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SPACE-TIME FRACTIONAL HEAT EQUATION'S SOLUTIONS WITH FRACTIONAL INNER PRODUCT

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Abstract. The main goal in this study is to determine the analytic solution of one-dimensional initial boundary value problem including sequential space-time fractional differential equation with boundary conditions in Neumann sense. The solution of the space-time fractional diffusion problem is accomplished in series form by employing the separation of variables method. To obtain coefficients in the Fourier series is utilized a fractional inner product. The obtained results are supported by an illustrative example. Moreover, it is observed that the implementation of the method is straightforward and smooth.

Keywords: Caputo derivative, Mittag-Leffler function, Fractional inner product.

AMS Subject Classification: 26A33, 65M70.

1. Introduction

As partial differential equations (PDEs) of fractional order play an essential role in modeling numerous processes and systems in various scientific research areas such as applied mathematics, physics chemistry, etc., the interest in this topic has become enormous. Since the fractional derivative is non-local, the model with a fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non-linear processes [1, 20, 21, 22, 25, 14, 16]. That is why this has attracted an increasing number of researchers. The derivatives in the sense of the Liouville-Caputo is one of the most common since modeling of physical processes with fractional differential equations, including the Liouville-Caputo derivative, is much better than other models. This result is supported by various researches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Especially there are various studies on fractional diffusion equations: Exact analytical solutions of heat equations are obtained by using operational method [27]. The existence, uniqueness, and regularity of the solution of the impulsive sub-diffusion equation are established utilizing eigenfunction expansion [19]. The anomalous diffusion models with non-singular power-law kernel have been investigated, and constructed [26]. Moreover, the Liouville-Caputo

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derivative of constant is zero, which is not held by many fractional derivatives. The solutions of fractional PDEs and ordinary differential equations (ODEs) are determined in terms of the Mittag-Leffler function. The diffusion problem, including fractional derivative in the Liouville-Caputo sense, has been studied by Sevindir, and Demir [18]. This study can be regarded as an extension of it.

2. Preliminary Results

In this part are given some basic descriptions and accomplished results of fractional derivative in the Liouville-Caputo sense.

Definition 2.1. *q^{th} order of the Liouville-Caputo fractional derivative of $u(t)$ is defined as:3*

$$D^q u\left(t\right) = \frac{1}{\Gamma\left(n-q\right)} \int_{t_0}^{t} \left(t-s\right)^{n-q-1} u^{(n)}(s) ds, t \in [t_0, t_0+T], \tag{1}$$

where $n-1 < q < n$ and $u^{(n)}\left(t\right) = \frac{d^n u}{dt^n}$. If q is an integer, then the Liouville-Caputo fractional derivative becomes the integer-order derivative.

Definition 2.2. *q^{th} order of the Liouville-Caputo fractional derivative of $u(t)$ is defined as*

$$D^q u\left(t\right) = \frac{1}{\Gamma\left(1-q\right)} \int_{t_0}^{t} \left(t-s\right)^{-q} u'(s) ds, t \in [t_0, t_0+T], \tag{2}$$

where $0 < q < 1$.

Definition 2.3. *The Mittag-Leffler function with the parameters α and β is given as follows* [24, 15]*:*

$$E_{\alpha,\beta}\left(\lambda\left(t-t_0\right)^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\left(\lambda\left(t-t_0\right)^{\alpha}\right)^k}{\Gamma\left(\alpha k+\beta\right)}, \alpha, \beta > 0, \lambda \in \mathbb{R}. \tag{3}$$

If $t_0 = 0, \alpha = \beta = q$, then we get

$$E_{q,q}\left(\lambda t^q\right) = \sum_{k=0}^{\infty} \frac{\left(\lambda t^q\right)^k}{\Gamma\left(qk+q\right)}, q > 0. \tag{4}$$

Moreover, substituting $q = 1$, in the equation (4) we have $E_{1,1}\left(\lambda t\right) = e^{\lambda t}$. If the reader wants more information, they should refer to [17, 23]. The following functions are used to obtain the solution of the problem discussed in this study.

$$\sin_q\left(\mu t^q\right) = \frac{E_{q,1}\left(i\mu t^q\right) - E_{q,1}\left(-i\mu t^q\right)}{2i} = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k \left(\mu t^q\right)^{2k+1}}{\Gamma\left((2k+1)q+1\right)}, \tag{5}$$

and

$$\cos_q\left(\mu t^q\right) = \frac{E_{q,1}\left(i\mu t^q\right) + E_{q,1}\left(-i\mu t^q\right)}{2} = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k \left(\mu t^q\right)^{2k}}{\Gamma\left(2kq+1\right)}. \tag{6}$$

When $q = 1$, equations (5) and (6) are $\sin\left(\mu t\right)$ and $\cos\left(\mu t\right)$ respectively.

