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A SEMI-PARAMETRIC ESTIMATION OF COPULA MODELS BASED ON MOMENTS METHOD UNDER RIGHT CENSORING

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Abstract. Based on the classical estimation method of moments, a new copula estimator was proposed for censored bivariate data. As theoretical results, general formulas were proved with analytical forms of the obtained estimators. Taking into account Lopez and Saint-Pierre's(2012)[19], Gribkova and Lopez's (2015)[10] results, the asymptotic normality of the empirical survival copula was established. The dependence structure between the bivariate survival times was modeled under the assumption that the underlying copula is Archimedean. Accounting for various censoring patterns (singly or doubly censored), a simulation study was performed enlighten the behavior of the procedure estimation method, shown the efficiency and robustness of the new estimator proposed.

Keywords: Archimedean copulas models, Bivariate censoring, Moment estimator, Survival copula, right censored data.

AMS Subject Classification: 62G05, 62G20.

1. Introduction

The modeling of bivariate or multivariate data in survival analysis has been discussed by several authors. Many approaches have been introduced for this modelisation, including Archimedean copula models, even their application (see [1], [3], [12], [13], [16], [24], [28]). Archimedean copula models arise naturally from bivariate frailty models ([18], [14]) in which the two failure times have given an unobserved frailty W and each follows proportional hazards model in W. However, in this aspect, an Archimedean copula is presented by:

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)),$$

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where, φ is a continuous, convex and decreasing function called the generator of C, defined on $I = [0,1] \to [0,\infty]$ and verifies $\varphi(1) = 0$. In the context of multivariate survival analysis, assume that T_1 and T_2 are two failure times conditionally independent, represented thereafter by the Archimedean copula C with the cumulative distribution function (CDF):

$$F(t_1,t_2) = P(T_1 \leq t_1, T_2 \leq t_2),$$

which can be identified according to a copula function as:

$$F(t_1,t_2) = C(F_1(t_1), F_2(t_2)),$$

where C is the associated copula function and F_1, F_2 are the margins. We noted the survival functions of T_1 and T_2 by $S_1(t_1) = P(T_1 > t_1)$ and $S_2(t_2) = P(T_2 > t_2)$ respectively and the joint survival function by:

$$S(t_1,t_2) = P(T_1 > t_1, T_2 > t_2).$$

Although this latter, can also be generated by an Archimedean copula (see [7], [6]) in the manner of the following:

$$S(t_1,t_2) = \varphi^{-1}(\varphi(S_1(t_1)) + \varphi(S_2(t_2))),$$

Besides, the function $\tilde{C}$ which couples S_1 and S_2 via $S(t_1,t_2) = \tilde{C}(S_1(t_1), S_2(t_2))$, called the survival copula of (T_1, T_2). Then, if we define $\tilde{C}$ from $I^2 \to I$ we obtain:

$$\tilde{C}(u,v) = u + v - 1 + C(1-u, 1-v), \tag{1}$$

where $(u,v) \in I^2$, see Nelsen (2006)[17]. Hence, it was demonstrated by Genest and Rivest (1993)[7] that if (T_1, T_2) follows an Archimedean copula with the marginal survival functions $S_1(t_1)$ and $S_2(t_2)$, then

$$U = \frac{\varphi\left(S_1\left(T_1\right)\right)}{\varphi\left(S_1\left(T_1\right)\right) + \varphi\left(S_2\left(T_2\right)\right)},$$

and

$$V = \tilde{C}(S_1\left(T_1\right), S_2\left(T_2\right)) = \varphi^{-1}\left(\varphi\left(S_1\left(T_1\right)\right) + \varphi\left(S_2\left(T_2\right)\right)\right),$$

are random variables distributed independently, where U distributed uniformity on I and V follows a so-called Kendall distribution with the density function:

$$k_C\left(t\right) = \frac{\varphi\left(t\right)\varphi''\left(t\right)}{\left(\varphi'\left(t\right)\right)^2},$$

defined on $(0,1]$, as a function of t depends on the unknown parameter θ. Assume that the two failure times T_1 and T_2 can be modeled by an Archimedean copula model and it is subject to dependence or independence right-censoring with the censoring vector (C_1, C_2), we also assume that the vector (C_1, C_2) follows an arbitrary bivariate continuous distribution. Therefore, if we denote $\delta_i = 1_{\{T_i \leq C_i\}_{i=1,2}}$ which represents the indicator function of censored data, that specifies if our variable of interest is observed or not. Then, we only observe the variable $Z_i = \min(T_i, C_i)$ if $T_i \leq C_i$ when $\delta_i = 1$, otherwise, if $T_i \geq C_i$ the variable in this case is censored and the indicator δ_i equal to zero $\delta_i = 0$. In this paper, we are interested by type one of censoring, where two models are presented, the first is for doubly censored variables (T_1 and T_2 both are right-censored) and the second for a singly censored when only T_1 (or T_2) is right-censored.

The issues of estimating copula parameters in literature are usually solved by maximum likelihood methods ([8], [4]). For example, if we consider the IFM method (Joe, 1997, 2005) Joe presented a two-stage procedure to estimate a copula, by maximizing the copula likelihood function. Even so, this maximization generally becomes very difficult to achieve when the dimension is large and the parameter numbers are also higher. For this reason, our main aim in this paper is to propose an alternative estimation method of a survival copula $\tilde{C}$, based on the moments method due to its simple mathematical form, given (T_1, T_2) as singly or doubly right-censored. General formulas were established when the considered variable $\tilde{C}$ defined under certain conditions.

The remainder of the paper is structured as follows: in section 2, our main theorems and corollary are presented where general forms of the survival copula estimator are established. As well as, the asymptotic normality of this estimator to be verified, by considering two types of right-censored models. However, in section 3 a semi-parametric estimation based on the classical moments method illustrated for a conditional distribution on $\tilde{C}$, followed by an application presented for the Gumbel model. A simulation study evaluates the performance of our estimator presented in Section 4. Our paper ends with some discussions in Section 5.

2. Main results

Interesting results to be proven, related by a semi-parametric estimation based on k^{th}-moments of a variable $V = \tilde{C}(u, v)$ conditionally distributed given T_1 and T_2 as singly or doubly censored. Moreover, the following theorems and corollary illustrate our main results.

