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A NEW FAMILY OF UNIT-DISTRIBUTIONS: DEFINITION, PROPERTIES AND APPLICATIONS

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Abstract. In this study, a new family of unit-distributions is introduced. Then, a unit-Gumbel distribution, member of the proposed family of unit-distributions, is obtained as an example, and some of its statistical properties are provided. The maximum likelihood method is used for estimating the shape parameter of the unit-Gumbel distribution. In addition, a new family of continuous distributions is defining by using the composition technique. Finally, real data sets are used for modeling purposes. The result shows that the unit-Gumbel distribution is preferable over some well-known unit-distributions such as the beta, Kumaraswamy, and Topp-Leone, and also the unit-Gompertz distribution, which is recently introduced.

Keywords: Beta distribution, a family of unit-distributions, Kumaraswamy distribution, maximum likelihood method, unit-Gumbel distribution.

AMS Subject Classification: 60E05, 62E10, 62F10.

1. Introduction

Researchers from different science areas usually aim to explore real phenomena by using the data extracted from them. In this context, statistical distributions are widely used in modeling data from various fields. In most of the cases, observed/obtained data are in $\mathbb{R}$ or $\mathbb{R}^+$. However, data from some specific experiments (different indices, rates, etc.) or real-life (infant mortality, human development index, etc.) may have a bounded range on the unit interval $(0, 1)$. Thus, well-known distributions such as normal, Laplace, gamma, and Weibull can not be used for modeling such data. In this regard, bounded distributions, e.g., the power, beta, Johnson [9], Topp-Leone [18], Kumaraswamy [12] distributions, are used in modeling data on the unit interval $(0, 1)$.

The unit-distribution also plays an essential role in defining the families of continuous distributions via the composition technique and constructing regression models for a variable having a distribution with unit support $(0, 1)$. See Alzaatreh et al. [1] and references therein for defining a new family of distributions. See also Ferrari and Cribari-Neto [6], Jodrá and Jiménez-Gamero [10] and Mazucheli et al. [17] in the context of regression model for bounded response.

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Recently, Mazucheli et al. [13, 14, 15, 16] introduced unit-Birnbaum-Saunders, unit-Weibull, unit-Lindley, and unit-Gompertz distirbutions, respectively. Also, Ghitany et al. [8] obtained unit-inverse Gaussian, and Korkmaz [11] derived unit-generalized half-normal distributions. For further information about recent literature dealing with the unit distributions, see Bantan et al. [3] and references given therein.

This study has the following contributions to the related literature. (i) A new family of unit-distributions is obtained. The resulting family of unit-distributions has not been introduced yet to the best of the author's knowledge. (ii) A unit-Gumbel (UG) distribution, member of the proposed family of unit-distributions, is obtained as an example, and some statistical properties of the UG distribution are shown. The maximum likelihood (ML) method is used for estimating a shape parameter α of the UG distribution. (iii) A new family of distributions called the UG-generated (UG-G) family of distributions is defined using the composition technique.

The paper is organized as follows. A new family of unit-distributions is introduced in Section 2. Section 3 is reserved for the UG distribution along with its statistical properties and the UG-G family of distributions. Real data sets are used in Section 4 to show the modeling capability of the UG distribution. The paper is finalized with some concluding remarks.

2. A new family of unit-distributions

Let X be a random variable on $\mathbb{R}$ and have probability density function (pdf) $f_X(x)$ and cumulative distribution function $F_X(x)$. Then, consider the variable-transformation

$$Z = [1 + \exp(-X)]^{-\frac{1}{\alpha}}$$

where power of the logistic function is used for the variable-transformation. Thus, the random variable Z has the pdf

$$f_Z(z;\alpha) = \alpha \left[z\left(1 - z^{\alpha}\right)\right]^{-1} f_X\left(-\ln\left(z^{-\alpha} - 1\right)\right); \quad z \in (0,1), \quad \alpha > 0 \tag{1}$$

and the cdf

$$\begin{aligned} F_Z(z;\alpha) =& P\left(Z \le z\right) = P\left(X \le -\ln\left(z^{-\alpha} - 1\right)\right) \\ =& F_X\left(-\ln\left(z^{-\alpha} - 1\right)\right). \end{aligned} \tag{2}$$

