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A MAINTENANCE MODEL FOR A DETERIORATING SYSTEM UNDER RANDOM ENVIRONMENT USING PARTIAL PRODUCT PROCESS

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ABSTRACT. In this paper, deteriorating system maintenance model under random environment using partial product process is studied. Up to time t, assume that the number of random shocks generated by the random environment is a counting process. Whenever a random shock occurs, the operating time of the system is reduced. The successive reductions in the operating time of the system are statistically independent and identically distributed random variables. Assume that the system's successive operating times after repairs form a decreasing partial product process. Assume that the system's consecutive repair times after failures constitute an increasing partial product process provided that the system suffers no random shock. A replacement policy N is applied. Afterwards, for minimizing the mean cost per unit time in long-run, an optimal policy N^* is determined analytically. A numerical illustration is provided to strengthen the theoretical results.

Keywords: Geometric Process, Partial Product Process, Replacement Policy, Renewal Process.

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1. Introduction

The mathematical theory of reliability has put forth a great effort to issues of life-testing, machine support, replacement, order statistics, and so on. The maintenance problems are concerned about the circumstance that emerges about the reduction of the productivity level of items or breakdown. The problem of replacement is to recognize the best policy which enables determination of ideal replacement time that is generally economical. One of the most interesting and critical topics to study in reliability is the study of maintenance problems.

A common assumption in the initial period of studying maintenance issues is that repair is perfect, a repairable framework after the repair is as good as new. This assumption clearly has the effect of a natal way. In practice, most repairable systems deteriorate

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because of the combined wear and tear impact. Barlow (1960) thusly presented a minimal repair model in which a system after the repair has the same failure rate and effective age as it was when it failed. Brown(1983) proposes an imperfect repair model, in which the repair is perfect with likelihood p, and the repair is minimal with a probability of $1 - p$.

Deteriorating systems have a different problem as the one portrayed above. For instance, in machine maintenance problems, after every repair, the working time of a machine will end up shorter and shorter, so the absolute working time or the existence of the machine must be limited. However, in perspective on the aging and aggregate wear, the repair time will turn out to be longer and tend to increase so that at the end the machine is non-repairable. Therefore, there is need to consider a repair replacement model for deteriorating systems, the progressive survival times are diminishing, while the consecutive repair times are expanding. Consequently, a monotone process model would be a most appropriate model for a deteriorating system.

Lam (1988) first presented a Geometric Process Repair model to model a deteriorating system with the above characteristics.

Definition 1.1. *For two random variables Y and Z, Z is called stochastically less than Y (or Y is stochastically larger than Z) if for all real α,*

$$P(Y > \alpha) \geq P(Z > \alpha).$$

This is written as $Z \leq_{st} Y$ or $Y \geq_{st} Z$.

Definition 1.2. *A stochastic process $\{Y_n,\ n = 1, 2, 3, ...\}$ is said to be stochastically increasing (decreasing) if*

$$Y_n \leq_{st} (\geq_{st}) Y_{n+1}$$

for all $n = 1, 2, 3, ...$

Definition 1.3. *Given a sequence of non-negative random variables $\{X_n,\ n = 1, 2, 3, ...\}$, if they are independent and the distribution function of X_n is given by $F\left(a^{n-1}\cdot\right)$ for $n = 1, 2, 3, ...$, where a is a positive constant, then $\{X_n,\ n = 1, 2, 3, ...\}$ is called a geometric process.*

Finkelstein (1993) proposes *particular deteriorating renewal process* in which the distribution function of X_n is $F(a_n\cdot)$ where a_n are scale parameters.

There is a need to estimate a large number of parameters in *particular deteriorating renewal process* which may be problematic in real applications as a large number of failure data are needed to estimate the parameters.

To overcome this, Babu etal (2018) introduced the *partial product process* in which the parameters a_n are related by $a_n = a_0 a_1 ... a_{n-1}$ for $n = 1, 2, 3, ...$.

Definition 1.4. *Let $\{Y_n,\ n = 1, 2, 3, ...\}$ be a sequence of independent and non-negative random variables and let $G(\cdot)$ be the distribution function of Y_1. Then $\{Y_n,\ n = 1, 2, 3, ...\}$ is called a partial product process, if the distribution function of Y_{k+1} is $G(\beta_k\cdot)$ for k=1,2,... where $\beta_k > 0$ are constants and $\beta_k = \beta_0\beta_1\beta_2...\beta_{k-1}$.*

The deterioration of a system is regularly brought about by an internal factor, for example, aging and accumulated wear and tear of the system. System deterioration can also be caused by external factors, for example, an ecological factor. For instance, if a virus attacks a computer, the operating time of the computer is likely to be reduced or the operating system of the computer can be crumbled. Therefore, when analyzing a maintenance problem for a repairable system, one ought to consider the internal reason as well as consider the impact that random shocks produced by the environment can bring to the system.

In Babu etal (2018) paper, the maintenance model for a deteriorating system using partial product process without any shocks is studied. In this paper, the maintenance model for a deteriorating under a random environment using partial product process is studied. The preliminary results about partial product process are given below.

Lemma 1.1. *If* $\beta_k = \beta_0\beta_1\beta_2...\beta_{k-1}$*, then* $\beta_k = {\beta_0}^{2^{k-1}}\,(k = 1, 2, 3, ...)$.

Then the distribution function of Y_{k+1} is $G\left({\beta_0}^{2^{k-1}}\cdot\right)$ for $k = 1, 2, 3, ...$.