Theorem 2.1. *Let (T_1, T_2) be a random pair whose distribution can be modeled by an Archimedean copula. Assuming that (T_1, T_2) is subject to dependent or independent right censoring by a censoring vector (C_1, C_2) that follows an arbitrary bivariate continuous distribution, then we have:*

(1) *The distribution function of $(V|\, T_1 > C_1 = c_1, T_2 > C_2 = c_2)$ is*

$$F_1(v, c_1, c_2) = \frac{1}{\tilde{C}(c_1, c_2)} \left\{ v - \frac{\varphi(v) - \varphi\left(\tilde{C}(c_1, c_2)\right)}{\varphi'(v)} \right\}, 0 \le v \le \tilde{C}(c_1, c_2).$$

(2) *The distribution function of $(V|\, T_1 > C_1 = c_1, T_2 = t_2)$ is*

$$F_2(v, c_1, t_2) = \frac{\varphi'\left(\tilde{C}(c_1, t_2)\right)}{\varphi'(v)}, 0 \le v \le \tilde{C}(c_1, t_2).$$

(3) *The distribution function of $(V|\, T_1 = t_1, T_2 > C_2 = c_2)$ is*

$$F_3(v, t_1, c_2) = \frac{\varphi'\left(\tilde{C}(t_1, c_2)\right)}{\varphi'(v)}, 0 \le v \le \tilde{C}(t_1, c_2).$$

Proof. See, Wang and Oakes (2008)[28]. Based on Theorem 2.1 we can show □

Corollary 2.1. *Under the same conditions given in Theorem 2.1, we have:*

(1) *The* k^{th} *moments of* $(V|\, T_1 > c_1, T_2 > c_2)$ *for* $k \geq 1$ *is*

$$\begin{aligned}\mathbb{E}(V^k \Big| T_1 &> c_1, T_2 > c_2) = \frac{\left(\tilde{C}\left(c_1, c_2\right)\right)^k}{k+1} \\ &- k\left(\tilde{C}\left(c_1, c_2\right)\right)^{k-1} \varphi\left(\tilde{C}\left(c_1, c_2\right)\right) \int_0^1 \frac{v^{k-1}}{\varphi'\left(v\tilde{C}\left(c_1, c_2\right)\right)} dv \\ &+ k\left(\tilde{C}\left(c_1, c_2\right)\right)^{k-1} \int_0^1 \frac{v^{k-1}\varphi\left(v\tilde{C}\left(c_1, c_2\right)\right)}{\varphi'\left(v\tilde{C}\left(c_1, c_2\right)\right)} dv.\end{aligned}$$

(2) *The* k^{th} *moments of* $(V|\, T_1 > c_1, T_2 = t_2)$ *for* $k \geq 1$ *is*

$$\begin{aligned}\mathbb{E}(V^k \Big| T_1 &> c_1, T_2 = t_2) = \left(\tilde{C}\left(c_1, t_2\right)\right)^k \\ &- k\left(\tilde{C}\left(c_1, t_2\right)\right)^{k} \varphi'\left(\tilde{C}\left(c_1, t_2\right)\right) \int_0^1 \frac{v^{k-1}}{\varphi'\left(v\tilde{C}\left(c_1, t_2\right)\right)} dv.\end{aligned}$$

(3) *The* k^{th} *moments of* $(V|\, T_1 = t_1, T_2 > c_2)$ *for* $k \geq 1$ *is*

$$\begin{aligned}\mathbb{E}(V^k \Big| T_1 &= t_1, T_2 > c_2) = \left(\tilde{C}\left(t_1, c_2\right)\right)^k \\ &- k\left(\tilde{C}\left(t_1, c_2\right)\right)^{k} \varphi'\left(\tilde{C}\left(t_1, c_2\right)\right) \int_0^1 \frac{v^{k-1}}{\varphi'\left(v\tilde{C}\left(t_1, c_2\right)\right)} dv.\end{aligned}$$

Proof. In order to prove the result of Corollary2.1 we need to use the results given in Theorem 2.1 and we start by equation1**,** using the conditional distribution of $(V|\, T_1 > c_1, T_2 > c_2)$. Then, for $k > 1$ the k^{th} moments is given by:

$$\mathbb{E}(V^k \Big| T_1 > c_1, T_2 > c_2) = \int_0^{\tilde{C}(c_1,c_2)} v^k dF_1\left(v, c_1, c_2\right)$$

$$\begin{aligned}\mathbb{E}(V^k \Big| T_1 &> c_1, T_2 > c_2) = \\ &= \frac{1}{\tilde{C}\left(c_1, c_2\right)} \int_0^{\tilde{C}(c_1,c_2)} v^k \left\{1 - \frac{\left(\varphi'(v)\right)^2 - \varphi''(v)\left(\varphi(v) - \varphi\left(\tilde{C}\left(c_1, c_2\right)\right)\right)}{\left(\varphi'(v)\right)^2}\right\} dv \\ &= \frac{1}{\tilde{C}\left(c_1, c_2\right)} \int_0^{\tilde{C}(c_1,c_2)} v^k dv \\ &- \frac{1}{\tilde{C}\left(c_1, c_2\right)} \int_0^{\tilde{C}(c_1,c_2)} v^k \frac{\left(\varphi'(v)\right)^2 - \varphi''(v)\left(\varphi(v) - \varphi\left(\tilde{C}\left(c_1, c_2\right)\right)\right)}{\left(\varphi'(v)\right)^2} dv \\ &= I_1 - I_2,\end{aligned}$$

by the way, I_1 have to simplify as follows: $I_1 = \frac{1}{\tilde{C}(c_1,c_2)} \int_0^{\tilde{C}(c_1,c_2)} v^k dv = \frac{\tilde{C}(c_1,c_2)^k}{k+1}$. On other hand, to simplify I_2 we pass directly to integration by parts, and we have:

$$\begin{aligned} I_2 &= \frac{1}{\tilde{C}(c_1,c_2)} \int_0^{\tilde{C}(c_1,c_2)} v^k \frac{(\varphi'(v))^2 - \varphi''(v)\left(\varphi(v) - \varphi\left(\tilde{C}(c_1,c_2)\right)\right)}{(\varphi'(v))^2} dv \\ &= \frac{1}{\tilde{C}(c_1,c_2)} \left(\left[v^k \frac{\varphi(v) - \varphi\left(\tilde{C}(c_1,c_2)\right)}{\varphi'(v)} \right]_0^{\tilde{C}(c_1,c_2)} \right. \\ &\quad \left. -k \int_0^{\tilde{C}(c_1,c_2)} v^{k-1} \frac{\varphi(v) - \varphi\left(\tilde{C}(c_1,c_2)\right)}{\varphi'(v)} dv \right) \\ &= -\frac{k}{\tilde{C}(c_1,c_2)} \int_0^{\tilde{C}(c_1,c_2)} v^{k-1} \frac{\varphi(v) - \varphi\left(\tilde{C}(c_1,c_2)\right)}{\varphi'(v)} dv. \end{aligned}$$

it follows after changing variables that:

$$\begin{aligned} I_2 &= -k\left(\tilde{C}(c_1,c_2)\right)^{k-1} \int_0^1 v^{k-1} \frac{\varphi(v\tilde{C}(c_1,c_2)) - \varphi\left(\tilde{C}(c_1,c_2)\right)}{\varphi'(v\tilde{C}(c_1,c_2))} dv \\ &= -k\left(\tilde{C}(c_1,c_2)\right)^{k-1} \int_0^1 v^{k-1} \frac{\varphi(v\tilde{C}(c_1,c_2)}{\varphi'(v\tilde{C}(c_1,c_2))} dv \\ &\quad +k\left(\tilde{C}(c_1,c_2)\right)^{k-1} \varphi\left(\tilde{C}(c_1,c_2)\right) \int_0^1 \frac{v^{k-1}}{\varphi'(v\tilde{C}(c_1,c_2))} dv. \end{aligned}$$

The same proof used previously can applies for equations 2 and 3 in the Corollary2.1. □

2.1. **Survival empirical copula for right-censored.** Initially, let us clarify that there are two models we are interested in, the first is for doubly censored variables (T_1 and T_2 both) and the second is for a singly censored, only T_1 (or T_2) is censored. Given the accessible observation $(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})_{1\le i\le n}$: the independent copies of a non-negative random variable of the vector $(Z_1, Z_2, \delta_1, \delta_2)$ and the survival copula $\tilde{C}$. Assuming that the survival copula $\tilde{C}$ is known and the following assumptions:

- $[H_1]$ The first and the second partial derivatives of $\tilde{C}$ are limited on I^2, where $\tilde{C}(u,v)$ is different to zero for $u \neq 0$ and $v \neq 0$.
- $[H_2]$ $\exists\ (\alpha,\beta) \in I^2$, where $\tilde{C}(u,v) \geq u^\alpha v^\beta$.
- $[H_3]$ The integral $\int \frac{dF(t_1,t_2)}{\tilde{C}(S_1(t_1),S_2(t_2))}$, is strictly less than infinity. For $\theta > 0$, where $\mathcal{F}_i(t) = \int_0^t \frac{dF_i(v)}{S_i(u)^2 S_{T_i}(u)}$, $i \in \{1,2\}$ we have

$$\int \{ \frac{S_1^{1-\alpha}(t_1)\,\mathcal{F}_1^{\frac{1}{2+\theta}}(t_1)}{S_2^{\beta}(t_2)} + \frac{S_2^{1-\beta}(t_2)\,\mathcal{F}_2^{\frac{1}{2+\theta}}(t_2)}{S_1^{\alpha}(t_1)} \} dF(t_1,t_2) < \infty.$$

- $[H_4]$ Suggesting that $\int \frac{dF(t_1,t_2)}{S_1(t_1^-)}$, is strictly less than infinity and for $\theta > 0$, we have

$$\int \left\{ \left(\int_0^{t_1} \frac{dF_1(v)}{S_1(v^-)^2 S_{T_1}(v)} \right)^{\frac{1}{2+\theta}} \right\} dF(t_1,t_2) < \infty.$$

Lopez and Saint-Pierre(2012)[19] have studied the first model, noting that F can be consistently estimated by an F_n estimator in the following form:

$$\tilde{F}_n(t_1,t_2) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{T_{1i} \le t_1, T_{2i} \le t_2\}},$$

that could not be used to estimate $F(t_1,t_2)$ since T_1 and T_2 are unobserved. Therefore, according to the proposition of Lopez and Saint-Pierre(2012)[19], the F estimate can be given in such form:

$$F_n(t_1,t_2) = \frac{1}{n} \sum_{i=1}^{n} \frac{\delta_{1i}\delta_{2i}}{\tilde{C}\left(\hat{S}_1\left(Z_{1i}\right), \hat{S}_2\left(Z_{2i}\right)\right)} 1_{\{Z_{1i} \le t_1, Z_{2i} \le t_2\}}, \tag{2}$$

where $\tilde{C}$ is the survival copula given by (1) and

$$\hat{S}_1(t) = \prod_{k/Z'_{1k} < t} \left(1 - \frac{\sum_{i=1}^{n} 1_{\{Z_{1i} = Z'_{1k}, \delta_{1i} = 0\}}}{\sum_{i=1}^{n} 1_{\{Z_{1i} \ge Z'_{1k}\}}}\right)$$

is the Kaplan-Meier estimate of S_1, for $((Z'_{1:k})_{1 \le k \le m}, m \le n)$, and $\hat{S}_2$, is the Kaplan-Meier estimate of S_2 defined by the same way. Noted Γ_{T_1} and Γ_{T_2} the support of T_1 and T_2 respectively and $l^\infty(W)$ all bounded real-valued functions space, identified on non-empty set W.

