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ON THE k-DISTANCE DIFFERENTIAL OF GRAPHS

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Abstract. Let $G = (V, E)$ be a graph and $X \subseteq V$. The differential of X is defined as $\partial(X) = |B(X)| - |X|$ where $B(X)$ is a set of all vertices in $V - X$ which has adjacent vertex in the set X. Also, the differential of a graph G written $\partial(G)$, is equal to $max\{\partial(X) : X \subseteq V\}$. In this paper, we initiate the study of k-distance differential of graphs which is a generalization of differential of graphs. In addition, we show that for any connected graph G of order $n \geq k+2$, the number $\frac{(2k-1)n}{2k+3}$ is a sharp lower bound for k-distance differential of G. We also obtain upper bounds for k-distance differential of graphs. Finally, we characterize the graphs whose k-distance differential belongs to $\{n-2, n-3, 1\}$.

Keywords: Differential of graphs, k-distance domination, k-distance differential of graph, domination number of graphs.

AMS Subject Classification: 05C69.

1. Introduction

Currently, social networks such as Facebook, Twitter and Instagram are Frecognized as important communication and information tools. Due to their far-reaching publicity, social networks are now broadly used in political geostrategies and viral marketing. Some authors have used maximization problems, utilizing their extensive applications in these topics as a fundamental algorithmic problem for disseminating information on social networks [7, 9, 12]. These problems require determining the best group of nodes to influence the rest. Assume that $G = (V, E)$ is a graph of order n. The study of the graph parameter $\partial(G)$ which is called the differential of G can be deduced from such scenarios. For every set of vertices $D \subseteq V$, suppose that $B(D)$ is the set of vertices in $V - D$ that have a neighbor in the vertex set D, and put $C(D) = V - (D \cup B(D))$. The *differential* of D is defined as $\partial(D) = |B(D)| - |D|$ and the *differential of a graph* G, written $\partial(G)$, is defined

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as $max\{\partial(D) : D \subseteq V\}$. A set D is also called a ∂-set or *differential set* when we have $\partial(D) = \partial(G)$. If D has minimum cardinality among all ∂-sets then D is called a minimum ∂-set. The graph parameter ∂ was presented in [13, 14], where several basic properties were obtained and it has been also studied in [1, 2, 3, 4, 5, 11, 15, 16, 17, 18]. More generally, the idea of viral marketing (as explained in [7, 9, 12]) tries to use customers acquired by specific marketing offers as multiplicators, influencing their immediate neighborhood to buy certain products. This pattern is a accidental one from the start but can be simplified to lead to the graph theoretical problem studied in this paper. Note that for a graph G of order $n \geq 2$, $0 \leq \partial(G) \leq n-2$. For each graph G with connected components $G_1, \cdots, G_k$, $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. Thus, we will only consider connected graphs. As described in [14] this parameter is related to the famous parameter $\gamma(G)$ denoting the minimum size of a vertex dominating set in G.

This paper is organized as follows: We initiate the study of k-distance differential of graphs which is a generalization of differential of graphs in Section 2. In Section 3, we show that for any connected graph G of order $n \geq k+2$, the number $\frac{(2k-1)n}{2k+3}$ is a sharp lower bound for the k-distance differential of G. We discuss the complexity of k-distance differential number in section 4. In Section 5, we obtain upper bounds for this parameter of graphs. Finally, in Section 6 we characterize all graphs G for which their k-distance differentials belong to $\{n-2, n-3, 1\}$.

2. Preliminary results

We refer the reader to [19, 10] for any terminology and notation which are not defined here. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E(G)\}$ while the *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$. The *closed neighborhood* of a vertex $v \in V$ is $N(v) \cup \{v\}$ while the *closed neighborhood* of a set $S \subseteq V$ is the set $N[S] = N(S) \cup S$. Let E_v be the set of edges incident with v in G, that is, $E_v = \{uv \in E(G) : u \in N(v)\}$. We denote the *degree* of v by $\deg_G(v) = |E_v|$. A *leaf* of G is a vertex with degree one, a *support vertex* is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves and an *end-support vertex* is a support vertex whose neighbors (at most except one) are leaves. The set of all leaves adjacent to a vertex v is denoted by $L(v)$. We denote *the minimum degree* of a graph G with $\delta(G)$ and *the maximum degree* with $\Delta(G)$. The *distance* between two vertices x and y in G denoted $d_G(x, y)$, is the length of the shortest (x, y)-path in G. The *diameter* of a graph G denoted by $diam(G)$, is the greatest distance between two vertices of G. For a set $S \subseteq V$, the *private neighborhood* $pn[v, S]$ of $v \in S$ is defined by $pn[v, S] = N[v] - N[S - \{v\}]$ equivalently, $pn[v, S] = \{u \in V : N[u] \cap S = \{v\}\}$. Every vertex in $pn[v, S]$ is named a *private neighbor* of v. The *external private neighborhood* $epn(v, S)$ of v with respect to S includes those private neighbors of v in $V - S$. Therefore, $epn(v, S) = pn[v, S] - \{v\}$ [13]. Let k be a positive integer. Given a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is equal to the set $\{u \in V(G) : u \neq v \text{ and } d(u, v) \leq k\}$ and the *closed k-neighborhood* $N_{k,G}[v]$ is equal to the set $N_{k,G}(v) \cup \{v\}$. The open k-neighborhood $N_{k,G}(S)$ of a set $S \subseteq V$ is equal to the set $\cup_{v \in S} N_{k,G}(v)$ and the closed k-neighborhood $N_{k,G}[S]$ of a set $S \subseteq V$ is equal to the set $N_{k,G}(S) \cup S$. The *k-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The *minimum* and *maximum* k-degree of a graph G are shown by $\delta_k(G)$ and $\Delta_k(G)$ respectively. For a nonempty subset $S \subseteq V$ and any vertex $v \in V$, we denote by $N_{k,S}(v)$ the set of k-neighbors of v in S that is, $N_{k,S}(v) := \{u \in S : 0 < d(u, v) \leq k\}$ and $d_{k,S}(v) = |N_{k,S}(v)|$. The graph G is called *k-distance regular* if $\delta_k(G) = \Delta_k(G)$. The k^{th} power G^k of a graph G is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{xy : 0 < d(x, y) \leq k\}$. Now clearly, we have $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$,