2.1. **Inner Product.** For $0 < \beta \leq 1,\ \mu \in \mathbb{R}$, let $V = span\{\sin_\beta\left(\mu(\frac{x}{b-a})^\beta\right), \cos_\beta\left(\mu(\frac{x}{b-a})^\beta\right)\}$ be the vector space in consideration. This vector space is made up of all linear combinations of functions $\sin_\beta\left(\mu(\frac{x}{b-a})^\beta\right), \cos_\beta\left(\mu(\frac{x}{b-a})^\beta\right)$ defined on $J = [a, b]$. As a result, the linear transformation $T : V \rightarrow\ span\{\sin\left(\frac{\mu x}{b-a}\right), \cos\left(\frac{\mu x}{b-a}\right)\}$ turn out to be onto and one-to-one. Hence its inverse transformation T^{-1} exists. The inner product $< \bullet, \bullet >: V \times V \rightarrow \mathbb{R}$ is defined as

$$< u\,(x;\beta)\,,\ v\,(x;\beta) > = T^{-1}\left(\int Tu\,(x;\beta)\,.Tv\,(x;\beta)\,dx\right)\Big|_{x=a}^{b}, \tag{7}$$

where $Tu\,(x;\beta) = u(x;1)$ and $Tv\,(x;\beta) = v(x;1)$ [18].
The sequential space-time fractional problem discussed in this study is as follows:

$$D_t^\alpha u\,(x,t) = \gamma^2 D_x^{2\beta} u\,(x,t)\,, \tag{8}$$

$$u\,(x,0) = f(x), \tag{9}$$

$$D_x^\beta u\,(0,t) = D_x^\beta u\,(l,t) = 0, \tag{10}$$

where $0 < \alpha < 1,\ 1 < 2\beta < 2, 0 \leq x \leq l,\ 0 \leq t \leq T, \gamma \in \mathbb{R}$.

3. Main Results

The generalized solution of the problem (8)-(10) is formed in an analytical form through the separation of variables method as follows:

$$u\,(x,t;\alpha,\beta) = X(x;\beta)\ T(t;\alpha,\beta), \tag{11}$$

where $0 \leq x \leq l, 0 \leq t \leq T$.
The functions X and T depend on orders of fractional derivatives with respect to x and t. Plugging (11) into (8) and adjusting it, we get

$$\frac{D_t^\alpha\,(T\,(t;\alpha,\beta))}{T\,(t;\alpha,\beta)} = \gamma^2\frac{D_x^{2\beta}\,(X\,(x;\beta))}{X\,(x;\beta)} = -\lambda^2(\beta). \tag{12}$$

With boundary condition (10) and the fractional differential equation acquired from Equation (12), the following problem is obtained:

$$D_x^{2\beta}\,(X\,(x;\beta)) + \lambda^2\,(\beta)\,X\,(x;\beta) = 0, \tag{13}$$

$$D_x^\beta X\,(0;\beta) = D_x^\beta X\,(l;\beta) = 0. \tag{14}$$

The solution of problem (13)-(14) is accomplished as follows:

$$X\,(x;\beta) = E_{\beta,1}(rx^\beta). \tag{15}$$

Therefore, the characteristic equation for equation (13) is calculated in the following form:

$$r^2 + \lambda^2(\beta) = 0. \tag{16}$$

Case 1: If $\lambda\,(\beta) = 0$, then the characteristic equation have coincident solutions $r_{1,2} = 0$, then the solution of (13)-(14) becomes

$$\begin{aligned} X\,(x;\beta) &= k_1\frac{x^\beta}{\beta} + k_2, \\ D_x^\beta X\,(x;\beta) &= \frac{k_1}{\beta}D_x^\beta x^\beta + D_x^\beta k_2 = \frac{k_1}{\beta}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\beta+1)}t^{\beta-\beta} = k_1\frac{\Gamma(\beta+1)}{\beta} \\ &= k_1\frac{\beta\Gamma(\beta)}{\beta} = k_1\Gamma\,(\beta)\,. \end{aligned}$$

By utilizing equation (14), we obtain

$$D_x^{\beta} X(0) = k_1 \Gamma(\beta) = 0 \Rightarrow k_1 = 0. \tag{17}$$

Hence the solution becomes

$$X(x;\beta) = k_2. \tag{18}$$

Likewise, from the last boundary condition, we get

$$D_x^{\beta} X(l) = k_1 \Gamma(\beta) = 0 \Rightarrow k_1 = 0, \tag{19}$$

which implies that

$$X_0(x;\beta) = k_2. \tag{20}$$

Case 2: If $\lambda(\beta) \neq 0$, then the characteristic equation have the solutions

$$r_{1,2} = \mp i\lambda(\beta), \tag{21}$$

which leads to the general solution of the problem (13)-(14) have the following form:

$$\begin{aligned} X(x;\beta) &= c_1 \cos_\beta\left(\lambda(\beta) x^{\beta}\right) + c_2 sin_\beta(\lambda(\beta)x^{\beta}), \\ D_x^{\beta} X(x;\beta) &= -c_1\lambda(\beta)\, sin_\beta\left(\lambda(\beta) x^{\beta}\right) + c_2\lambda(\beta)\cos_\beta(\lambda(\beta)x^{\beta}). \end{aligned}$$