Lemma 1.2. *The partial product process* $\{Y_n,\ n = 1, 2, 3, ...\}$ *is*

(i) *stochastically decreasing, if* $\beta_0 > 1$.
(ii) *stochastically increasing, if* $0 < \beta_0 < 1$.

It is clear that if $\beta_0 = 1$, then the partial product process is a renewal process.

Lemma 1.3. *Let* $E(Y_1) = \mu,\ Var(Y_1) = \sigma^2$*. Then for* $k = 1, 2, 3, ...$,

$$E(Y_{k+1}) = \frac{\mu}{{\beta_0}^{2^{k-1}}} \text{ and } Var(Y_{k+1}) = \frac{\sigma^2}{{\beta_0}^{2^k}}.$$

The proof of the above Lemmas may be found in Babu, Govindaraju and Babu (2018).

2. Model Descriptions

We consider the maintenance model for a deteriorating system and make the following assumptions.

2.1 A new simple repairable system is used at the start. At whatever point the system fails, it might be repaired or replaced by a new and similar one.

2.2 Given that there is no random shock, let X_1 be the operating time before the first failure and let $F(\cdot)$ be the distribution function of X_1. Assume that $\lambda_1 = E(X_1) = \lambda > 0$. Let X_{i+1} be the operating time after the i-th repair for $i = 1, 2, 3, ...$ Then, following Babu etal (2018), the distribution function of $X_{i+1}$is $F({\beta_0}^{2^{i-1}}\cdot)$ where $\beta_0(\geq 1)$ is a constant and $\lambda_{i+1} = E(X_{i+1}) = \dfrac{\lambda}{{\beta_0}^{2^{i-1}}}$ for $i = 1, 2, 3, ...$ That is the successive operating times $\{X_j,\ j = 1, 2, 3, ...\}$ after repair constitute a decreasing partial product process if $\beta_0 > 1$ and a renewal process if $\beta_0 = 1$.

2.3 After the first failure, let Y_1 be the repair time and let $G(\cdot)$ be the distribution function of Y_1. Assume that $\mu_1 = \mu \geq 0$. Here $\mu = 0$ means that the expected repair time is negligible. For $i = 1, 2, 3, ...,$ let Y_{i+1} be the repair time after the $(i + 1)$-st failure. Regardless of whether or not there is a random shock, the distribution function of Y_{i+1} is $G({\gamma_0}^{2^{i-1}}\cdot)$ where $0 < \gamma_0 \leq 1$ is a constant and $\mu_{i+1} = E(Y_{i+1}) = \dfrac{\mu}{{\gamma_0}^{2^{i-1}}}$ for $i = 1, 2, 3, ...$. That is, the consecutive repair times $\{Y_j,\ j = 1, 2, 3, ...\}$ form an stochastically increasing partial product process if $0 < \gamma_0 < 1$ and a renewal process if $\gamma_0 = 1$.

2.4 Let the random variable R denote the replacement time with mean ξ.

2.5 Assume that $\{\omega(t),\ t \geq 0\}$, the number of random shocks by time t produced by the random environment, form a counting process having stationary and independent increment. The operating time of the system will be shortened when a shock occurs. After the j-th random shock, let W_j be the reduction in the operating time. Then $\{W_j,\ j = 1, 2, ...\}$ are independent identically distributed random variables. The consecutive reductions in the system operating time are additive. If a system fails, it is shut in the meaning that the failed system is not affected by any shock that happens during the fix time.

2.6 The processes $\{X_j,\ j=1,2,...\}$, $\{Y_j,\ j=1,2,...\}$ and random variable R are independent. The processes $\{X_j,\ j=1,2,...\}$, $\{W_j,\ j=1,2,...\}$ and $\{\omega(t),\ t\geq 0\}$ are also independent.

2.7 The reward rate is τ, the repair cost rate is κ and the basic replacement cost is ζ. Let ρ be the proportional cost associated with the duration of the replacement time R.

Let X'_j be the actual operating time following the $(j-1)$-st repair. First, we study its distribution. Denote t_{j-1} as the completion time of the $(j-1)$-st repair. Then the number of shocks due to environment in $(t_{j-1},\ t_{j-1}+t]$ is given by

$$\omega(t_{j-1},\ t_{j-1}+t]=\omega(t_{j-1}+t)-\omega(t_{j-1}). \tag{1}$$

Thus, in the operating time in $(t_{j-1},\ t_{j-1}+t]$, the total reduction is given by

$$\begin{aligned}\Delta X_{(t_{j-1},\ t_{j-1}+t]} &= \sum_{i=\omega(t_{j-1})+1}^{\omega(t_{j-1}+t)} W_i \\ &= \sum_{i=1}^{\omega(t_{j-1},\ t_{j-1}+t)} W_i\ . \end{aligned} \tag{2}$$

Let $R_j(t)$ be the residual time at $t_{j-1}+t$ under random environment. Then,

$$R_j(t)=X_j-t-\Delta X_{(t_{j-1},t_{j-1}+t]} \tag{3}$$

subject to $R_j(t)\geq 0$. This implies that

$$X'_j=\inf_{t\geq 0}\left\{t\ \middle|\ R_j(t)\leq 0\ \right\}. \tag{4}$$

To study the monotonicity of $E(X'_j)$, we need following two lemmas.