$N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $deg_{k,G}(v) = deg_{1,G^k}(v) = deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. A vertex v is called *k-adjacent* to (or *k-neighbor of*) a vertex w if $d(v, w) = k$.

A set $S \subseteq V$ is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set $S \subseteq V$ is called a $\gamma(G)$-set if $|S| = \gamma(G)$ [10]. Assume that $k \geq 1$ is an integer. A set $D \subseteq V$ is called a *k-distance dominating set* of G if every vertex in $V(G) - D$ is within distance k of at least one vertex in D. The *k-distance domination number* $\gamma^k(G)$ of G is the minimum cardinality among all k-distance dominating sets of G [8].

Assume that $G = (V, E)$ is an arbitrary graph and notice the following game. You are allowed to buy as many tokens from a bank as you like at a cost of 1 dollar each. For example suppose you buy s tokens. You then place the tokens on some subset of s vertices of V. For every vertex of G which has no token on it, but is k-adjacent to a vertex with a token on it, you received 1 dollar from the bank. Your target is to maximize your profit, that is the total value received from the bank minus the cost of the tokens bought. According to this game we define the *k-distance differential* of a set X to be $\partial^k(X) = |B^k(X)| - |X|$ and the k-distance differential of a graph G to be equal to $max\{\partial^k(X)\}$ for any subset X of V as follows.

Definition 2.1. *Let $G = (V, E)$ be a graph, $X \subseteq V$ and $B^k(X)$ be the set of vertices in $V - X$ that have a k-neighbor in the set X. For a nonempty subset $X \subseteq V$ we say $C^k(X) = V - (X \cup B^k(X))$. We define the k-distance differential of a set X to be $\partial^k(X) = |B^k(X)| - |X|$ and the k-distance differential of a graph G to be equal to $\partial^k(G) =max\ \{\partial^k(X) : X \subseteq V\}$. A set D satisfying $\partial^k(D) = \partial^k(G)$ is called a ∂^k-set or k-distance differential set. A graph G is said to be a k-distance dominant differential if it contains a ∂^k-set which is also a k-distance dominating set.*

An alternative way to study the k-distance differential of a graph is as follows, which is based on the notion of a *k-distance big subtree*, i.e. some subtree $S_d = T_{1,d}$ with $d \geq 2$ such that $T_{1,d}$ is a *rooted tree* with root c such that $d = deg(c) = max\{deg(v) : v \in V(T_{1,d})\}$ and the maximum distance between the root c and other vertices of subtree S_d is k. For the graph $G = (V, E)$, a *k-distance big subtree packing(kDBSP)* is given by a vertex-disjoint collection $\mathcal{C} = \{X_i : 1 \leq i \leq t\}$ of (not necessarily induced) k-distance big subtrees $X_i \subseteq V$, the graph induced by X_i, written $G[X_i]$ for short, contains some S_d with $d = |X_i| - 1 \geq 2$. If $\mathcal{C}$ is a collection of k-distance big subtree packing of G, we also show this property by $\mathcal{C} \in kDBSP(G)$. For a set $S \subseteq V$, the *private k-neighborhood* $pn_{k,G}[v, S]$ of $v \in S$ is defined by $pn_{k,G}[v, S] = N_{k,G}[v] - N_{k,G}[S - \{v\}]$ equivalently, $pn_{k,G}[v, S] = \{u \in V : N_{k,G}[u] \cap S = \{v\}\}$. Every vertex in $pn_{k,G}[v, S]$ is called a *private k-neighbor* of v. The *external private k-neighborhood* $epn_{k,G}(v, S)$ of v with respect to S includes those private k-neighbors of v in $V - S$. Therefore $epn_{k,G}(v, S) = pn_{k,G}[v, S] - \{v\}$. We will say that $v \in V - S$ is an *S-external private k-neighbor(S-epkn)* of u if $N_{k,G}(v) \cap S = \{u\}$.