Here, α is a shape parameter. The moments of the random variable Z is

$$\begin{aligned} \mathbb{E}\left[Z^r\right] &= \int_0^1 z^r \alpha \left[z\left(1 - z^{\alpha}\right)\right]^{-1} f_X\left(-\ln\left(z^{-\alpha} - 1\right)\right) dz \\ &= \int_0^1 \left[1 + \exp\left(-F_X^{-1}(u;\alpha)\right)\right]^{-\frac{r}{\alpha}} du \end{aligned} \tag{3}$$

where $F^{-1}(\cdot)$ is inverse function of the cdf, i.e., quantile function of the corresponding distribution. Noted that second line in $\mathbb{E}\left[Z^r\right]$ is obtianed by using the transformation $F_X(-\ln\left(z^{-\alpha} - 1\right)) = u$. The closed form of the expression for $\mathbb{E}[Z^r]$ given in (3) may not straightforward depending on form of the $F_X^{-1}(u;\alpha)$.

3. A new unit-distribution

In this section, a new unit-distribution called unit-Gumbel is introduced as an alternative to some well-known unit-distributions. The ML method is then considered to estimate its parameter. A small Monte-Carlo simulation study is conducted to show the efficiency of the ML estimator of the α, i.e., $\hat{\alpha}_{ML}$. Also, note that a new family of distributions called the UG-generated family of distributions (UG-G) is introduced. Then, the UG-generated normal distribution (UG-N), a member of the UG-G family of distributions, is obtained as an example.

3.1. **The unit-Gumbel distribution.** Let X follows Gumbel distribution having the pdf

$$f_X(x) = \exp\left(-x - \exp(-x)\right); \quad x \in \mathbb{R}. \tag{4}$$

Then, random variable Z defined by $Z = \left[1 + \exp(-X)\right]^{-\frac{1}{\alpha}}$ follows the UG distribution having the pdf

$$f_{\mathrm{UG}}(z;\alpha) = \alpha z^{-(\alpha+1)} \exp\left(-\left(z^{-\alpha} - 1\right)\right); \quad z \in (0,1), \quad \alpha > 0 \tag{5}$$

and the cdf

$$F_{\mathrm{UG}}(z;\alpha) = \exp\left(-\left(z^{-\alpha} - 1\right)\right) \tag{6}$$

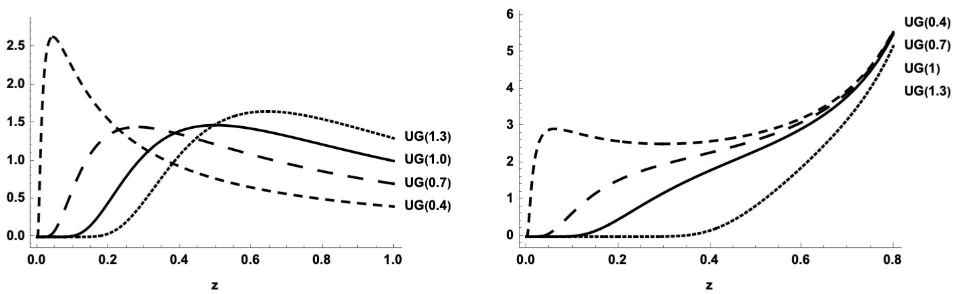
where α is the shape parameter. The hrf of the UG distribution is

$$h_{\mathrm{UG}}(z;\alpha) = \frac{\alpha z^{-(\alpha+1)} \exp\left(-\left(z^{-\alpha} - 1\right)\right)}{\exp\left(-\left(z^{-\alpha} - 1\right)\right)} \tag{7}$$

and quantile function of the UG distribution is

$$Q_p = (1 - \ln p)^{-\frac{1}{\alpha}}; \quad 0 < p < 1. \tag{8}$$

The pdf and hrf of the UG distibution are plotted for different values of the shape parameter α in Figure 1(a)-1(b), respectively.