Assuming that the assumptions $[H_1] - [H_3]$ hold, Lopez's and Saint Pierre's(2012) have concluded that the processes $n^{\frac{1}{2}}(F_n - F)$ converges weakly in $l^\infty(\Gamma_{T_1} * \Gamma_{T_2})$ to a centred Gaussian process (theorem 3.4[19]). Otherwise, indicating that we are in the second case, this model was studied by Stute (1993)[24], who suggested G_n the empirical distribution function given by:

$$G_n(t_1,t_2) = \frac{1}{n} \sum_{i=1}^{n} \frac{\delta_{1i}}{\hat{S}_1\left(Z_{1i}^-\right)} 1_{\{Z_{1i} \le t_1, Z_{2i} \le t_2\}} \tag{3}$$

Which is a particular model situation from the first case, where $\tilde{C}(u,v) = uv$ (see [16]). Following the theorem 3.4, of Lopez and Saint Pierre (2012)[19], the weak convergence of G_n has proved under the assumptions $[H_4]$. By the way, in the event of complete data, the copula C can be estimated by:

$$\hat{C}(u,v) = F_n(F_{1n}^{-1}(u), F_{2n}^{-1}(v)),$$

where $(u,v) \in I^2$, $F_{1n}(t_1) = \lim_{t_2 \to \infty} F_n(t_1,t_2)$ and $F_{2n}(t_2) = \lim_{t_1 \to \infty} F_n(t_1,t_2)$, Gribkova and Lopez (2015)[10] proposed the empirical copula of C in the case of incomplete data given by

$$C_n(u,v) = \frac{1}{n} \sum_{i=1}^{n} \frac{\delta_{1i}\delta_{2i}}{\tilde{C}\left(\hat{S}_1\left(Z_{1i}\right), \hat{S}_2\left(Z_{2i}\right)\right),} 1_{\{F_{1n}(Z_{1i}) \le u, F_{2n}(Z_{2i}) \le v\}}, \tag{4}$$

when the two variables are both right-censored (first model). By analogy, using (1) and (4), the empirical survival copula via:

$$\tilde{C}_n(u,v) \quad = \quad u+v-1+\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_{1i}\delta_{2i}}{\tilde{C}\left(\hat{S}_1\left(Z_{1i}\right),\hat{S}_2\left(Z_{2i}\right)\right)}1_{\{1-F_{1n}(Z_{1i})\geq u,1-F_{2n}(Z_{2i})\geq v\}} \quad (5)$$

As a result, for a singly censored (second case), it is possible to define empirical survival copulas $\tilde{C}_n$ in the same manner as seen above. The reader is invited to take a look on the references mentioned below ([16] and [10]). Observe that for both models

$$\sup_{(u,v)\in I^2}|C_n(u,v)-\hat{C}(u,v)|=O_p\left(\frac{1}{n}\right),$$

which means that the process $n^{\frac{1}{2}}(C_n-C)$ converges weakly in $l^{\infty}(I^2)$ to the limiting approach L (centered Gaussian process), that either have been proven by Gribkova and Lopez (2015)[10] in theorem 2. Hence, this weak convergence allows us to prove the asymptotic normality of a statistics given by the form $\int_{I^2}g(u,v)dC_n(u,v)$, noted g as a function that has a real value defined on I^2. Fermanian, Radulovic, and Wegkamp (2004) have proven this asymptotic normality in the case of complete data. By the way, thanks to theorem1 of M. Boukeloua (2020), who proved that under some assumption when $n\to\infty$ the quantity $n^{\frac{1}{2}}\left\{\int_{I^2}g(u,v)d\left(C_n(u,v)-C(u,v)\right)\right\}$ converges in distribution to a Gaussian random variable $G=\int_{I^2}g(u,v)d\left(L(u,v)\right)$, where $g\in R_2(I^2)$ the set of all real-valued functions defined on I^2. Based on these results and if we assume the assumptions $[H_1]-[H_4]$ hold we can show the next theorem.

Theorem 2.2. *Assuming the function $g\in R_2(I^2)$, $\tilde{C}$ and $\tilde{C}_n$ the survival copula and its empirical version respectively, then when $n\to\infty$ we have*

$$n^{\frac{1}{2}}\left\{\int_{I^2}g(u,v)d\left(\tilde{C}_n(u,v)-\tilde{C}(u,v)\right)\right\}\xrightarrow{D}\int_{I^2}g(u,v)d\left(L(u,v)\right),$$

where the limiting is a Gaussian random variable and $(u,v)\in I^2$.

This theorem proved the asymptotic normality of the empirical survival copula, which remains valid for both models considered.

Proof. If we consider the survival copula $\tilde{C}$ and its empirical version $\tilde{C}_n$, we have

$$\begin{aligned}\tilde{C}_n(u,v)-\tilde{C}(u,v) &= u+v-1+C_n(1-u,1-v)-\tilde{C}(u,v)\\ &= C_n(1-u,1-v)-C(1-u,1-v),\end{aligned}$$

hence by a change of variables $w_1=1-u$ and $w_2=1-v$, we get

$$\tilde{C}_n(u,v)-\tilde{C}(u,v)=C_n(w_1,w_2)-C(w_1,w_2),$$

where (w_1,w_2) remain belongs to the interval I^2. So, we can concluded that $n^{\frac{1}{2}}\left(\tilde{C}_n-\tilde{C}\right)$ also converges weakly in $l^{\infty}(I^2)$ to the limiting approach L. Let the set among all functions $R_2(I^2)$ defined on $[0,1]^2$ and we assume the application ζ represented on $R_2(I^2)$ and given by

$$\zeta(h)=\int_{I^2}g(w_1,w_2)dh(w_1,w_2),$$

which is Hadamard differentiable on $R_2(I^2)$, see van der Vaart and Wellner (1996). Beacause $n^{\frac{1}{2}}(C_n - C)$ converges weakly to the limiting approach L, then, by using delta method we get

$$\begin{aligned} n^{\frac{1}{2}}\left\{\int_{I^2} gdC_n(w_1,w_2) - \int_{I^2} gdC(w_1,w_2)\right\} &= n^{\frac{1}{2}}\left\{\int_{I^2} gd\tilde{C}_n(u,v) - \int_{I^2} gd\tilde{C}(u,v)\right\} \\ &= n^{\frac{1}{2}}\left\{\zeta(\tilde{C}_n(u,v)) - \zeta(\tilde{C}(u,v))\right\} = \bar{\zeta} \\ &\Leftrightarrow \ \bar{\zeta} \ \underset{\rightarrow}{D} \ \zeta_c'(L) \end{aligned}$$

where $\zeta_c'(L) = \int_{I^2} g(u,v)d\left(L(u,v)\right)$ is the derivative of ζ in the point c. See M. Boukeloua Theorem1's proof (2020). $\square$

3. Moments estimator for right-censoring

Either the following figure, T_1 and T_2 represent the survival time point and (C_1, C_2) the censoring time point. The display contains four data kinds points, including observed points (T_1, T_2), two types of singly censored points (T_1, C_2), (C_1, T_2) and doubly censored points (C_1, C_2).