Lemma 2.1. *For a graph G, $\partial^k(G) = max\{\sum_{S \in \mathcal{C}}(|S| - 2) : \mathcal{C} \in kDBSP(G)\}$.*

Proof. For every $\mathcal{C} = \{X_1, \cdots, X_t\} \in kDBSP(G)$, if we consider the set D consisting of all centers $x_1, \cdots, x_t$ of $X_1, \cdots, X_t$, then we have

$$\partial^k(G) \geq \partial^k(D) = |B^k(D)| - |D| \geq \sum_{j=1}^{t}(|N_{k,G}(x_j) \cap X_j| - 1) = \sum_{j=1}^{t}(|X_j| - 2).$$

Thus, $\partial^k(G) \geq max\{\sum_{S\in\mathcal{C}}(|S|-2) : \mathcal{C} \in kDSP(G)\}$. Conversely, if we take a minimum ∂^k-set $D = \{v_1, \cdots, v_t\} \subseteq V$, then each vertex $v \in D$ has at least two D-*epkn*'s because of if there exists $v \in D$ having less than two D-*epkn*, $D' = D - \{v\}$ satisfies $\partial^k(D') \geq \partial^k(D)$ contracting the minimality of D. Then the family of sets $X_1 = N_{k,G}[v_1] - \{v_2, \cdots, v_t\}$ and $X_j = N_{k,G}[v_j] - (\cup_{i=1}^{j-1} X_i \cup \{v_{j+1}, \cdots, v_t\})$ for every $j = 2, \cdots, t$ is a k-distance subtree packing of G and

$$\partial^k(D) = \sum_{j=1}^{t}(|N_{k,G}(v_j) - (\cup_{i=1}^{j-1}(X_i \cup \{v_{j+1}, \cdots, v_t\})| - 1)$$

$$= \sum_{j=1}^{t}(|X_i| - 2) \leq max\{\sum_{S\in\mathcal{C}}(|S|-2) : \mathcal{C} \in kDBSP(G)\}.$$

Thus the proof ends. □

We now present the following lemma which provides the basic background for the proof of the next theorem.

Lemma 2.2. *If D is minimum ∂^k-set of G, then the set $\{D, B^k(D), C^k(D)\}$ is a partition(not necessarily nonempty) of V such that:*
(a) *for all $v \in D$, $\deg_{k,B^k(D)}(v) \geq 2$.*
(b) *for all $v \in B^k(D)$, $\deg_{k,C^k(D)}(v) \leq 2$.*
(c) *for all $v \in C^k(D)$, $\deg_{k,C^k(D)}(v) \leq 1$.*

Proof. (a) If there exists $v \in D$ such that $\deg_{k,B^k(D)}(v) \leq 1$, then we put $S = D - \{v\}$. Thus we have $\partial^k(S) \geq \partial^k(D)$ and $|S| < |D|$, that is a contradiction.
(b) If there exists $v \in B^k(D)$ such that $\deg_{k,C^k(D)}(v) \geq 3$, then we put $S = D \cup \{v\}$. Thus we obtain a greater k-distance differential, that is a contradiction.
(c) If there exists $v \in C^k(D)$ such that $\deg_{k,C^k(D)}(v) \geq 2$, then taking $S = D \cup \{v\}$, the k-distance differential would be greater, that is a contradiction. □

3. Existence results and complexity

Lemma 3.1. *Let G be a connected graph. If D is a ∂^k-set of G and D is not a k-distance dominating set, then $B^k(D) = X = V - D - S$ where $S \neq \emptyset$, $\partial^k(D) = |V| - 2|D| - |S|$ and every vertex in S has at most one k-neighbor in S.*

Proof. Let G be a graph and $\partial^k(G) = \partial^k(D) =$ be a k-distance differential set of G. Then it is clear $B^k(D) = X = V - D - S$ where $S \neq \emptyset$. On the contrary, suppose that a vertex v in S has at least two k-neighbors vertices in S like S'. Let $D' = D \cup \{v\}$. Then $B^k(D') = V - D - S \cup S'$ and $\partial^k(D') = |B^k(D')| - |D'| = |V| - |D| - |S| + |S'| - |D| - 1 = |V| - 2|D| - |S| + |S'| - 1 > \partial^k(D)$. □

By Lemma 3.1 we have.

Proposition 3.1. *Let G be a connected graph.*
1. If D is a ∂^k-set of G, then D is a $k+1$-distance dominating set of G. Therefore a subset D' of D is a γ^{k+1}-set of G.
2. If D is a γ^k-set of G, then $\partial^k(D) = |V(G)| - 2|D| \leq \partial^k(G)$. Therefore there exists a set X with $|X| \leq |D|$ such that $\partial^k(X) = \partial^k(G)$.

Proof. 1. Let $B^k(D) = V - D - S$. If $S = \emptyset$, then D is a k-distance dominating set of G and the proof is trivial. Let $S \neq \emptyset$. Then Lemma 3.1 shows that every vertex in S has distance at most 1 of a vertex in $V - D$. Therefore every vertex in S has distance at most $k+1$ of a vertex in D. Therefore, if D is a ∂^k-set of G, then D is a $k+1$ distance dominating set and there is a subset D' of D for which γ^{k+1}-set of G.
2. Let D be a γ^k-set of G. Then $\partial^k(D) = |V(G)| - 2|D|$ and $\partial^k(G) \geq |V(G)| - 2|D|$. Therefore there is a set X of $V(G)$ such that $\partial^k(G) = \partial^k(G)$ and $|V(G)| - 2|D| \leq |V(G)| - 2|X| - |S|$. This inequality shows that $|X| \leq |D|$. □

Now we discuss on the complexity of k-distance differential set of a graph G. The rest of this section we want to focus on the complexity of the ∂^k problem. So we interest to know the decision problem whether an arbitrary graph admits a k-distance differential number of a graph G (k-DDN) with at least m. We give the following decision problem as.

m(k-DDN) Problem:
INSTANCE: A connected graph G and a positive integer $m \leq |V(G)|$.
QUESTION: Is $\partial^k \geq m$?