(a) pdfs (b) hrfs

Figure 1. Plots for the pdf and hrf of the UG distribution.

Figure 1(b) shows that for different values of the parameter α, the hrf of the UG distribution can form a variety of shapes.

3.1.1. *Moments.* Moments of the random variable Z having the UG distribution is obtained by using the formula

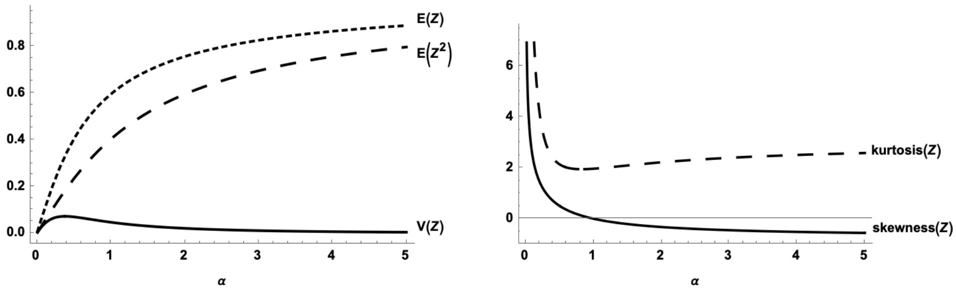
$$\mathbb{E}[Z^r] = \mathbf{e}\mathbf{E}_{\frac{r}{\alpha}}(1) \tag{9}$$

where $\mathbf{E}_a(b)$ represents the exponential integral function $\int_1^\infty z^{-a}\exp(-bz)dz$ and $\mathbf{e}$ is the exponential constant ($\simeq 2.71828$). The $\mathbb{E}[Z]$, $\mathbb{V}[Z]$, skewness, and kurtosis measures of the UG distribution are calculated via `Mathematica` for certain values of shape parameter α and tabulated in Table 1.

TABLE 1. The values of some characteristic measures of the UG distribution.

α	0.1	0.3	0.6	0.9	1.0	1.1	1.4	1.7	2.0	5.0
$\mathbb{E}[Z]$	0.0989	0.2735	0.4545	0.5680	0.5963	0.6213	0.6810	0.7246	0.7579	0.8906
$\mathbb{V}[Z]$	0.0401	0.07159	0.0669	0.0523	0.0480	0.0440	0.0343	0.0272	0.0219	0.0053
skewness	2.5460	1.0312	0.3724	0.0819	0.0171	-0.0378	-0.1620	-0.2476	-0.3103	-0.5432
kurtosis	8.9939	2.9731	2.0352	1.9666	1.9815	2.003	2.0821	2.1616	2.2331	2.5989

The $\mathbb{E}[Z]$, $\mathbb{E}[Z^2]$, $\mathbb{V}[Z]$, skewness, and kurtosis measures of the UG distribution are also plotted in Figure 2 for an illustration.



(a) $E(Z), E(Z^2)$, and $V(Z)$ (b) skewness(Z) and kurtosis(Z)

FIGURE 2. Plots of some characteristic measures of the UG distribution.

It is clear from Table 1 and Figure 2 that the pdf of the UG distribution is skewed to the right if $\alpha < 1.0295$ and to the left if $\alpha > 1.0295$. It can also be symmetric when $\alpha = 1.0295$.

3.1.2. *Order statistics.* Let $Z_{(i)}$ be a i-th order statistics of Z in a random sample of size n from the UG distribution where $i = 1, 2, \cdots, n$. Then, the pdf of the $Z_{(i)}$ is

$$\begin{aligned} f_{Z_{(i)}}(z;\alpha) =& \frac{n!}{(i-1)!(n-i)!} f_{\mathrm{UG}}(z;\alpha) F_{\mathrm{UG}}(z;\alpha)^{i-1}\left[1 - F_{\mathrm{UG}}(z;\alpha)\right]^{n-i} \\ =& \frac{n!}{(i-1)!(n-i)!} \alpha z^{-(\alpha+1)} \exp\left(-i\left(z^{-\alpha}-1\right)\right)\left[1-\exp\left(-\left(z^{-\alpha}-1\right)\right)\right]^{n-i} \end{aligned} \tag{10}$$

where $z \in (0,1)$ and $\alpha > 0$. From (10), the pdf of the minimum and maximum order statistics, i.e., $Z_{(1)}$ and $Z_{(n)}$, are obtained for $i = 1$ and $i = n$, respectively.

