'(\varsigma)(\varsigma)\mathcal{G}_2(1,\varsigma)d\varsigma=\frac{1}{3}\int_0^1 e^{\frac{\varsigma}{3}}[e^{\frac{1}{3}}-e^{\frac{\varsigma}{3}}]^{\frac{1}{2}}d\varsigma=\frac{2}{3}\left(e^{\frac{1}{3}}-1\right)^{\frac{3}{2}}\approx 0.2<1.$$

Hence, suppositions of Theorem 1.3 hold. So, (12) has a solution on $\mathcal{J}$.

2. Conclusion

This paper, intend to examine some BVPs for a nonlinear fractional differential equation involving a general form of Caputo fractional derivative operator with respect to new function ψ in $b - MS$s.The obtained results in this article are more general and cover many of the parallel problems that contain special cases of function , because our proposed method contains investigating of the existence of solutions for some BVPs with the global fractional derivative that extends many BVP with classic fractional derivatives.

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