By utilizing equation (14), we obtain

$$D_x^{\beta} X(0) = 0 = c_2\lambda(\beta) \Rightarrow c_2 = 0. \tag{22}$$

Hence the solution becomes

$$\begin{aligned} X(x;\beta) &= c_1 \cos_\beta\left(\lambda(\beta) x^{\beta}\right), \\ D_x^{\beta} X(x;\beta) &= -c_1\lambda(\beta)\, sin_\beta\left(\lambda(\beta) x^{\beta}\right). \end{aligned}$$

Similarly last boundary condition leads to

$$D_x^{\beta} X(l) = -c_1\lambda(\beta)\, sin_\beta\left(\lambda(\beta) l^{\beta}\right) = 0, \tag{23}$$

which implies that

$$sin_\beta\left(\lambda(\beta) l^{\beta}\right) = 0. \tag{24}$$

Let $w_n(\beta) = \sqrt{\lambda(\beta)} l^{\beta}$. Therefore, with the help of $w_n(\beta)$, the eigenvalues are obtained as follows.

$$\lambda_n(\beta) = \frac{w_n^2(\beta)}{l^{2\beta}}, 0 < w_1(\beta) < w_2(\beta) < w_3(\beta) < \dots \tag{25}$$

$$X_n(x;\beta) = c_n \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right) = \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right), n = 1,2,3,\dots \tag{26}$$

is the solution of the problem (13)-(14).
The following equation is obtained from the left of (12) for each eigenvalue $\lambda_n(\beta)$:

$$\frac{D_t^{\alpha}(T(t;\alpha,\beta))}{T(t;\alpha,\beta)} = -\gamma^2\lambda^2(\beta). \tag{27}$$

The solution for equation (20) is as follows:

$$T_n(t;\alpha,\beta) = k_1 E_{\alpha,1}\left(-\gamma^2\lambda_n^2(\beta)t^{\alpha}\right) = E_{\alpha,1}\left(-\gamma^2\frac{w_n^2(\beta)}{l^{2\beta}}t^{\alpha}\right), n = 1,2,3,\dots \tag{28}$$

$$u_n(x,t;\alpha,\beta) = E_{\alpha,1}\left(-\gamma^2\frac{w_n^2(\beta)}{l^{2\beta}}t^{\alpha}\right)\cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^{\beta}\right), n = 1,2,3,\dots \tag{29}$$

is the solution for each $\lambda_n(\beta)$ and so we get

$$u(x,t;\alpha,\beta) = d_0 + \sum_{n=1}^{\infty} d_n \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right) E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2(\beta)}{l^{2\beta}} t^\alpha\right), \tag{30}$$

that satisfies both equation (8) and equation (10).
Equation (7) is used so as to establish the solution which satisfies equation (9). In (30), replacing t by 0 and utilizing equation (7), we have

$$u(x,0) = f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right).$$

$$\begin{aligned} < f(x), \cos_\beta\left(w_k(\beta)\left(\frac{x}{l}\right)^\beta\right) >&=< d_o, \cos_\beta\left(w_k(\beta)\left(\frac{x}{l}\right)^\beta\right) > \\ &+ \sum_{n=1}^{\infty} d_n < \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right), \cos_\beta\left(w_k(\beta)\left(\frac{x}{l}\right)^\beta\right) > . \end{aligned}$$

$$< \cos_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right), \cos_\beta\left(w_k(\beta)\left(\frac{x}{l}\right)^\beta\right) >= T^{-1}\left(\int \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{k\pi x}{l}\right)dx\right)\Bigg|_{x=0}^{x=l}.$$

$$\begin{aligned} T^{-1}\left(\int \cos\left(\frac{k\pi x}{l}\right)f(x)dx\right)\Bigg|_{x=0}^{x=l} &= d_0\, T^{-1}\left(\int \cos\left(\frac{k\pi x}{l}\right)dx\right)\Bigg|_{x=0}^{x=l} \\ &+ \sum_{n=1}^{\infty} T^{-1}\left(\int \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{k\pi x}{l}\right)dx\right)\Bigg|_{x=0}^{x=l}. \end{aligned}$$