3.1.3. *Stochastic ordering.* Let $Z_1 \sim \mathrm{UG}(\alpha_1)$ and $Z_2 \sim \mathrm{UG}(\alpha_2)$. The stochastic ordering

$$[Z_1 \leq_{\text{lro}} Z_2], \quad [Z_1 \leq_{\text{hro}} Z_2], \quad [Z_1 \leq_{\text{mrlo}} Z_2], \quad \text{and} \quad [Z_1 \leq_{\text{sto}} Z_2]$$

are satisfied when $\alpha_1 < \alpha_2$, since

$$\frac{d}{dz} \ln \left(\frac{f_{Z_1}(z;\alpha_1)}{f_{Z_2}(z;\alpha_1)} \right) \leq 0.$$

Here, $\leq_{\text{lro}}$, $\leq_{\text{hro}}$, $\leq_{\text{mrlo}}$, and $\leq_{\text{sto}}$ represent the likelihood ratio order, hazard rate order, mean residual life order, and stochastic order, respectively. See Erdogan et al. [4] and reference therein for definitions of them.

3.1.4. *The UG generated family of distributions.* As stated in Introduction, unit-distributions can be used to generate a new family of distributions. For example, Eugene et al. [5] used the beta distribution to obtain a new family of distributions called as beta-generated family of distributions, i.e., $F_{\text{Beta-G}}(t) = F_{\text{Beta}}(G(t))$ where $G(t)$ is the cdf of the any random variable, $F_{\text{Beta}}(\cdot)$ is the cdf of the beta distribution, and $F_{\text{Beta-G}}(\cdot)$ is the cdf of the beta-generated distribution.

By following the same line as in Eugene et al. [5], a new family of distributions having the cdf

$$\begin{aligned} F_{\text{UG-G}}(t) =& F_Z(G(t);\alpha) \\ =& F_X\left(-\ln\left(G(t)^{-\alpha} - 1\right)\right) \end{aligned}$$

can be obtained. Here, $F_Z(\cdot)$ is the cdf of the family of unit distributions given in (2), $F_X(\cdot)$ is the cdf of any random variable on $\mathbb{R}$, and $G(\cdot)$ is the cdf of a baseline distribution.

For example, a unit-Gumbel-Normal (UG-N) distribution have the cdf

$$\begin{aligned} F_{\text{UG-N}}(t;\alpha) =& F_{\text{UG}}(\Phi(t);\alpha) \\ =& \exp\left(-\left(\Phi(t)^{-\alpha} - 1\right)\right); \quad t \in \mathbb{R}, \quad \alpha > 0 \end{aligned}$$

and the pdf

$$f_{\text{UG-N}}(t;\alpha) = \alpha\phi(t)\Phi(t)^{-(\alpha+1)} \exp\left(-\left(\Phi^{-\alpha} - 1\right)\right)$$

where the $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of the standard normal distribution, respectively. Note that possible future works can be conducted about the construction of general families of distributions by using the cdf given in (2).

3.2. **Parameter estimation.** In this subsection, the ML estimation of the parameter of the UG distribution is provided. Finally, a Monte-Carlo simulation study is carried out to show the ML estimator's efficiency of the parameter α in terms of the mean squared error (MSE) criterion.

3.2.1. *The ML estimation.* Let $z_1, z_2, \ldots, z_n$ be a random sample from the UG distribution. Then, the ML estimate of the parameter α is the point on which the log-likelihood function ($\ln L$)

$$\ln L(\alpha; z) = n(1 + \ln\alpha) - (\alpha+1)\sum_{i=1}^{n} \ln z_i - \sum_{i=1}^{n} z_i^{-\alpha} \tag{10}$$

attains its maximum.