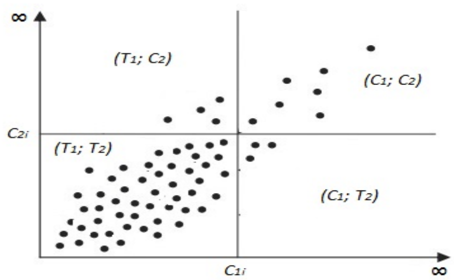

Figure 1. Censored data samples

From now, we are only interested by the first model presented before. Let (T_1, T_2) a random variables whose distribution can be modeled by an Archimedean copula and is subject to dependent or independent right censoring, $V = \tilde{C}\left(S_1\left(Z_{1i}\right), S_2\left(Z_{2i}\right)\right)$ is a conditionally distributed variable follows a so-called Kendall distribution K_C with the density function: $k_C(t) = \frac{\varphi(t)\varphi''(t)}{(\varphi'(t))^2}$, defined on $(0, 1]$.
We define the k^{th}-moments of V for $k \geq 1$ by: $M_k(V|H) = E(V^k|H)$, where $H = h_{(c_1,c_2)}$ indicate the first case of censoring (T_1 and T_2 are both right-censoring). Then, relying on the results obtained above in Corollary 2.1, we have:

$$\begin{aligned} M_k(V|H) &= \frac{\left(\tilde{C}\left(c_1,c_2\right)\right)^k}{k+1} - k\left(\tilde{C}\left(c_1,c_2\right)\right)^{k-1}\varphi\left(\tilde{C}\left(c_1,c_2\right)\right)\int_0^1 \frac{v^{k-1}}{\varphi'\left(v\tilde{C}\left(c_1,c_2\right)\right)}dv \\ &\quad + k\left(\tilde{C}\left(c_1,c_2\right)\right)^{k-1}\int_0^1 \frac{v^{k-1}\varphi\left(v\tilde{C}\left(c_1,c_2\right)\right)}{\varphi'\left(v\tilde{C}\left(c_1,c_2\right)\right)}dv. \end{aligned} \tag{6}$$

Suppose now V belongs to a parametric family $V_\theta = \tilde{C}_\theta(u,v)$, it follows that $\varphi = \varphi_\theta$, $\tilde{C} = \tilde{C}_\theta$ and $K_C = K_\theta$, where $u = S_1(t_1) = \bar{F}_1(t_1)$ and $v = S_2(t_2) = \bar{F}_2(t_2)$, mentioned that F_1 and F_2 are completely known. Noted that $M_k(V|H) = M_k(\theta|H)$, then, we can distinguish the following form of the k^{th}-moments:

$$\begin{aligned} M_k(\theta|h_{(c_1,c_2)}) &= \frac{\left(\tilde{C}_\theta\left(c_1,c_2\right)\right)^k}{k+1} \\ &\quad -k\left(\tilde{C}_\theta\left(c_1,c_2\right)\right)^{k-1}\varphi_\theta\left(\tilde{C}_\theta\left(c_1,c_2\right)\right)\int_0^1 \frac{v_\theta^{k-1}}{\varphi'_\theta\left(v_\theta\tilde{C}_\theta\left(c_1,c_2\right)\right)}dv_\theta \\ &\quad +k\left(\tilde{C}_\theta\left(c_1,c_2\right)\right)^{k-1}\int_0^1 \frac{v_\theta^{k-1}\varphi_\theta\left(v_\theta\tilde{C}_\theta\left(c_1,c_2\right)\right)}{\varphi'_\theta\left(v_\theta\tilde{C}_\theta\left(c_1,c_2\right)\right)}dv_\theta, \end{aligned}$$

for unknown $\theta \in \mathbb{R}^d$. Given the empirical version of moment estimator under doubly censored presented by:

$$\hat{M}_k = \hat{M}_k(\hat{V}|h_{(c_1,c_2)}) = \frac{1}{n}\sum_{i=1}^{n}\left\{\tilde{C}_n\left(\hat{S}_i(t_i)|H\right)\right\}^k,$$

for $k \geq 1$ where $\hat{V} = \tilde{C}_n$ is the survival empirical copula given by formula (5). By analogy, as the natural estimators of moments copula it is necessary to solve the equation system given below:

$$\begin{cases} M_1(\theta|h_{(c_1,c_2)}) = \hat{M}_1 \\ M_2(\theta|h_{(c_1,c_2)}) = \hat{M}_2 \\ \quad\vdots \\ M_d(\theta|h_{(c_1,c_2)}) = \hat{M}_d. \end{cases}$$

To obtained the unique solution $\hat{\theta}^{CCM} = (\hat{\theta}_1, ..., \hat{\theta}_d)$ called the censored copula moment (CCM) estimator of θ.

3.1. **Application: illustrative example.** In particular, in the bivariate case, the Gumbel model of one-parameter is given by:

$$C_\alpha\left(u,v\right) = \exp\left(-\left((-\ln u)^\alpha + (-\ln v)^\alpha\right)^{\frac{1}{\alpha}}\right),$$

with the generator: $\varphi_\alpha(t) = (-\ln t)^\alpha$, $\alpha \in [1, +\infty[$. Consequently, by considering the case of two parameters, the preceding model becomes:

$$C_{\alpha,\beta}\left(u,v\right) = \left(\left(\left(u^{-\alpha}-1\right)^\beta + \left(v^{-\alpha}-1\right)^\beta\right)^{\frac{1}{\beta}} + 1\right)^{-\frac{1}{\alpha}}, \tag{7}$$

with the generator: $\varphi_{\alpha,\beta}(t) = (t^{-\alpha} - 1)^\beta$, where $\alpha > 0$ and $\beta \geq 1$ (see [2]). Obviously, by the use of (1), we obtain the survival copula of the Gumbel family given by:

$$\tilde{C}_{\alpha,\beta}\left(u,v\right) = u + v - 1 + \left(\left(\left((1-u)^{-\alpha}-1\right)^\beta + \left((1-v)^{-\alpha}-1\right)^\beta\right)^{1/\beta} + 1\right)^{-1/\alpha} \tag{8}$$

Hence, as an application of our results proved previously we can reach the following bivariate censoring models using equation 1 in Corollary2.1.