Lemma 2.1. *If f_j and l_m are the density functions of X_j and $\sum_{i=1}^{m} W_i$ respectively, then*

$$P(X'_j>t'\mid\omega(t_{j-1},t_{j-1}+t']=m)=\int_0^\infty\left[\int_{t'+w}^\infty f_j(x)dx\right]l_m(w)dw. \tag{5}$$

Proof.

$$\begin{aligned}&P(X'_j>t'\mid\omega(t_{j-1},t_{j-1}+t']=m)\\ &= P(\inf_{t\geq 0}\left\{t\ \middle|\ R_j(t)\leq 0\right\}>t'\mid\omega(t_{j-1},t_{j-1}+t']=m)\\ &= P(R_j(t)=X_j-t-\Delta X_{(t_{j-1},t_{j-1}+t]}>0,\ \forall t\in[0,t']\mid\omega(t_{j-1},t_{j-1}+t']=m)\\ &= P(X_j-\Delta X_{(t_{j-1},t_{j-1}+t']}>t'\mid\omega(t_{j-1},t_{j-1}+t']=m)\\ &= P(X_j-\sum_{i=1}^{m}W_i>t')\\ &= \int_0^\infty\left[\int_{t'+w}^\infty f_j(x)dx\right]l_m(w)dw.\end{aligned}$$

∎

Lemma 2.2. *If F_j and L_m are the distribution functions of X_j and $\sum_{i=1}^{m} W_i$ respectively, then*

$$P(X'_j > t') = 1 - \sum_{m=0}^{\infty} \left[\int_0^{\infty} F_j(t'+w) dL_m(w) \right] P(\omega(t') = m). \tag{6}$$

Proof. First, note that using Lemma 2.1, we have

$$\begin{aligned} P(X'_j > t' \mid \omega(t_{j-1}, t_{j-1}+t'] = m) &= \int_0^{\infty} \left[\int_{t'+w}^{\infty} f_j(x)dx \right] l_m(w)dw \\ &= \int_0^{\infty} \left[1 - F_j(t'+w) \right] dL_m(w) \\ &= 1 - \int_0^{\infty} F_j(t'+w) dL_m(w). \end{aligned} \tag{7}$$

Thus,

$$\begin{aligned} P(X'_j > t') &= \sum_{m=0}^{\infty} P(X'_j > t' \mid \omega(t_{j-1}, t_{j-1}+t'] = m) P(\omega(t_{j-1}, t_{j-1}+t'] = m) \\ &= \sum_{m=0}^{\infty} \left[1 - \int_0^{\infty} F_j(t'+w) dL_m(w) \right] P(\omega(t_{j-1}, t_{j-1}+t'] = m) \\ &= 1 - \sum_{m=0}^{\infty} \left[\int_0^{\infty} F_j(t'+w) dL_m(w) \right] P(\omega(t') = m) \end{aligned}$$

(since $\{\omega(t),\ t \geq 0\}$ has stationary increments). ■

Lemma 2.3. *$\lambda'_j = E(X'_j)$ is non-increasing in j.*

Proof. The distribution function of X'_j is

$$P(X'_j \leq x) = \begin{cases} \sum_{m=0}^{\infty} \int_0^{\infty} F(x+w) dL_m(w) P(\omega(x) = m), & \text{if } j = 1, \\ \sum_{m=0}^{\infty} \int_0^{\infty} F({\beta_0}^{2^{j-2}}(x+w)) dL_m(w) P(\omega(x) = m), & \text{if } j = 2, 3, 4, \ldots . \end{cases} \tag{8}$$

Now, as $\{X_j,\ j = 1, 2, 3, ...\}$ form an decreasing partial product process, and F_j is the distribution of X_j, then from equation (8) for all real x, we have

$$P(X'_j > x) \geq P(X'_{j+1} > x).$$

This implies that,

$$E(X'_j) \geq E(X'_{j+1}).$$

Hence the proof is completed. ■

3. The Mean Cost in Long-run

Let $\mathscr{C}(N)$ be the mean cost per unit time in long-run under N-policy. Then,

$$\mathscr{C}(N) = \frac{\kappa \sum_{j=1}^{N-1} E(Y_j) + \zeta + \rho E(R) - \tau \sum_{j=1}^{N} E(X'_j)}{\sum_{j=1}^{N} E(X'_j) + \sum_{j=1}^{N-1} E(Y_j) + E(R)}$$

$$= \frac{\kappa(\ \mu + \sum_{j=2}^{N-1} \frac{\mu}{\gamma_0{}^{2^{j-2}}}\ ) + \zeta_1 - \tau \sum_{j=1}^{N} \lambda'_j}{\sum_{j=1}^{N} \lambda'_j + (\ \mu + \sum_{j=2}^{N-1} \frac{\mu}{\gamma_0{}^{2^{j-2}}}\ ) + \xi} \tag{9}$$

where $\zeta_1 = \zeta + \rho\xi$. Let

$$D(N) = \sum_{j=1}^{N} \lambda'_j + \mu(\ 1 + \sum_{j=2}^{N-1} \frac{1}{\gamma_0{}^{2^{j-2}}}\ ) + \xi.$$

Then, from equation (9), we have,

$$\mathscr{C}(N) = \frac{(\kappa+\tau)\mu(\ 1 + \sum_{j=2}^{N-1} \frac{1}{\gamma_0{}^{2^{j-2}}}\ ) + \zeta_1 + \tau\xi}{D(N)} - \tau\ . \tag{10}$$