By taking the derivative of the $\ln L$ with respect to the parameter α and then setting it to 0, the likelihood equation

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln z_i + \sum_{i=1}^{n} z_i^{-\alpha} \ln z_i = 0 \tag{11}$$

is obtianed. It is clear that ML estimate of the unknown parameter of the UG distribution $\hat{\alpha}_{ML}$ can be obtained by using the Newton-Raphson (NR) technique. The one-dimensional NR algorithm consists of the following steps.

i. Set $k = 0$, then give the initial values of the parameter, i.e., α_0.

ii. Obtain the values α_{k+1} by using the equation

$$\alpha_{k+1} = \alpha_k - \left[\frac{\partial^2 \ln L}{\partial \alpha^2}(\alpha_k)\right]^{-1} \frac{\partial \ln L}{\partial \alpha}(\alpha_k)$$

iii. Repeat (i) and (ii) for $(k = 1, 2, \dots)$ until $|\alpha_{k+1} - \alpha_k| \le \varepsilon$.

Here, second partial derivates of the $\ln L$ concerning the α is

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} z_i^{-\alpha} \ln^2 z_i.$$

3.2.2. *Monte-Carlo simulation.* The ML estimator's efficiency of the parameter α is shown via the Monte-Carlo simulation study. All the simulations are conducted for $1,000$ Monte-Carlo runs via `MATLAB2015b` software. Here, the sample size n is considered as 30 (small), 50 (moderate) and 100 (large) and shape parameter α is taken to be $0.5, 1.0, 1.5, 2.0$, and 2.5. To generate the random variates from the UG distribution, equation

$$z = (1 - \ln p)^{-\frac{1}{\alpha}}; \quad p \sim \text{Uniform}(0, 1)$$

is used. For each generated sample, the ML estimate of the parameter α is obtained by using the NR procedure given in subsection 3.2.1. Then, simulated mean, variance, and MSE values for the ML estimate of the parameter α are calculated. The results of the Monte-Carlo simulation study are given in Table 2 .

TABLE 2. The simulated mean, variance, and MSE values of the ML estimates.

n		$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 2.5$
	Mean	0.5123	1.0297	1.5486	2.0581	2.5587
30	Variance	0.0034	0.0145	0.0339	0.0651	0.0922
	MSE	0.0035	0.0154	0.0362	0.0684	0.0956
	Mean	0.5089	1.0152	1.5214	2.0326	2.5401
50	Variance	0.0020	0.0080	0.0184	0.0298	0.0512
	MSE	0.0021	0.0082	0.0188	0.0309	0.0527
	Mean	0.5046	1.0071	1.5122	2.0116	2.5137
100	Variance	0.0010	0.0034	0.0091	0.0145	0.0248
	MSE	0.0010	0.0035	0.0093	0.0146	0.0249

It can be seen from Table 2 that the simulated mean value of $\hat{\alpha}$ is closed to the exact value of α. Also, simulated variance values for the $\hat{\alpha}$, and therefore the MSE values are small for the moderate and large sample sizes. Also, note that the MSE values for the ML estimate of the parameter α decrease when the sample size increases, as expected.

4. Applications

In this subsection, the UG distribution is used to model three data sets from the related literature. Data Set-I and Data Set-II have the values obtained by two different algorithms, which are used for estimating unit capacity factor; see Genc [7] and references therein. Data Set-III includes 30 measurements of tensile strength of polyester fibers; see Mazuchli et al. [16]. These data sets are provided in Table 3.

Table 3. Data sets which are modeled by the UG distribution.