Our aim is to show that the problem is NP-complete for arbitrary graph G. To this end we make use of the well-known k-DISTANCE DOMINATION PROBLEM (k-DISDN problem) which is known to be NP-complete for $k \geq 1$.

Theorem 3.1. *For $k \geq 1$, the (k-DDN) Problem is NP-hard.*

Proof. By Proposition 3.1 every k-DDN leads to a $k+1$-distance dominating set and any k-distance dominating set leads to a k-DDN. Since (k-DISDN problem) is NP-hard, hence k-DDN Problem is NP-hard. □

4. Lower bound of k-distance differential

This section prepares the main part of our paper. We put out one core result, giving lower bound on connected graphs in general in terms of the order of the graph. We complement this result by representing infinite families of graphs that attain the given bound.

Theorem 4.1. *For every connected graph G of order $n \geq k+2$, $\partial^k(G) \geq \frac{(2k-1)n}{2k+3}$. This bound is sharp.*

Proof. By Lemma 2.1,

$$\partial^k(G) = max_{\{S_2,\cdots,S_\Delta\}-packing}\{\sum_{d=2}^{\Delta}(d-1)k_d\},$$

where k_d is the number of k-distance S_d subtrees in the k-distance big subtree packing. We assume that D is the set of vertices which are the centers of the k-distance subtrees in a packing $\mathcal{C}$ giving the k-distance differential of G with minimum size, that is minimum number of k-distance stars. We are going to find the maximum value for $|C^k(D)|$. For each vertex $v \in D$ which is a center of a k-distance S_d subtree X where $d \geq 3$, we consider the subgraph induced by X. Plus slightly abusing notation, let $B^k(\{v\})$ denote $X - \{v\}$ and let $C^k(\{v\})$ be the C^k-vertices(vertices in $C^k(D)$ that are k-neighbors of $B^k(\{v\})$. Let us note that it is possible that a vertex u belongs to two k-distance different sets $C^k(\{v_1\})$ and $C^k(\{v_2\})$, but it does not matter because we are looking for the maximum cardinality of $C^k(D)$. Since by Lemma 2.2, the maximum number of k-neighbors that every vertex

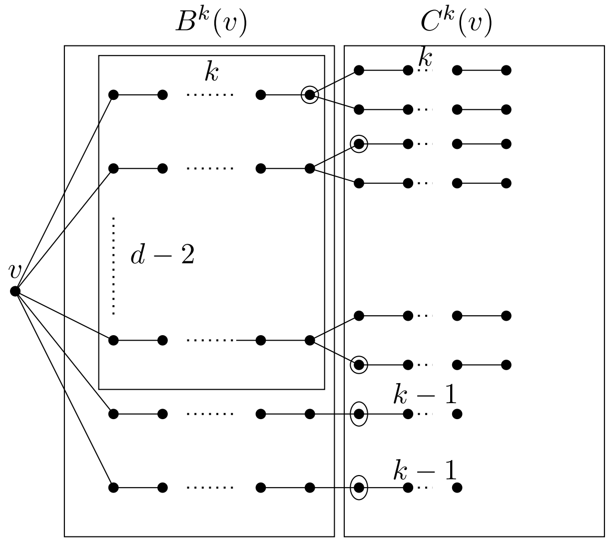

FIGURE 1. A local worst-case situation for big subtree.

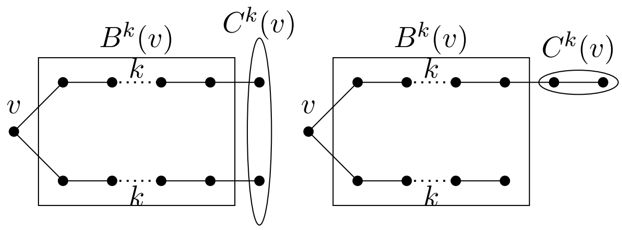

FIGURE 2. A local worst-case situation for small subtree.

in $B^k(\{v\})$ has in $C^k(\{v\})$ is two and $C^k(\{v\})$ has only P_1 and P_{k+1} components, the maximum cardinality of $C^k(\{v\})$ is attained in the case depicted in Figure 1.

We cannot have more than $d-2$ vertices in $B^k(\{v\})$ having two private k-neighbors in $C^k(\{v\})$ because of taking all these vertices, we would obtain a bigger k-distance differential. None of the rectangular vertices in $C^k(\{v\})$ can have a k-neighbor in $C^k(\{v\})$, since in such a case, taking this vertex and circular vertices, we would also obtain a bigger k-distance differential. Therefore the maximum number of vertices we can have in $C^k(\{v\})$ for any k-distance subtree S_d is $(2k+2)|B^k(v)|-(2k+4)=(2k+2)d-(2k+4)$. If $v\in D$ is a center of a k-distance subtree S_2, then the set $C^k(\{v\})$ cannot have more than two vertices since in such a case, we can choose one or two vertices in $\{v\}\cup B^k(\{v\})\cup C^k(\{v\})$ giving a bigger k-distance differential; see Figure 2.