For $k \geq 1$, $\alpha > 0$ and $1 \leq \beta \leq 2$, the k^{th} moments of the Gumbel's survival copula, is given by:

$$\begin{aligned} M_k((\alpha,\beta)\,|H) &= E(V^k \Big| h_{(c_1,c_2)}) \\ &= \frac{m^k}{k+1} + \frac{k\,(m^{-\alpha}-1)^{\beta}}{\alpha^2\beta m}\beta_{m^\alpha}\left(\beta+\frac{k+1}{\alpha},2-\beta\right) \\ &\quad + \frac{km^{k-1}}{\alpha\beta}\left(\frac{m^{\alpha+1}}{k+\alpha+1}-\frac{m}{k+1}\right), \end{aligned} \tag{9}$$

in which $\beta_{m^\alpha}(x,y)$ is the Beta function and $m=\tilde{C}\,(c_1,c_2)$ is the ordinary copula. If we simplify more the previous formula we will obtain the following writing:

$$\begin{aligned} M_k((\alpha,\beta)\,|h_{(c_1,c_2)}) &= \frac{m^k}{k+1} + \frac{k}{\alpha\beta} \\ \times\Bigg(\frac{m^{k+\alpha}}{k+\alpha+1} &- \frac{m^k}{k+1} - \frac{(\beta-1)\,(m^{-\alpha}-1)^{\beta}}{\alpha m^{\alpha+1}}\frac{\Gamma\,(1-\beta)\,\Gamma\left(\frac{1}{\alpha}\,(k+\alpha\beta+1)\right)}{\Gamma\left(\frac{1}{\alpha}\,(k+2\alpha+1)\right)}\Bigg), \end{aligned} \tag{10}$$

where $\Gamma\,(x)$ is the Gamma function. In particular, the two first moments are given by:

$$\begin{cases} M_1((\alpha,\beta)\,|h_{(c_1,c_2)}) = \frac{1}{2}m + \frac{(m^{-\alpha}-1)^{\beta}}{\alpha^2\beta m}\beta_{m^\alpha}\left(\beta+\frac{2}{\alpha},2-\beta\right) + \frac{1}{\alpha\beta}\left(\frac{m^{\alpha+1}}{\alpha+2}-\frac{m}{2}\right) \\ M_2((\alpha,\beta)\,|h_{(c_1,c_2)}) = \frac{1}{3}m^2 + \frac{2(m^{-\alpha}-1)^{\beta}}{\alpha^2\beta m}\beta_{m^\alpha}\left(\beta+\frac{3}{\alpha},2-\beta\right) + \frac{1}{\alpha\beta}\left(\frac{m^{\alpha+1}}{\alpha+3}-\frac{m}{3}\right) \end{cases}$$

Which can further simplify as well:

$$\begin{cases} M_1((\alpha,\beta)\,|h_{(c_1,c_2)}) = \frac{1}{2}m + \frac{1}{\alpha\beta}\left\{\frac{m^{\alpha+1}}{\alpha+2} - \frac{1}{2}m - \frac{(\beta-1)(m^{-\alpha}-1)^{\beta}}{\alpha m^{\alpha+1}}\frac{\Gamma(1-\beta)\Gamma\left(\frac{1}{\alpha}(\alpha\beta+2)\right)}{\Gamma\left(\frac{2}{\alpha}(\alpha+1)\right)}\right\} \\ M_2((\alpha,\beta)\,|h_{(c_1,c_2)}) = \frac{1}{3}m^2 + \frac{2}{\alpha\beta}\left\{\frac{m^{\alpha+2}}{\alpha+3} - \frac{1}{3}m^2 - \frac{(\beta-1)(m^{-\alpha}-1)^{\beta}}{\alpha m^{\alpha+1}}\frac{\Gamma(1-\beta)\Gamma\left(\frac{1}{\alpha}(\alpha\beta+3)\right)}{\Gamma\left(\frac{1}{\alpha}(2\alpha+3)\right)}\right\} \end{cases}$$

However, the CCM estimator of $\theta=(\alpha,\beta)$ is the unique solution of the system:

$$\begin{cases} M_1(\theta|h_{(c_1,c_2)}) = \hat{M}_1 \\ M_2(\theta|h_{(c_1,c_2)}) = \hat{M}_2 \end{cases}$$

4. Simulation study

To illustrate the performances of the proposed estimator, a simulation study is carried out based on the Monte Carlo method for right-censored sampling. First, we generate a bivariate survival distribution of the Gumbel copula model where the margins are assumed to be Pareto(λ), $F(t)=1-t^{-\lambda},\ t\geq 0$. The distribution of survival times T_1 , T_2, and the censoring times C_1, C_2 are all assumed to be Pareto of parameters $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ respectively. If we suppose that the corresponding percentage of observed data is equal to $p_1=\frac{\lambda_2}{\lambda_1+\lambda_2}$ for the first sample, then we can choose the values 0.3 for λ_1 and $0.95, 0.90, 0.85, 0.80$ for p_1, next we solve the equation $p_1=\frac{\lambda_2}{\lambda_1+\lambda_2}$ to get the pertaining λ_2-values. In this path, we fix λ_3 and $p_2=\frac{\lambda_4}{\lambda_3+\lambda_4}$ by the same previous values to find λ_4 by the same way. Since the quality of the estimate is assessed by evaluating the bias (relative Bias) and the root mean square error (RMSE), then for the two samples both we generate 1000 replicas for each common size n varied for $n=30, 50, 100, 500, 1000{,}2000$, to pick our final performance as empirical evidence of the results gained across all replicates. Besides, for a wide set of parameters of the true survival copula $\widetilde{C}_{\alpha,\beta}$ the simulation procedure based on Section 3 is repeated for each sample. The selection of true survival copula parameter values (α,β)

must be significant, i.e. each couple of parameters consists a value of one of the dependency measurements. So, if we consider Kendall's τ as an association index then, it can be expressed as a function of the dependency parameter in Archimedean copula models. In this case, we should select the parameter values of $\tilde{C}$ that correspond to specified values of τ by using the transformed of the underlying survival copula. Since the link between Kendall's τ and $\tilde{C}$ is usually formulated by $\tau_{\alpha,\beta} = 4E(V_{\alpha,\beta}) - 1$, where $V_{\alpha,\beta} = \tilde{C}_{\alpha,\beta}(u, v)$, then to generate data, we select values for survival copula parameters that corresponding to Kendall's tau values 0.05 (low association), 0.5 (mean association) and 0.7 (high positive association), summarized in Tables $1 - 3$.