Data Set-I ($n = 22$)											
0.853	0.759	0.874	0.800	0.716	0.557	0.503	0.399	0.334	0.207	0.118	0.118
0.097	0.078	0.067	0.056	0.044	0.036	0.026	0.019	0.014	0.010		
Data Set-II ($n = 23$)											
0.853	0.759	0.866	0.809	0.717	0.544	0.492	0.403	0.344	0.213	0.116	0.116
0.092	0.070	0.059	0.048	0.036	0.029	0.021	0.014	0.011	0.008	0.006	
Data Set-III ($n = 30$)											
0.023	0.032	0.054	0.069	0.081	0.094	0.105	0.127	0.148	0.169	0.188	0.216
0.255	0.277	0.311	0.361	0.376	0.395	0.432	0.463	0.481	0.519	0.529	0.567
0.642	0.674	0.752	0.823	0.887	0.926						

The beta and Kumaraswamy distributions are the natural rivals of the UG distribution. Thus, they are included in the applications to compare the modeling capabilities of the UG, beta, and Kumaraswamy distributions. However, Genc [7] modeled Data Set-I and Data Set-II by using the Topp-Leone distribution having one shape parameter. Also, Mazucheli et al. [16] proposed to use the unit-Gompertz distribution to model the Data Set-III. Therefore, the Topp-Leone and unit-Gompertz distributions are also considered in applications to make comparisons complete.

In the comparisons, Akaike Information Criterion (AIC) and some well-known goodness-of-fit statistics, such as Kolmogorov-Smirnov (KS), Cramér-von Mises (CvM), and root-mean-squared error (RMSE), are considered.

The KS, CvM, RMSE, and AIC values are calculated by using the equations

$$\mathrm{KS} = \max\left|F(x_{(i)};\hat{\Theta}) - \frac{i}{n+1}\right|, \quad \mathrm{CvM} = \sum_{i=1}^{n}\left[F(x_{(i)};\hat{\Theta}) - \frac{2i-1}{2n}\right]^2 + \frac{1}{12n},$$

$$\mathrm{RMSE} = \left[\frac{1}{n}\sum_{i=1}^{n}\left(F(x_{(i)};\hat{\Theta}) - \frac{i}{n+1}\right)^2\right]^{1/2} \quad \text{and} \quad \mathrm{AIC} = -2\ln L + 2k,$$

respectively. Here, $x_{(i)}$ is the i-th ordered observation and $\hat{\Theta}$ denotes the estimated parameter vector. The minimum value of the AIC, KS, CvM, and RMSE imply the best modeling performance; see Arslan et al. [2] and Erdogan et al. [4].

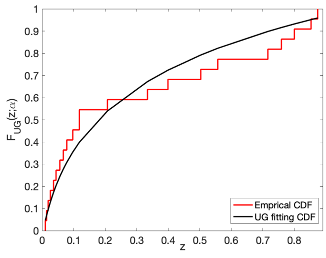
The unknown parameter α of the UG distribution is estimated by using the NR given in subsection 3.2.1. Estimates of parameters of the beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions are also obtained via the ML method by using the optimization tool "`fminsearch`" function, which is available in `MATLAB2015b` software. Parameter estimates of the UG, beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions, along with goodness-of-fit statistics for them, are given in Table 4.

Table 4. Estimation results along with the goodness-of-fit statistics for the UG, Topp-Leone, beta, Kumaraswamy, and unit-Gompertz distributions.

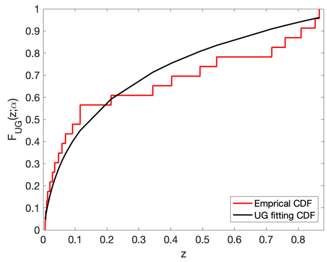
	Fitting results for the Data Set-I						
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\ln L$	AIC	KS	CvM	RMSE
UG	0.3050	—	8.5052	-15.0104	0.1466	0.0893	0.0691
Topp-Leone	0.6777	—	5.4983	-8.9965	0.1848	0.1777	0.0955
Beta	0.5539	1.2198	6.7819	-9.5628	0.2002	0.1264	0.0778
Kumaraswamy	1.2305	0.5718	6.8436	-9.6872	0.1963	0.1230	0.0771
unit-Gompertz	0.7264	0.3666	8.5660	-13.1320	0.1370	0.0907	0.0703
	Fitting results for the Data Set-II						
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\ln L$	AIC	KS	CvM	RMSE
UG	0.2727	—	11.3099	-20.6198	0.1268	0.0853	0.0668
Topp-Leone	0.5943	—	8.1151	-14.2302	0.1690	0.1735	0.0926
Beta	0.4869	1.1679	9.6075	-15.2149	0.1836	0.1189	0.0739
Kumaraswamy	1.1862	0.5044	9.6708	-15.3416	0.1789	0.1159	0.0734
unit-Gompertz	0.8115	0.3080	11.3364	-18.6728	0.1336	0.0880	0.0681
	Fitting results for the Data Set-III						
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\ln L$	AIC	KS	CvM	RMSE
UG	0.4306	—	3.9475	-5.8950	0.0766	0.0168	0.0237
Topp-Leone	1.1091	—	2.9039	-3.8078	0.0665	0.0332	0.0388
Beta	0.9666	1.6205	3.3051	-2.6101	0.0669	0.0221	0.0288
Kumaraswamy	1.6084	0.9627	3.3110	-2.6221	0.0650	0.0207	0.0278
unit-Gompertz	1.0381	0.4212	3.9488	-3.8976	0.0733	0.0155	0.0225