4. The Optimal Policy N^*

For minimizing $\mathscr{C}(N)$, an optimal policy N^* is determined analytically in this section.
Now, from equation (10), we have,

$$\mathscr{C}(N+1) - \mathscr{C}(N)$$

$$= \frac{\left\{ \begin{array}{c} \left[\sum_{j=1}^{N} \lambda'_j + \mu(1 + \sum_{j=2}^{N-1} \frac{1}{\gamma_0{}^{2^{j-2}}}) + \xi\right] \left[(\kappa+\tau)\mu(1 + \sum_{j=2}^{N} \frac{1}{\gamma_0{}^{2^{j-2}}}) + \zeta_1 + \tau\xi\right] \\ - \left[\sum_{j=1}^{N+1} \lambda'_j + \mu(1 + \sum_{j=2}^{N} \frac{1}{\gamma_0{}^{2^{j-2}}}) + \xi\right] \left[(\kappa+\tau)\mu(1 + \sum_{j=2}^{N-1} \frac{1}{\gamma_0{}^{2^{j-2}}}) + \zeta_1 + \tau\xi\right] \end{array} \right\}}{D(N)D(N+1)}$$

$$= \frac{\left\{ \begin{array}{c} (\kappa+\tau)\mu \left[\sum_{j=1}^{N} \lambda'_j (1 + \sum_{j=2}^{N} \frac{1}{\gamma_0{}^{2^{j-2}}}) - \sum_{j=1}^{N+1} \lambda'_j (1 + \sum_{j=2}^{N-1} \frac{1}{\gamma_0{}^{2^{j-2}}})\right] - (\zeta_1 + \tau\xi)\lambda'_{N+1} \\ -(\zeta_1 + \tau\xi)\mu(\frac{1}{\gamma_0{}^{2^{N-2}}}) + (\kappa+\tau)\mu\xi(\frac{1}{\gamma_0{}^{2^{N-2}}}) \end{array} \right\}}{D(N)D(N+1)}$$

$$= \frac{\begin{bmatrix} (\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j (1 + \sum\limits_{j=2}^{N} \frac{1}{{\gamma_0}^{2^{j-2}}}) - \sum\limits_{j=1}^{N+1} \lambda'_j (1 + \sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) + \xi (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \\ -(\zeta_1 + \tau\xi) \left[\lambda'_{N+1} + \mu (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \end{bmatrix}}{D(N)D(N+1)}$$

$$= \frac{\begin{Bmatrix} (\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j (\sum\limits_{j=2}^{N} \frac{1}{{\gamma_0}^{2^{j-2}}}) - \sum\limits_{j=1}^{N+1} \lambda'_j (\sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) - \lambda'_{N+1} + \xi (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \\ -(\zeta_1 + \tau\xi) \left[\lambda'_{N+1} + \mu (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \end{Bmatrix}}{D(N)D(N+1)}$$

$$= \frac{\begin{Bmatrix} (\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j (\frac{1}{{\gamma_0}^{2^{N-2}}}) - (\lambda'_{N+1}) (\sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) - \lambda'_{N+1} + \xi (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \\ -(\zeta_1 + \tau\xi) \left[\lambda'_{N+1} + \mu (\frac{1}{{\gamma_0}^{2^{N-2}}}) \right] \end{Bmatrix}}{D(N)D(N+1)}$$

$$= \frac{\begin{Bmatrix} (\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j - (\lambda'_{N+1}){\gamma_0}^{2^{N-2}} (\sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) - {\gamma_0}^{2^{N-2}} \lambda'_{N+1} + \xi \right] \\ -(\zeta_1 + \tau\xi)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}} + \mu) \end{Bmatrix}}{{\gamma_0}^{2^{N-2}} D(N)D(N+1)}. \quad (11)$$

For optimality, define the auxiliary function $\mathscr{A}(N)$ from equation (11) as follows.

$$\mathscr{A}(N) = \frac{(\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j - (\lambda'_{N+1}){\gamma_0}^{2^{N-2}} (\sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) - {\gamma_0}^{2^{N-2}} \lambda'_{N+1} + \xi \right]}{(\zeta_1 + \tau\xi)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}} + \mu)}. \quad (12)$$

Now, we prove the following lemma.

Lemma 4.1. *For $\mathscr{C}(N)$ given by the equation (10) and $\mathscr{A}(N)$ by the equation (12), we have $\mathscr{C}(N)$ is increasing(decreasing) if and only if $\mathscr{A}(N) > 1(\mathscr{A}(N) < 1)$.*

Proof. It is clear that, ${\gamma_0}^{2^{N-2}} D(N)D(N+1) > 0$.
Thus, from equations (11) and (12),

$\mathscr{C}(N)$ is increasing if and only if $\mathscr{C}(N+1) - \mathscr{C}(N) > 0$.

This implies that

$$\begin{bmatrix} (\kappa+\tau)\mu \left[\sum\limits_{j=1}^{N} \lambda'_j - (\lambda'_{N+1}){\gamma_0}^{2^{N-2}} (\sum\limits_{j=2}^{N-1} \frac{1}{{\gamma_0}^{2^{j-2}}}) - {\gamma_0}^{2^{N-2}} \lambda'_{N+1} + \xi \right] \\ -(\zeta_1 + \tau\xi)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}} + \mu) \end{bmatrix} > 0.$$

From this, we have $\mathscr{A}(N) > 1$. In a similar way, we can prove that $\mathscr{C}(N)$ is decreasing if and only if $\mathscr{A}(N) < 1$. ■

Remark: $\mathscr{C}(N+1) = \mathscr{C}(N)$ if and only if $\mathscr{A}(N) = 1$.