For the Gumbel survival copula of two parameters, the performance of the estimator proposed is presented in Tables $1-3$. The results obtained for different values of Kendall's τ are quite good in the three cases of dependence considered (0.05, 0.5, 0.7) and by considering different censoring percentages. In each table, τ_1 and τ_2 are represent respectively the Kendall's tau value before and after censoring. From the three tables, we deduce that the estimator proposed have a good performance and works quite well if we compare it by other methods used before on the copulas estimation. By the way, the performance of survival copula estimate based on the moments method is justified, through the adoption of relative bias (Re.Bais) and RMSE discourse, when we can see all their values are sufficiently decreased for each case of small and even large samples (are almost close to zero). Even so, the value of Kendall's tau after censoring (τ_2) remains close to its original theoretical value given by τ_1 , which means that the variables remain dependent despite the censorship.

5. Discussion

In this paper, we elaborate a semi-parametric estimation method of a survival copula based on Archimedean models, but in specific conditions on the data. Indeed, under different censoring (singly or doubly), the results of our estimator were presented with an analytical form which overcame the problem that occurs usually by other methods. As an application of the considered method we have chosen the Gumbel model, given T_1 and T_2 as doubly right-censored variables. In the simulation part, three cases of dependence are considered, where the results can validate the use of the method proposed. Consequently, this method is preferable if we compare it with the maximum likelihood method, because of its easy mathematical form. Our main result for these studies is based on the copula approaches and the survival analysis, in which the correlation between two survival time variables was detected. Therefore, our research results open a vast area of application, notably in real life, when there are two related events defined under specific situations. This will be discussed in an interesting new paper that we are currently working on. Based on the outcomes of Gripkova and Lopez's (2015)[10], Lopez and Saint-Pierre's (2012)[19] research, our results can be applied for left and right censoring. This is one of our current research topics and the idea has been developed in another paper that is also under preparation.

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$\tau = 0.05$, $\alpha = 0.1 \to \beta = 1.00$

N	$n=30$		$n=50$		$n=100$		$n=500$		$n=1000$		$n=2000$	
$(\hat{\alpha}, \hat{\beta})$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
1% of censoring												
Re.Bias	-0.0563	0.2852	-0.0537	0.2119	-0.0539	0.2515	-0.0536	0.2403	-0.051	0.2461	-0.0546	0.1972
RMSE	0.0649	0.0116	0.0624	0.0117	0.0629	0.0119	0.0620	0.0117	0.0606	0.0118	0.0631	0.0116
τ_1	0.04336		0.05059		0.0477		0.04733		0.04892		0.04791	
τ_2	0.04384		0.04992		0.0477		0.04711		0.04838		0.03699	
c_1	0.03188		0.01975		0.00974		0.00208		0.00105		0.00043	
c_2	0.03134		0.01956		0.00947		0.00195		0.00104		0.00041	
5% of censoring												
Re.Bias	-0.0539	0.2072	-0.0526	0.2490	-0.0551	0.2450	-0.0544	0.2339	-0.0520	0.2457	-0.0540	0.2475
RMSE	0.0625	0.0115	0.0613	0.0115	0.0638	0.0118	0.0628	0.0115	0.0610	0.0115	0.0629	0.0116
τ_1	0.04685		0.04920		0.04682		0.04851		0.05031		0.04969	
τ_2	0.04240		0.04914		0.04524		0.04656		0.04771		0.04782	
c_1	0.03090		0.01852		0.00932		0.00195		0.00100		0.00053	
c_2	0.03135		0.01953		0.00948		0.00192		0.00100		0.00051	
10% of censoring												
Re.Bias	-0.0526	0.2387	-0.0538	0.2264	-0.0548	0.2455	-0.0526	0.2294	-0.0547	0.2380	-0.0546	0.2186
RMSE	0.0616	0.0117	0.0627	0.0119	0.0637	0.0117	0.0617	0.0120	0.0634	0.0118	0.0635	0.0116
τ_1	0.05670		0.04865		0.04794		0.05103		0.04924		0.04989	
τ_2	0.04909		0.04385		0.04431		0.04669		0.04515		0.04565	
c_1	0.02813		0.01722		0.00890		0.00179		0.00098		0.00052	
c_2	0.02860		0.01670		0.00867		0.00175		0.00098		0.00048	
20% of censoring												
Re.Bias	-0.0531	0.2136	-0.0532	0.2003	-0.0530	0.1926	-0.0524	0.2049	-0.0532	0.2038	-0.0518	0.1965
RMSE	0.0619	0.0117	0.0617	0.0118	0.0620	0.0118	0.0615	0.0114	0.0623	0.0115	0.0611	0.0117
τ_1	0.0524		0.05195		0.05116		0.04761		0.04974		0.05035	
τ_2	0.03684		0.04077		0.04059		0.03969		0.04155		0.04125	
c_1	0.02514		0.01573		0.00828		0.00171		0.00088		0.00045	
c_2	0.02540		0.01537		0.00785		0.00169		0.00081		0.00045	
25% of censoring												
Re.Bias	-0.0518	0.1861	-0.0531	0.2058	-0.0526	0.2109	-0.0519	0.1782	-0.0534	0.1950	-0.0546	0.1972
RMSE	0.0610	0.0116	0.0625	0.0115	0.0612	0.0114	0.0606	0.0118	0.0622	0.0116	0.0631	0.0116
τ_1	0.04636		0.04423		0.04691		0.04842		0.04888		0.04791	
τ_2	0.03840		0.03729		0.03637		0.03693		0.03795		0.03699	
c_1	0.02441		0.01444		0.00754		0.00148		0.00078		0.00043	
c_2	0.02402		0.01445		0.00698		0.00161		0.00078		0.00041	

TABLE 1. Moments estimator performance based on Gumbel survival copula generated from 1000 replications with Pareto margins and shape parameter 0.3. Re.Bias and RMSE of the estimators are calculated for different censoring values and weak dependence.