It is clear from Table 4 that the UG distribution has the smallest CvM and RMSE values for Data Set-I and Data Set-II. Concerning the KS criterion, the UG distribution has the smallest value only for the Data Set-II. The UG and unit-Gompertz distribution have more or less the same KS, CvM, and RMSE values for the Data Set-III. Thus, the UG distribution shows better goodness-of-fit to the data than the beta and Kumaraswamy distributions. Although the UG distribution has only one shape parameter, it models the corresponding data sets as good as the unit-Gompertz distribution with two shape parameters. Also, note that the UG distribution has the smallest AIC values among the beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions. Overall, the UG distribution is preferable over the beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions in modeling the data having bounded range on the unit interval $(0, 1)$.

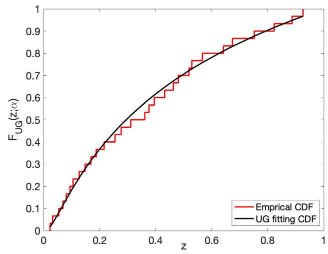
The fitting performance of the UG distribution is also illustrated in Figure 3(a)-(c). Also, plots for the $\ln L$ function of the UG distribution for the data sets considered in the application are given in Figure 3(d)-(f). It is clear from Figure 3(d),(e),(f) that the ML estimate of parameter α is the point on which the $\ln L$ function of the UG distribution attains its maximum.



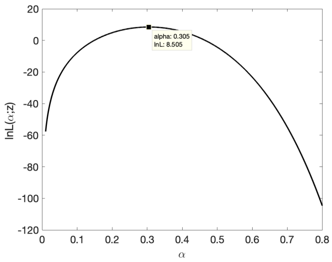
(a) The cdf fitting for Data Set-I



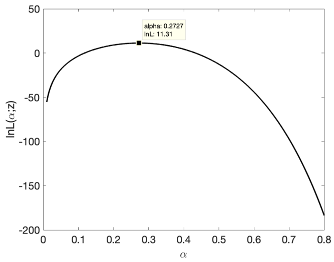
(b) The cdf fitting for Data Set-II



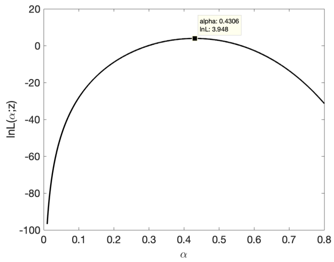
(c) The cdf fitting for Data Set-III



(d) The $\ln L$ plot for Data Set-I



(e) The $\ln L$ plot for Data Set-II



(f) The $\ln L$ plot for Data Set-III

Figure 3. The cdf fitting and $\ln L$ plots of the UG distribution.

5. Conclusion

In this paper, a new family of unit-distributions is introduced. The UG distribution is obtained as an alternative to the beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions. The ML estimate of its parameter α is also provided. Three real data sets are modeled by the UG, beta, Kumaraswamy, Topp-Leone, and unit-Gompertz distributions. The results show that the UG distribution is preferable over the beta, Kumaraswamy, and Topp-Leone distributions in terms of the goodness-of-fit criteria; see Table 4. Although the UG distribution has a smaller number of parameters than the beta, Kumaraswamy, and unit-Gompertz distribution, it shows better modeling performance in modeling the data having bounded range on the unit interval $(0, 1)$. Also, note that obtaining new distributions using the cdf given in Section 2 can be carried out as future work.