By utilizing equation (7) we obtain the coefficients d_n for $n = 1, 2, 3, \ldots$ in the following form:

$$\begin{aligned} d_0 &= \frac{1}{l}\, T^{-1}\left(\int f(x)dx\right)\Bigg|_{x=0}^{x=l}, \\ d_n &= \frac{2}{l}\, T^{-1}\left(\int \cos\left(\frac{n\pi x}{l}\right)f(x)dx\right)\Bigg|_{x=0}^{x=l}. \end{aligned}$$

4. Illustrative Example

We first take into account the following initial boundary value problem in this part:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \\ u_x(0,t) &= 0, u_x(1,t) = 0, \\ u(x,0) &= \cos(\pi x), \end{aligned} \tag{31}$$

where $0 \le x \le 1, 0 \le t \le T$. Solution to the problem (31) is as follows:

$$u(x,t) = \cos(\pi x)e^{-\pi^2 t}. \tag{32}$$

Now we take into account the following fractional heat-like problem:

$$D_t^\alpha u(x,t) = D_x^{2\beta} u(x,t), \tag{33}$$

$$u(x,0) = cos(\pi x), \tag{34}$$

$$u_x(0,t) = u_x(1,t) = 0, \tag{35}$$

where $0 < \alpha < 1,\ 1 < 2\beta < 2, 0 \leq x \leq 1,\ 0 \leq t \leq T$.
By means of (30), the solution of problem (33)-(35) is represented in the following form:

$$u(x,t;\alpha,\beta) = d_0 + \sum_{n=1}^{\infty} d_n \cos_\beta \Big(w_n(\beta)\, x^\beta \Big) E_{\alpha,1}\left(-w_n^2(\beta)\, t^\alpha\right). \tag{36}$$

Notice that boundary conditions (35) and the fractional equation (33) is satisfied by equation (36). Equation (7) is used so as to establish the solution which satisfies equation (34). In (36), replacing t by 0 and utilizing equation (7), we have

$$u(x,0;\alpha,\beta) = d_0 + \sum_{n=1}^{\infty} d_n \cos_\beta \Big(w_n(\beta)\, x^\beta \Big). \tag{37}$$

By utilizing equation (7), we obtain the coefficients d_n for $n = 1, 2, 3, \dots$ in the following form:

$$\begin{aligned} d_0 = \frac{1}{l}\, T^{-1} \left(\int f(x)dx \right) \bigg|_{x=0}^{x=l} &= T^{-1} \left(\int \cos(\pi x) dx \right) \bigg|_{x=0}^{x=1} = T^{-1} \left(\frac{1}{\pi} \sin(\pi x) dx \right) \bigg|_{x=0}^{x=1} \\ &= \frac{1}{\pi} sin_\beta \Big(w_1(\beta)\, x^\beta \Big) \bigg|_{x=0}^{x=1} = \frac{1}{\pi} \left[sin_\beta(w_1(\beta)) - sin_\beta(0) \right] = 0. \end{aligned}$$

$$d_n = \frac{2}{l}\, T^{-1} \left(\int \cos\left(\frac{n\pi x}{l}\right) f(x) dx \right) \bigg|_{x=0}^{x=l} = 2\, T^{-1} \left(\int \cos(n\pi x) \cos(\pi x)) dx \right) \bigg|_{x=0}^{x=1}.$$

Thus $d_n = 0$ for $n \neq 1$. For $n = 1$, we get

$$\begin{aligned} d_1 = 2\, T^{-1} \left(\int \cos^2(\pi x)) dx \right) \bigg|_{x=0}^{x=1} &= 2\, T^{-1} \left(\frac{x}{2} + \frac{1}{4\pi} \sin(2\pi x) \right) \bigg|_{x=0}^{x=1} \\ &= 2 \left[\frac{x^\beta}{2} + \frac{1}{w_4(\beta)} \sin_\beta\left(w_2(\beta)\, x^\beta\right) \right] = 1^\beta - 0^\beta = 1. \end{aligned}$$

Thus

$$u(x,t;\alpha,\beta) = \cos_\beta \Big(w_1(\beta)\, x^\beta \Big) E_{\alpha,1}\left(-w_1^2(\beta)\, t^\alpha\right). \tag{38}$$

When $\alpha = \beta = 1$, solution (38) is the same as solution (32), which verifies the correctness of the method we implement.

5. Conclusion

This research focuses on constructing the one-dimensional exact solution of sequential space-time fractional diffusion problem in the Liouville-Caputo sense in series form. Taking the separation of variables into account, the solution is formed in the form of a Fourier series concerning the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem, including fractional derivative in the Liouville-Caputo sense. Because of the solution's structure, the inner product is used to ascertain the coefficients effectively. Based on the analytic solution, we conclude that diffusion processes decay exponentially until the initial condition is reached. As α tends to 0, the rate of decay increases. This implies that in the mathematical model for diffusion of the matter, which has low diffusion rate, the value of α must be close to 0. This model can account for various diffusion processes of various methods.

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Suleyman Cetinkaya for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.12, N.3.

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