Lemma 4.2. *$\mathscr{A}(N)$ given by the equation (12) is non-decreasing in N.*

Proof. Let

$$K(N) = \frac{(\kappa+\tau)\mu}{(\zeta_1+\tau\xi)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}+\mu)(\lambda'_{N+2}{\gamma_0}^{2^{N-1}}+\mu)}.$$

From equation (12),

$\mathscr{A}(N+1)-\mathscr{A}(N)$

$$=K(N)\times\left[\begin{array}{l}(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}+\mu)\left[\sum_{j=1}^{N+1}\lambda'_j-(\lambda'_{N+2}){\gamma_0}^{2^{N-1}}(\sum_{j=2}^{N}\frac{1}{{\gamma_0}^{2^{j-2}}})-{\gamma_0}^{2^{N-1}}\lambda'_{N+2}+\xi\right]\\ -(\lambda'_{N+2}{\gamma_0}^{2^{N-1}}+\mu)\left[\sum_{j=1}^{N}\lambda'_j-(\lambda'_{N+1}){\gamma_0}^{2^{N-2}}(\sum_{j=2}^{N-1}\frac{1}{{\gamma_0}^{2^{j-2}}})-{\gamma_0}^{2^{N-2}}\lambda'_{N+1}+\xi\right]\end{array}\right]$$

$$=K(N)\times\left[\begin{array}{l}(\sum_{j=1}^{N+1}\lambda'_j)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}-\lambda'_{N+2}{\gamma_0}^{2^{N-1}})+(\sum_{j=2}^{N}\frac{1}{{\gamma_0}^{2^{j-2}}})\mu(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}-\lambda'_{N+2}{\gamma_0}^{2^{N-1}})\\ +\mu(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}-\lambda'_{N+2}{\gamma_0}^{2^{N-1}})+\xi(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}-\lambda'_{N+2}{\gamma_0}^{2^{N-1}})\end{array}\right]$$

$$= K(N)\times{\gamma_0}^{2^{N-2}}\left[(\lambda'_{N+1}-\lambda'_{N+2}{\gamma_0}^{2^{N-2}})(\sum_{j=1}^{N+1}\lambda'_j+\mu\sum_{j=2}^{N}\frac{1}{{\gamma_0}^{2^{j-2}}}+\mu+\xi)\right]$$

$$= \frac{(\kappa+\tau)\mu{\gamma_0}^{2^{N-2}}\left[(\lambda'_{N+1}-\lambda'_{N+2}{\gamma_0}^{2^{N-2}})(\sum_{j=1}^{N+1}\lambda'_j+\mu\sum_{j=2}^{N}\frac{1}{{\gamma_0}^{2^{j-2}}}+\mu+\xi)\right]}{(\zeta_1+\tau\xi)(\lambda'_{N+1}{\gamma_0}^{2^{N-2}}+\mu)(\lambda'_{N+2}{\gamma_0}^{2^{N-1}}+\mu)}.$$

This implies that $\mathscr{A}(N+1)-\mathscr{A}(N)\geq 0$, because $0<\gamma_0\leq 1$ and λ'_j is non-increasing in j. ■

Now, we prove the main theorem of this section.

Theorem 4.1. *For $\mathscr{A}(N)$ given in equation (12),*

$$N^* = \min\{N|\ \mathscr{A}(N)\geq 1\} \tag{13}$$

is the optimal replacement policy . Moreover, N^* is unique if and only if $\mathscr{A}(N)>1$.

Proof. The proof of this theorem follows from Lemma (4.1) and Lemma (4.2). ■

5. Numerical Example

In this section, we provide an example to illustrate the theoretical results. Assume that $\{\omega(t),\ t\geq 0\}$ follows a Poisson process with rate υ.
Then,

$$P(\omega(t)=k)=\frac{(\upsilon t)^k}{k!}e^{-\upsilon t},\ \ k=0,1,2,... . \tag{14}$$

Assume that $W_1, W_2, ..., W_m, ...$ follows a gamma distribution $\Gamma(a,b)$ which are independent and identically distributed with density function l given by

$$l(w)=\begin{cases}\dfrac{b^a}{\Gamma(a)}w^{a-1}e^{-bw}, & \text{if } w>0,\\ 0, & \text{elsewhere.}\end{cases} \tag{15}$$

Thus $\sum_{i=1}^{m} W_i$ follows a Gamma distribution $\Gamma(ma,b)$ and $\Delta X_{(0,t]}=\sum_{i=1}^{\omega(t)} W_i$ constitute a compound Poisson process. Moreover, let X_1 follows an exponential distribution with parameter $\frac{1}{\lambda}$. Then for $x>0$,

$$F_j(x)=\begin{cases}1-e^{\frac{-x}{\lambda}}, & \text{if } j=1,\\ 1-e^{\frac{-\beta_0{}^{2^{j-2}}x}{\lambda}}, & \text{if } j=2,3,4,...\end{cases} \tag{16}$$

where $F_1(x)=F(x)$. Now, from Equation (8), we have,

$P\left(X_j{}'\leq x\right)$

$$=\begin{cases}\displaystyle\sum_{m=0}^{\infty}\left[\int_0^{\infty}\left[1-e^{\frac{-(x+w)}{\lambda}}\right]\frac{b^{ma}}{\Gamma(ma)}w^{ma-1}e^{-bw}dw\right]\frac{(\upsilon x)^m}{m!}e^{-\upsilon x}, & \text{if } j=1,\\ \displaystyle\sum_{m=0}^{\infty}\left[\int_0^{\infty}\left[1-e^{\frac{-\beta_0{}^{2^{j-2}}(x+w)}{\lambda}}\right]\frac{b^{ma}}{\Gamma(ma)}w^{ma-1}e^{-bw}dw\right]\frac{(\upsilon x)^m}{m!}e^{-\upsilon x}, & \text{if } j=2,3,...\end{cases}$$