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References

[1] Alzaatreh, A., Lee, C., and Famoye, F., (2013), A new method for generating families of continuous distributions, Metron, 71, pp. 63–79.

[2] Arslan, T., Acitas, S., Senoglu, S., Parameter estimation for the two-parameter Maxwell distribution under complete and censored samples, REVSTAT, 19(2), pp. 237-253.

[3] Bantan, R.A.R., Chesneau, C., Jamal, F., Elgarhy, M., Tahir, M.H., Ali, A., Zubair, M., and Anam, S., (2020), Some new facts about the unit-Rayleigh distribution with applications, Mathematics, 8, 1954. doi:10.3390/math8111954

[4] Erdogan, N., Bagci, K., Arslan, T., Celik, H.E., Alpha power Maxwell distribution: Properties and application, Journal of Mathematical Modeling, (Article in Press). DOI: 10.22124/jmm.2021.17987.1553

[5] Eugene, N., Lee, C., and Famoye, F., (2002), The beta-normal distribution and its applications, Communications in Statistics - Theory and Methods, 31(4), pp. 497–512.

[6] Ferrari, S., and Cribari-Neto, F., (2004), Beta regression for modelling rates and proportions, Journal of Applied Statistics, 31(7), pp. 799–815.

[7] Genc, A.I., (2013), Estimation of $P(X > Y)$ with Topp-Lone distribution, Journal of Statistical Computation and Simulation, 83(2), pp. 326–339.

[8] Ghitany, M.E., Mazucheli, J., Menezes, A.F.B., and Alqallaf, F., (2019), The unit-inverse Gaussian distribution: A new alternative to two- parameter distributions on the unit interval, Communications in Statistics - Theory and Methods, 48, pp. 3423–3438.

[9] Johnson, N.L., (1949), Systems of frequency curves generated by methods of translation, Biometrika, 36, pp. 149–176.

[10] Jodrá, P., and Jiménez-Gamero, M.D., (2020), A quantile regression model for bounded response based on the exponential-geometric distribution, REVSTAT – Statistical Journal, 18(4), pp. 415–436.

[11] Korkmaz, M.C. (2020) The unit generalized half-normal distribution: A new bounded distribution with inference and application. Univ. Politeh. Buchar. Sci. Bull. Ser. Appl. Math. Phys., 82(2), pp. 133-–140.

[12] Kumaraswamy, P., (1980), A generalized probability density functions for double-bounded random processes, Journal of Hydrology, 46, pp. 79–88.

[13] Mazucheli, J., Menezes, A.F., and Dey, S, (2018a), The unit-Birnbaum-Saunders distribution with applications, Chilean Journal of Statistics, 9, pp. 47–57.

[14] Mazucheli, J., Menezes, A.F.B., and Ghitany, M.E., (2018b), The unit-Weibull distribution and associated inference, Journal of Applied Probability and Statistics, 13, pp. 1–22.

[15] Mazucheli, J., Menezes, A.F.B., and Chakraborty, S., (2019a), On the one parameter unit-Lindley distribution and its associated regression model for proportion data, Journal of Applied Statistics, 46, pp. 700–714.

[16] Mazucheli, J., Menezes, A.F.B., and Sanku, D., (2019b), Unit-Gompertz distribution with applications, Statistica, 79(1), pp. 25–43.

[17] Mazucheli, J., Menezes, A.F.B., Fernandes, L.B., de Oliveira, R.P., and Ghitany, M.E., (2020), The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates, Journal of Applied Statistics, 47(6), pp. 954–974.

[18] Topp, C.W., Leone, F.C., (1955), A family of J-shaped frequency functions, Journal of the American Statistical Association, 50, pp. 209–219.

Talha Arslan for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.10, N.4.