Hence, if the number of P_{k+1}, k-distance paths, is p_{k+1}, then

$$\begin{aligned}|C^k(D)| &\le (2k+2)(|B^k(D)|-2p_{k+1})-(2k+4)(|D|-p_{k+1})+2p_{k+1}\\ &=(2k+2)|B^k(D)|-(2k+4)|D|-2kp_{k+1}+2p_{k+1}.\end{aligned}$$

On the other hand

$$n=|C^k(D)|+|B^k(D)|+|D|\le$$
$$(2k+2)|B^k(D)|-(2k+4)|D|-2kp_{k+1}+2p_{k+1}+|B^k(D)|+|D|$$

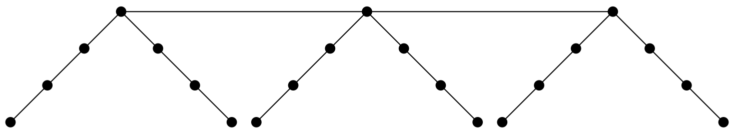

FIGURE 3. Graph $G_{3,2}$.

$$= (2k+3)(|B^k(D)| - |D|) + p_{k+1}(2-2k) \leq (2k+3)(|B^k(D)| - |D|) + n(2-2k).$$

Therefore

$$n - n(2-2k) \leq (2k+3)(|B^k(D)| - |D|), \text{ and then } |B^k(D)| - |D| \geq \frac{(2k-1)n}{2k+3}.$$

Since $\partial^k(G) \geq \partial^k(D) \geq |B^k(D)| - |D|$, $\partial(G) \geq \frac{(2k-1)n}{2k+3}$.

For seeing the sharpness of the lower bound of $\partial^k(G)$, let k be a positive integer and let P^i_{2k+3} be a path with vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_{2k+3}}$ and with center $c_i = v_{i_{k+2}}$. Let $G_{t,k}$ be a graph consists of t disjoint P^i_{2k+3}, $(1 \leq i \leq t)$ and a path of P_t with vertex set $V(P_t) = \{c_1, c_2, \cdots, c_t\}$, see $G_{3,2}$ in Figure 3. Then the connected graphs $G_{t,k}$, $(t \geq 1)$ of order $n = (2k+3)t$ is a graph with k-distance differential of $\frac{(2k-1)n}{2k+3}$. Thus the proof ends. □

5. Upper bounds of k-distance differential

Theorem A [2]**.** Let G be a graph of order n and $\Delta(G) \geq 1$. Then

$$\partial(G) \leq \frac{n(\Delta(G)-1)}{\Delta(G)+1}.$$

From Theorem A and the definition of G^k we have:

Theorem 5.1. *Let $k \geq 1$ be an integer and G be a graph of order n with $\Delta_k(G) \geq 1$, then*

$$\partial^k(G) \leq \frac{n(\Delta_k(G)-1)}{\Delta_k(G)+1}.$$

Proof. Firstly, we have clearly $\partial^k(G) = \partial(G^k)$ and $\Delta_k(G) = \Delta(G^k)$. Thus by Theorem A, we obtain

$$\partial^k(G) = \partial(G^k) \leq \frac{n(\Delta(G^k)-1)}{\Delta(G^k)+1} = \frac{n(\Delta_k(G)-1)}{\Delta_k(G)+1}.$$

□

Theorem 5.2. *Let $k \geq 1$ be an integer and G be a graph of order $n \geq 2$. If $diam(G) \leq k$, then $\partial^k(G) = n-2$.*

Proof. If $diam(G) \leq k$, then we put $D = \{v\}$. Then $\partial^k(G) \geq \partial^k(D) = |B^k(D)| - |D| = (n-1) - 1 = n-2$. On the other hand, for any nonempty set $X \subseteq G$, $\partial^k(X) \leq n-2$. Therefore $\partial^k(G) = n-2$. □

Theorem 5.3. *For any graph G of order $n \geq k+2$, $\frac{(2k+1)n}{2k+3} \leq \partial^k(G) + \gamma^k(G) \leq n-1$.*

Proof. If D is a $\partial^k(G)$-set, since $\gamma^k(G) \leq |D| + |C^k(D)|$, we have
$\partial^k(G) + \gamma^k(G) \leq |B^k(D)| - |D| + |D| + |C^k(D)| = |B^k(D)| + |C^k(D)| = n - |D| \leq n-1.$
Now we prove the lower bound. If $\gamma^k(G) \geq \frac{2n}{2k+3}$, since $\partial^k(G) \geq \frac{(2k-1)n}{2k+3}$, we deduce $\partial^k(G) + \gamma^k(G) \geq \frac{(2k+1)n}{2k+3}$. If $\gamma^k(G) < \frac{2n}{2k+3}$ and we take a k-distance dominating set $S \subseteq V$ such that $|S| = \gamma^k(G)$, we obtain $\partial^k(G) + \gamma^k(G) \geq \partial^k(S) + \gamma^k(G)$

$$= |V-S| - |S| + \gamma^k(G) = n - 2\gamma^k(G) + \gamma^k(G) = n - \gamma^k(G) > n - \frac{2n}{2k+3} = \frac{(2k+1)n}{2k+3}.$$
□