$$=\begin{cases}\displaystyle 1-\sum_{m=0}^{\infty}e^{\frac{-x}{\lambda}}\frac{b^{ma}}{\Gamma(ma)}\frac{(\upsilon x)^m}{m!}e^{-\upsilon x}\int_0^{\infty}e^{\frac{-w}{\lambda}}w^{ma-1}e^{-bw}dw, & \text{if } j=1,\\ \displaystyle 1-\sum_{m=0}^{\infty}e^{\frac{-\beta_0{}^{2^{j-2}}x}{\lambda}}\frac{b^{ma}}{\Gamma(ma)}\frac{(\upsilon x)^m}{m!}e^{-\upsilon x}\int_0^{\infty}e^{\frac{-\beta_0{}^{2^{j-2}}w}{\lambda}}w^{ma-1}e^{-bw}dw, & \text{if } j=2,3,...\end{cases}$$

$$=\begin{cases}1-\exp\left[-\left[\upsilon\left[1-\dfrac{b^a}{\left[b+\frac{1}{\lambda}\right]^a}\right]+\dfrac{1}{\lambda}\right]x\right], & \text{if } j=1,\\ 1-\exp\left[-\left[\upsilon\left[1-\dfrac{b^a}{\left[b+\frac{\beta_0{}^{2^{j-2}}}{\lambda}\right]^a}\right]+\dfrac{\beta_0{}^{2^{j-2}}}{\lambda}\right]x\right], & \text{if } j=2,3,...\end{cases}$$

Thus,

$$E\left[X_j'\right] = \lambda_j' = \begin{cases} \left[\upsilon \left[1 - \dfrac{b^a}{\left[b + \dfrac{1}{\lambda} \right]^a} \right] + \dfrac{1}{\lambda} \right]^{-1}, & \text{if } j = 1, \\ \left[\upsilon \left[1 - \dfrac{b^a}{\left[b + \dfrac{{\beta_0}^{2^{j-2}}}{\lambda} \right]^a} \right] + \dfrac{{\beta_0}^{2^{j-2}}}{\lambda} \right]^{-1}, & \text{if } j = 2, 3, \ldots . \end{cases} \tag{17}$$

If $a = 1$, then each W_i follows an *exponential* distribution. Then equation (17) becomes,

$$E(X_j') = \lambda_j' = \begin{cases} \dfrac{\lambda}{1 + \dfrac{\upsilon}{b + \dfrac{1}{\lambda}}}, & \text{if } j = 1, \\ \dfrac{\lambda}{{\beta_0}^{2^{j-2}} \left[1 + \dfrac{\upsilon}{b + \dfrac{{\beta_0}^{2^{j-2}}}{\lambda}} \right]}, & \text{if } j = 2, 3, 4, \ldots . \end{cases} \tag{18}$$

Let the parameter values be $\kappa = 45,\ \tau = 90,\ \rho = 10,\ \lambda = 40,\ \mu = 10,\ ,\zeta = 5000, \xi = 10, \upsilon = 5, b = 4$.

The results of $\mathscr{C}(N)$ and $\mathscr{A}(N)$ for different values of $\beta_0 = 1.05, 1.1, 1.15, 1.5$ and $\gamma_0 = 0.8, 0.85, 0.9, 0.95$ are provided in Table 1 to Table 4 and plotted in Figure 1 to Figure 8.

For all these values, it can be seen that $\mathscr{C}(N)$ attains its minimum when $\mathscr{A}(N) \geq 1$. This accords with the Theorem 4.1.

TABLE 1. Values of $\mathscr{C}(N)$ and $\mathscr{A}(N)$ when $\beta_0 = 1.05$ for different values of γ_0

N	$\gamma_0 = 0.8$		$\gamma_0 = 0.85$		$\gamma_0 = 0.9$		$\gamma_0 = 0.95$	
	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$
1	129.0876	0.2646	128.0477	0.2557	127.1180	0.2473	126.2809	0.2395
2	44.0456	0.3126	44.0456	0.2943	44.0456	0.2772	44.0456	0.2612
3	18.2094	0.4624	17.9715	0.4140	17.7564	0.3689	17.5610	0.3271
4	7.9248	0.8549	7.0856	0.7341	6.3393	0.6167	5.6728	0.5052
5	**6.0679**	**1.5662**	**3.9442**	**1.3894**	**2.1203**	**1.1738**	0.5526	0.9311
6	13.0917	2.1083	7.8693	2.0446	3.5507	1.9121	**0.0811**	**1.6615**
7	32.0539	2.2446	22.0732	2.2420	11.8900	2.2308	3.8227	2.1800
8	44.4253	2.2644	41.4560	2.2644	30.3482	2.2643	12.5720	2.2632
9	44.9995	2.2657	44.9771	2.2657	44.1561	2.2657	29.8264	2.2657
10	45.0000	2.2657	45.0000	2.2657	44.9989	2.2657	44.0011	2.2657