Theorem 5.4. *For any graph G of order n with $\Delta(G) \geq 3$ and for every positive integer k, we have*

$$\partial^k(G) \leq \frac{n(\Delta-1)(\Delta(\Delta-1)^{k-1}-1)}{(\Delta-2)(\Delta+1)}.$$

Proof. Since each vertex $v \in V(G)$ dominates at most Δ vertices at distance 1 from v, at most $\Delta(\Delta-1)$ vertices at distance 2 from v, at most $\Delta(\Delta-1)^2$ vertices at distance 3 from v, and so on, then we have

$$\Delta_k(G) \leq \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots + \Delta(\Delta-1)^{k-1} = \Delta\frac{(\Delta-1)^k - 1}{\Delta-2}.$$

On the other hand, Theorem 5.1 yields that $\partial^k(G) \leq \frac{n(\Delta_k(G)-1)}{\Delta(G)+1}$. Therefore

$$\partial^k(G) \leq \frac{n(\Delta\frac{(\Delta-1)^k-1}{\Delta-2} - 1)}{\Delta+1} \leq n\frac{\Delta(\Delta-1)^k - \Delta}{(\Delta-2)(\Delta+1)} \leq \frac{n(\Delta-1)(\Delta(\Delta-1)^{k-1}-1)}{(\Delta-2)(\Delta+1)}.$$
□

The next example has straightforward proof. We only sketch the proof of one of the cases, the other cases are left.

Example 5.1. *For all paths P_n and cycles C_n,*

$$\partial^k(P_n) = \partial^k(C_n) = \begin{cases} \frac{(2k-1)n}{2k+1}, & n \equiv 0 \ (mod\ 2k+1); \\ n - 2\lfloor\frac{n}{2k+1}\rfloor - 1, & n \equiv 1 \ (mod\ 2k+1); \\ n - 2\lfloor\frac{n}{2k+1}\rfloor - 2, & otherwise. \end{cases}$$

Proof. Let $n \equiv 0$ (mod $2k+1$) and $n = t(2k+1)$. For any $X \subseteq V(C_n)$ ($X \subseteq V(P_n)$), $B^k(X)$ has maximum size if any two vertices in X like x_i, x_j has distance $d(x_i, x_j) \geq 2k+1$. In particular if we choose the set S as follows $\{v_{k+1}, v_{3k+2}, \cdots, v_{(2t-1)k+t}\}$, then $|B^k(S)|$ has maximum value and $|S|$ minimum value for covering all vertices between $V-S$ and S. It is easy to see that if $n \equiv 0$ (mod $2k+1$), then $\partial^k(C_n) = \frac{(2k-1)n}{2k+1} = \partial^k(P_n)$. The other cases are similarly proved. □

Given the positive integers $2m < n$, place n vertices around a circle, equally spaced. For $2m$, form $H_{2m,n}$ by making each vertex adjacent to the nearest m vertices in each direction around the circle. This graph is called the first type of Harary graph.

Example 5.2. *If $G = H_{2m,n}$ is of the first type of Harary graph, then we have:*

$$\partial^k(G) = \begin{cases} (2km-1)\lfloor\frac{n}{2km+1}\rfloor & \text{if } n \equiv 0,\ 1,\ 2\ (mod\ 2km+1); \\ (2km-1)\lfloor\frac{n}{2km+1}\rfloor + l - 2 & \text{if } n \equiv l\ (mod\ 2km+1),\ (3 \le l \le 2km). \end{cases}$$

Proof. If x_i is a vertex, then x_i k-distance dominates $2km$ vertices except itself. If x_i and x_j are two vertices such that $|i-j| = 2km+1$, then the $4km$ distinct vertices are k-distance dominated by x_i and x_j. Therefore, if then $n \equiv t$ (mod $2km+1$) and $t \in \{0,1,2\}$, we say $X = \{v_{km+1}, v_{3km+2}, v_{5km+3} \cdots, v_{(2r+1)km+r+1}\}$, where $r = \lfloor\frac{n}{2km+1}\rfloor - 1$.

In this case $B^k(X) = V - X - S$ is such away $|B^k(X)| = n - |X| - |S|$ and $\partial^k(X) = n - 2|X| - |S| = \partial^k(G)$ for $|S| = t$. If $n \equiv t$ (mod $2km+1$) and $3 \le t \le 2km$, we say $X' = \{v_{km+1}, v_{3km+2}, v_{5km+3} \cdots, v_{(2r+1)km+r+1}, v_{((2r+2)km+r+1)\lfloor\frac{t}{2}\rfloor}\}$. In this case $B^k(X') = V - X'$ is such away $|B^k(X')| = n - |X'|$ and $\partial^k(X') = n - 2|X'| = \partial^k(G)$.

If $Y \subseteq V(G)$ and $|Y| < |X|$, then it is easy to see that $|B^k(Y)| \le |B^k(X)|$ or if $|Y| = |X|$, but the indices of some vertices like y_i and y_j in Y are such that $|i-j| < 2km+2$, then it is easy to see that $|B^k(Y)| \le |B^k(X)|$ and so $\partial^k(Y) \le \partial^k(X)$.
Now let $n \equiv 0$ (mod $2km+1$). Then $\partial^k(X) = n - 2|X| = n - 2(r+1) = n - 2\lfloor\frac{n}{2km+1}\rfloor = (2km-1)\lfloor\frac{n}{2km+1}\rfloor$. Let $n \equiv l$ (mod $2km+1$) where $1 \le l \le 2km$. If we consider X same as above, then $|B^k(X)| = n - l - |X|$ and $\partial^k(X) = n - l - 2|X| = (2km-1)\lfloor\frac{n}{2km+1}\rfloor$.
If we add a vertex $v_{n+\lceil\frac{l}{2}\rceil}$ to the set X and obtain the set X_1, then $|B^k(X_1)| = n - |X_1| = n - |X| - 1$ and $\partial^k(X_1) = n - 2|X_1| = n - 2|X| - 2 = (2km-1)\lfloor\frac{n}{2km+1}\rfloor + l - 2$. Therefore, for $l = 1, 2$, the set X gives us a maximum $\partial^k(X)$ and for $l \ge 3$, the set X_1 gives us a maximum $\partial^k(X)$. Thus the proof ends. □

6. Exact value of k-distance differential

In this section we study the exact value of k-distance differential of graphs.

Theorem 6.1. *For any graph G with maximum k-degree $\Delta_k(G)$, $\partial^k(G) \ge \Delta_k(G) - 1$. This bound is sharp.*

Proof. Let $D = \{v\}$ where v is a vertex of maximum k-degree $\Delta_k(G)$. Then we have $\partial^k(G) \ge \partial^k(D) = |B^k(D)| - |D| = \Delta_k(G) - 1$.
For seeing the sharpness of the bound, assume that $k \ge 1$ is an integer and L is a graph with $\Delta_k(L) = n(L) - 1 \ge 2$. Now put $G = tK_1 \cup t'K_2 \cup L$ for two integers $t, t' \ge 0$. Then $\Delta_k(G) = \Delta_k(L)$ and $\partial^k(G) = (n - t - 2t' - 1) - 1 = \Delta_k(G) - 1$. □

Theorem 6.2. *Suppose that $k \ge 1$ is an integer and G is a graph of order $n \ge 2$. Then $\partial^k(G) = n - 2$ if and only if $n = 2$ or $n \ge 3$ and $\Delta_k(G) = n - 1$.*

Proof. If $n = 2$, then $\partial^k(G) = 2 - 2 = n - 2$. If $n \ge 3$ and $\Delta_k(G) = n - 1$, then by Theorem 6.1 we conclude

$$\Delta_k(G) - 1 = n - 1 - 1 = n - 2 \le \partial^k(G) \le n - 2.$$

Therefore $\partial^k(G) = n - 2$. Conversely, we suppose that $\partial^k(G) = n - 2$. If $\Delta_k(G) = 0$, then it follows that $n = 2$. If $\Delta_k(G) \ge 1$, then we deduce from Theorem 5.1 that $\partial^k(G) = n - 2 \le \frac{(\Delta_k - 1)n}{\Delta_k + 1}$. Therefore $\Delta_k(G) \ge n - 1$. This leads to $\Delta_k(G) = n - 1$. □

Theorem 6.3. *Let $k \ge 1$be an integer, and let G be a graph of order n. Then $\partial^k(G) = 0$ if and only if $G = tK_1 \cup t'K_2$ for some integers $t, t' \ge 0$.*

Proof. If $G = tK_1 \cup t'K_2$ for some integers t, t' with $t + t' \geq 1$, then clearly, $\partial^k(G) = 0$. Conversely, suppose that $\partial^k(G) = 0$. If $\Delta_k(G) \geq 2$, then from Theorem 6.1 we have a contradiction. Therefore $\Delta_k(G) \leq 1$ and so $G = tK_1 \cup t'K_2$ for some integers t, t' with $t + t' \geq 1$. □

Theorem 6.4. *Let $k \geq 1$ be an integer and G be a graph of order $n \geq 4$. Then $\partial^k(G) = n - 3$ if and only if $\Delta_k(G) = n - 2$.*

Proof. If $\Delta_k(G) = n-2$, then by Theorem 6.2, we have $\partial^k(G) \leq n-3$. On the other hand, from Theorem 6.1, $\partial^k(G) \geq \Delta_k(G) - 1 = n - 2 - 1 = n - 3$, so we deduce $\partial^k(G) = n - 3$. Conversely, If $\partial^k(G) = n - 3$, then by Theorem 6.2, we have $\Delta_k(G) \leq n - 2$. Now let D be a $\partial^k(G)$-set. Since $\partial^k(G) = n - 3 \geq 1$, it follows that $|D| \geq 1$. If $|D| \geq 2$, then $|B^k(D)| \leq n - 2$. Therefore $\partial^k(G) \leq n - 4$, a contradiction. So, $D = \{x\}$ for some vertex x. This shows that $deg_{k,G}(x) = n - 2$, implying $\Delta_k(G) \geq n - 2$. On the other hand, $\Delta_k(G) = n - 1$ is impossible by Theorem 6.2. So, $\Delta_k(G) = n - 2$. □

Theorem 6.5. *Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq 3$. Then $\partial^k(G) = 1$ if and only if $G = K_3 \cup tK_1 \cup t'K_2$ or $G = P_3 \cup tK_1 \cup t'K_2$ for some integers $t, t' \geq 0$.*