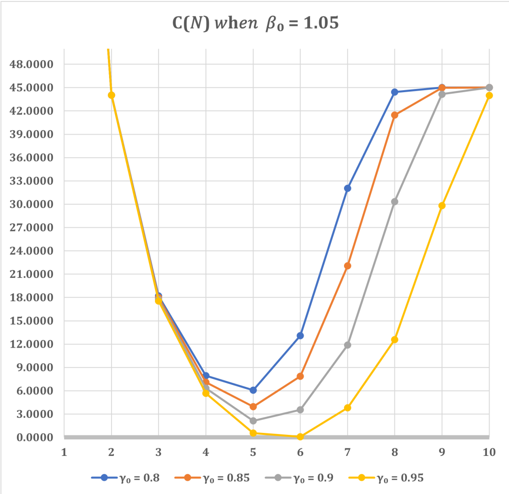

FIGURE 1

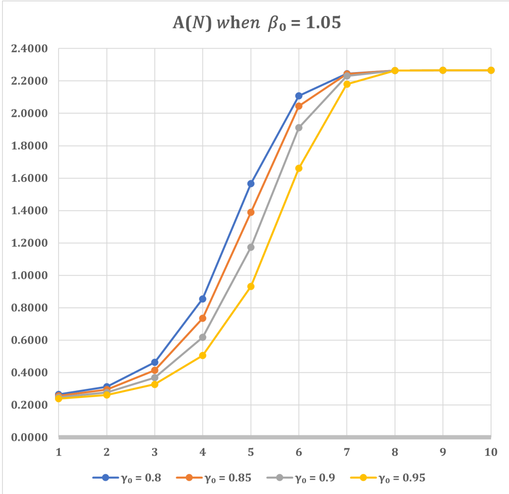

FIGURE 2

TABLE 2. Values of $\mathscr{C}(N)$ and $\mathscr{A}(N)$ when $\beta_0 = 1.1$ for different values of γ_0

N	$\gamma_0 = 0.8$		$\gamma_0 = 0.85$		$\gamma_0 = 0.9$		$\gamma_0 = 0.95$	
	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$
1	129.0876	0.2713	128.0477	0.2624	127.1180	0.2541	126.2809	0.2462
2	45.9538	0.3329	45.9538	0.3146	45.9538	0.2975	45.9538	0.2815
3	21.1400	0.5210	20.9222	0.4738	20.7253	0.4294	20.5464	0.3877
4	12.1287	0.9718	11.3520	0.8694	10.6600	0.7656	10.0409	0.6629
5	**11.7711**	**1.5884**	**9.8447**	**1.4940**	**8.1802**	**1.3690**	**6.7420**	**1.2124**
6	19.4648	1.8627	14.9817	1.8483	11.2047	1.8171	8.1232	1.7524
7	35.4424	1.8946	27.7040	1.8944	19.4516	1.8935	12.6416	1.8898
8	44.5922	1.8961	42.4688	1.8961	34.2764	1.8961	20.2874	1.8961
9	44.9997	1.8961	44.9838	1.8961	44.4014	1.8961	33.8924	1.8961
10	45.0000	1.8961	45.0000	1.8961	44.9993	1.8961	44.2912	1.8961

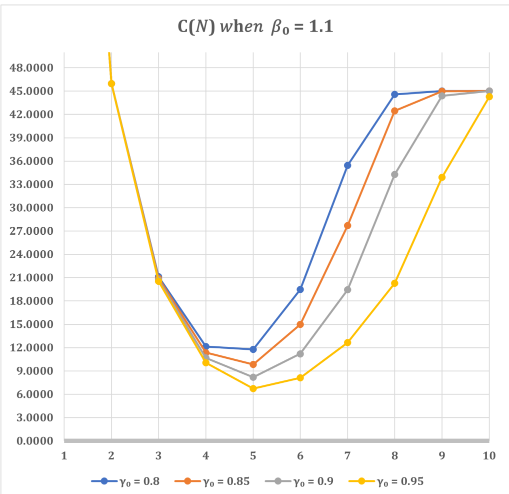

FIGURE 3

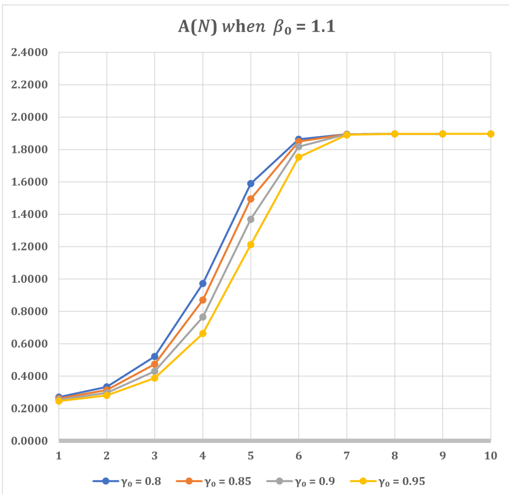

FIGURE 4

TABLE 3. Values of $\mathscr{C}(N)$ and $\mathscr{A}(N)$ when $\beta_0 = 1.15$ for different values of γ_0

N	$\gamma_0 = 0.8$		$\gamma_0 = 0.85$		$\gamma_0 = 0.9$		$\gamma_0 = 0.95$	
	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$
1	129.0876	0.2777	128.0477	0.2689	127.1180	0.2606	126.2809	0.2528
2	47.7441	0.3521	47.7441	0.3340	47.7441	0.3169	47.7441	0.3009
3	23.8742	0.5731	23.6767	0.5277	23.4980	0.4846	23.3355	0.4437
4	**15.9440**	**1.0505**	15.2311	0.9674	14.5950	0.8803	14.0250	0.7910
5	16.5874	1.5446	**14.8575**	**1.4960**	**13.3551**	**1.4287**	**12.0513**	**1.3391**
6	24.1210	1.6826	20.2707	1.6787	16.9816	1.6704	14.2669	1.6526
7	37.4923	1.6914	31.2315	1.6914	24.3671	1.6914	18.5507	1.6911
8	44.6850	1.6916	43.0383	1.6916	36.5716	1.6916	25.0973	1.6916
9	44.9997	1.6916	44.9875	1.6916	44.5376	1.6916	36.2641	1.6916
10	45.0000	1.6916	45.0000	1.6916	44.9994	1.6916	44.4524	1.6916