Proof. If $G = K_3 \cup tK_1 \cup t'K_2$ or $G = P_3 \cup tK_1 \cup t'K_2$ for some integers $t, t' \geq 0$, then clearly $\partial^k(G) = 1$.
Conversely, we suppose that $\partial^k(G) = 1$. If $\Delta_k(G) \geq 3$, then Theorem 6.1, implies the contradiction $1 = \partial^k(G) \geq \Delta_k(G) - 1 \geq 2$. Thus $\Delta_k(G) \leq 2$.
If $\Delta_k(G) \leq 1$, then we have $\partial^k(G) = 0$ by Theorem 5.1, which is a contradiction. Consequently, $\Delta_k(G) = 2$. If G contains at least two components L_1 and L_2 with $\Delta_k(L_1) = \Delta_k(L_2)$, then $\partial^k(G) \geq 2 - 1 + 2 - 1 = 2$, a contradiction. Hence G has exactly one component L with $\Delta_k(L) = 2$ and the remaining components are isolated vertices or isomorphic to K_2. If $|V(L)| \geq 4$, then the assumption $k \geq 2$ shows that $\Delta_k(G) = \Delta_k(L) \geq 3$, a contradiction. Hence $|V(L)| = 3$ and so $G = K_3 \cup tK_1 \cup t'K_2$ or $G = P_3 \cup tK_1 \cup t'K_2$ for some integers $t, t' \geq 0$. □

A classical result from [6] states that, for a simple graph G, if $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$. It follows that, if $diam(G) \geq 4$, then $diam(\overline{G}) \leq 2$. From this result we have:

Theorem 6.6. *Let $k \geq 3$ be an integer and let G be a graph of order $n \geq 2$. Then $\partial^k(G) = n - 2$ or $\partial^k(\overline{G}) = n - 2$.*

Proof. If $diam(G) \leq 3$, then it follows from Theorem 5.2 that $\partial^k(G) = n-2$. If $diam(G) \geq 4$, then $diam(\overline{G}) \leq 2$. Applying again Theorem 5.2 for $\overline{G}$, we see that $\partial^k(\overline{G}) = n - 2$. □

Theorem 6.7. *Let G be a graph of order $n \geq 2$. If $diam(G) \neq 3$, then $\partial^2(G) = n - 2$ or $\partial^2(\overline{G}) = n - 2$.*

Proof. If $diam(G) \leq 2$, then it follows from Theorem 5.2 that $\partial^2(G) = n-2$. If $diam(G) \geq 3$, then the assumption $diam(G) \neq 3$ implies that $diam(G) \geq 4$. Same as above we deduce that $diam(\overline{G}) \leq 2$, and Theorem 5.2 leads to $\partial^2(\overline{G}) = n - 2$. □

Theorem 6.8. *Let $k \geq 1$ be an integer and G be a connected graph of order n with $n - \Delta(G) - k \geq 0$, then*

$$\partial^k(G) \geq \Delta(G) + k - 2.$$

Proof. Let v be a vertex of G such that $deg_G(v) = \Delta(G)$. If $d(u,v) \leq k$ for each $u \in V(G)$, then obviously $\partial^k(G) = n-2$ and we are done. If $d(w,v) > k$ for some $w \in V(G)$, then choose a vertex u in G such that $d(u,v) = k+1$. Let P be a shortest (u,v)-path. Then $d(v,z) \leq k$ for each $z \in (V(P) - \{u\}) \cup N_G(v)$. By setting $D = \{v,u\}$, we have $\partial^k(D) = |B^k(D)| - |D| \geq (\Delta(G)+k-1)+1-2 = \Delta(G)+k-2$. □

Theorem 6.9. *Let G be a t-regular graph with $diam(G) \geq 3$. For any integer $1 \leq k \leq \lfloor \frac{diam(G)}{2} \rfloor$, $\partial^k(G) \geq (k+1)(t-1)$.*

Proof. Suppose that $P = x_1x_2\cdots x_{diam(G)+1}$ be a diametral path of graph G. Clearly, we have $N_k(x_1) \cap N_k(x_d) = \emptyset$. Now we put $D = \{x_1, x_d\}$. Since G is t-regular, $|B^k(D)| \geq 2t + (k-1)(t-1)$ and hence $\partial^k(G) \geq \partial^k(D) = |B^k(D)| - |D| \geq 2t + (k-1)(t-1) - 2 = 2(t-1) + (k-1)(t-1)$. □

7. Conclusions

The concept of k-distance differential in graphs was initially investigated in this paper. We studied the computational complexity of this concept and proved some bounds on k-distance differential of graphs. We now conclude the paper with some problems suggested by this research.

- For every connected graph G of order $n \geq k+2$, $\partial^k(G) \geq \frac{(2k-1)n}{2k+3}$ as already noted in Theorem 4.1. It is worthwhile to characterize all graphs G with $\partial^k(G) = \frac{(2k-1)n}{2k+3}$.
- The decision problem k-DISTANCE DIFFERENTIAL NUMBERS is NP-complete for arbitrary graph G, as proved in Theorem 3.1. Is it possible to construct a polynomial-time algorithm to compute $\partial^k(G)$ for some well-known families of graphs, like trees?

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