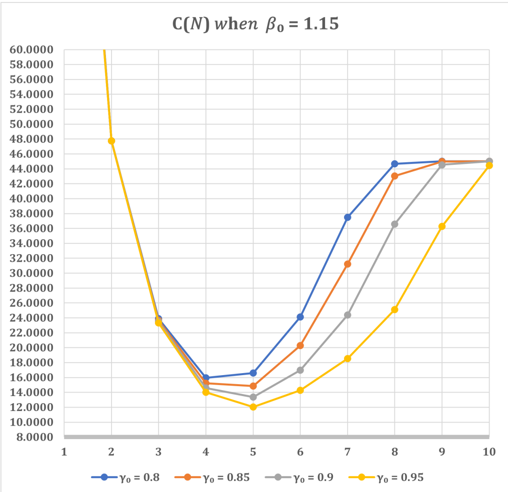

FIGURE 5

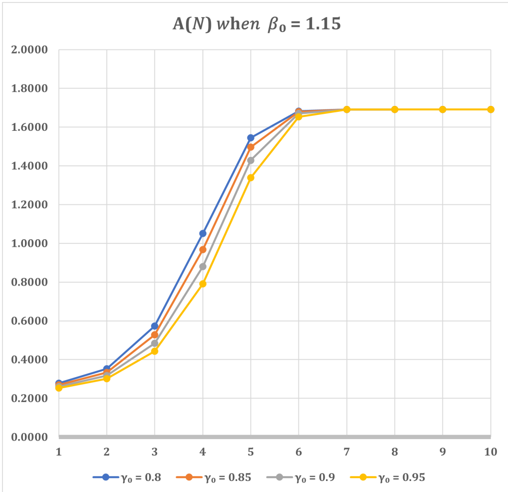

FIGURE 6

TABLE 4. Values of $\mathscr{C}(N)$ and $\mathscr{A}(N)$ when $\beta_0 = 1.5$ for different values of γ_0

N	$\gamma_0 = 0.8$		$\gamma_0 = 0.85$		$\gamma_0 = 0.9$		$\gamma_0 = 0.95$	
	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$	$\mathscr{C}(N)$	$\mathscr{A}(N)$
1	129.0876	0.3170	128.0477	0.3085	127.1180	0.3005	126.2809	0.2930
2	57.7308	0.4588	57.7308	0.4426	57.7308	0.4271	57.7308	0.4123
3	38.7114	0.7796	38.6448	0.7529	38.5845	0.7260	38.5295	0.6992
4	**34.6676**	**1.0939**	**34.3680**	**1.0802**	**34.0984**	**1.0644**	**33.8551**	**1.0464**
5	36.0518	1.1688	35.4002	1.1677	34.8223	1.1661	34.3115	1.1639
6	39.0776	1.1720	37.8117	1.1720	36.6795	1.1720	35.7076	1.1720
7	43.0512	1.1720	41.2912	1.1720	39.2023	1.1720	37.2859	1.1720
8	44.9215	1.1720	44.5068	1.1720	42.8008	1.1720	39.4339	1.1720
9	44.9999	1.1720	44.9969	1.1720	44.8847	1.1720	42.7165	1.1720
10	45.0000	1.1720	45.0000	1.1720	44.9999	1.1720	44.8634	1.1720

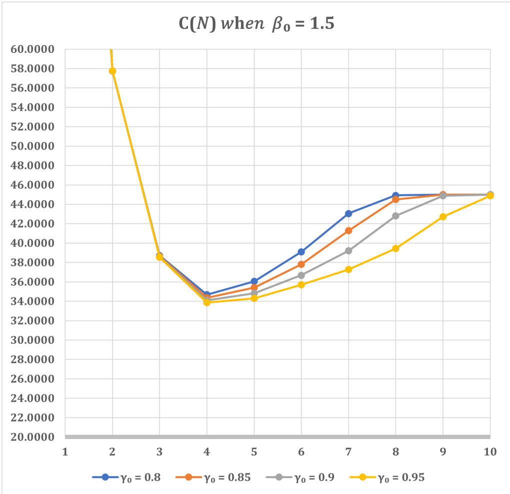

FIGURE 7

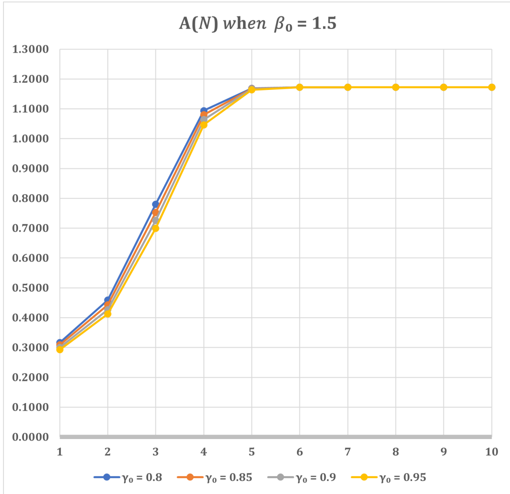

FIGURE 8

6. Conclusion

The deteriorating system maintenance model under a random environment utilizing partial product process is discussed in detail. The long-run mean cost per unit time under N-policy is derived explicitly and an optimal policy N^* is determined analytically and is explained with numerical example.

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