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$\tau = 0.5$, $\alpha = 0.2 \rightarrow \beta = 1.82$

1% of censoring												
N	$n = 30$		$n = 50$		$n = 100$		$n = 500$		$n = 1000$		$n = 2000$	
$(\hat{\alpha}, \hat{\beta})$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
Re.Bias	-0.0263	0.3759	-0.0263	0.3735	-0.0259	0.3480	-0.0256	0.3648	-0.0252	0.3649	-0.0250	0.3695
RMSE	0.0302	0.0063	0.0302	0.0063	0.0299	0.0064	0.0299	0.0064	0.0293	0.0063	0.0292	0.0064
τ_1	0.49995		0.50077		0.49672		0.50080		0.50048		0.50033	
τ_2	0.49148		0.49273		0.48865		0.49225		0.49224		0.49208	
c_1	0.03163		0.01859		0.00977		0.00195		0.00102		0.00005	
c_2	0.03296		0.01876		0.00972		0.00194		0.00103		0.00053	
5% of censoring												
Re.Bias	-0.0267	0.3538	-0.0266	0.3554	-0.0255	0.3759	-0.0257	0.3415	-0.0255	0.3500	-0.0265	0.3566
RMSE	0.0309	0.0064	0.0306	0.0062	0.0296	0.0062	0.0299	0.0063	0.0298	0.0065	0.0306	0.0064
τ_1	0.50274		0.50408		0.49962		0.50013		0.50015		0.50143	
τ_2	0.46311		0.46417		0.45925		0.46042		0.45939		0.46095	
c_1	0.02847		0.01856		0.00946		0.00183		0.00099		0.00051	
c_2	0.02926		0.01803		0.00940		0.00183		0.00094		0.00049	
10% of censoring												
Re.Bias	-0.0259	0.3875	-0.0261	0.3658	-0.0255	0.3319	-0.0261	0.3312	-0.0262	0.3260	-0.0265	0.3301
RMSE	0.0300	0.0065	0.0303	0.0065	0.0298	0.0063	0.0302	0.0065	0.0302	0.0065	0.0304	0.0062
τ_1	0.49346		0.50026		0.50191		0.49985		0.50019		0.49999	
τ_2	0.41385		0.41922		0.42175		0.42134		0.42191		0.42137	
c_1	0.02790		0.01841		0.00902		0.00178		0.00092		0.00049	
c_2	0.02938		0.01794		0.00901		0.00177		0.00089		0.00046	
20% of censoring												
Re.Bias	-0.0261	0.3413	-0.0256	0.2964	-0.0252	0.3214	-0.0259	0.3021	-0.0250	0.3065	-0.0264	0.3017
RMSE	0.0303	0.0063	0.0298	0.0065	0.0295	0.0063	0.0302	0.0063	0.0292	0.0065	0.0304	0.0063
τ_1	0.50244		0.49760		0.49889		0.50007		0.50027		0.49999	
τ_2	0.35571		0.34813		0.35141		0.35171		0.35227		0.35153	
c_1	0.02586		0.01614		0.00838		0.00164		0.00081		0.00041	
c_2	0.02487		0.01648		0.00797		0.0016		0.00081		0.00042	
25% of censoring												
Re.Bias	-0.0251	0.2793	-0.0259	0.3205	-0.0266	0.2833	-0.0254	0.2982	-0.0253	0.2869	-0.0256	0.2838
RMSE	0.0293	0.0063	0.0299	0.0062	0.0305	0.0064	0.0296	0.0064	0.0295	0.0064	0.0298	0.0065
τ_1	0.49657		0.50095		0.50224		0.50036		0.50043		0.50059	
τ_2	0.31482		0.32157		0.31815		0.31958		0.32107		0.32089	
c_1	0.02483		0.01532		0.00737		0.00156		0.00079		0.00040	
c_2	0.02584		0.01444		0.00729		0.00152		0.00074		0.00040	

TABLE 2. Moments estimator performance based on Gumbel survival copula generated from 1000 replications with Pareto margins and shape parameter 0.3. Re.Bias and RMSE of the estimators are calculated for different censoring values and moderate dependence.

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$\tau = 0.7$, $\alpha = 0.4 \to \beta = 2.78$

N	$n = 30$		$n = 50$		$n = 100$		$n = 500$		$n = 1000$		$n = 2000$	
$(\hat{\alpha}, \hat{\beta})$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
1% of censoring												
Re.Bias	-0.0131	0.4582	-0.0129	0.4197	-0.0128	0.4053	-0.0127	0.4127	-0.0128	0.4195	-0.0125	0.4171
RMSE	0.0150	0.0041	0.0149	0.0042	0.0147	0.0040	0.0147	0.0040	0.0148	0.0430	0.0145	0.0042
τ_1	0.70344		0.69913		0.70007		0.70007		0.70013		0.69997	
τ_2	0.69002		0.68812		0.68855		0.68788		0.68815		0.68794	
c_1	0.03001		0.01890		0.00982		0.00190		0.00096		0.00005	
c_2	0.03035		0.01927		0.01000		0.00191		0.00950		0.00048	
5% of censoring												
Re.Bias	-0.0126	0.4252	-0.0127	0.4063	-0.0127	0.3972	-0.0126	0.3973	-0.0126	0.4056	-0.0124	0.4001
RMSE	0.0146	0.0041	0.0147	0.0042	0.0147	0.0042	0.0146	0.0042	0.0146	0.0042	0.0144	0.0041
τ_1	0.69937		0.69845		0.69828		0.70019		0.70010		0.70065	
τ_2	0.64116		0.63732		0.6397		0.64107		0.64168		0.64204	
c_1	0.03068		0.02042		0.00955		0.00194		0.00098		0.00049	
c_2	0.03050		0.01966		0.00974		0.00186		0.00095		0.00049	
10% of censoring												
Re.Bias	-0.0123	0.3847	-0.0125	0.3756	-0.0127	0.3768	-0.0127	0.3927	-0.0129	0.3889	-0.0121	0.3860
RMSE	0.0144	0.0041	0.0145	0.0042	0.0146	0.0041	0.0147	0.0042	0.0149	0.0041	0.0142	0.0043
τ_1	0.69714		0.70026		0.69879		0.69974		0.70095		0.70013	
τ_2	0.58936		0.58693		0.58613		0.58613		0.58814		0.58743	
c_1	0.03007		0.01752		0.00928		0.00183		0.00088		0.00045	
c_2	0.02926		0.01711		0.00886		0.00186		0.00092		0.00047	
20% of censoring												
Re.Bias	-0.0128	0.3923	-0.0125	0.3671	-0.0125	0.3364	-0.0132	0.3458	-0.0130	0.3441	-0.0127	0.3445
RMSE	0.0148	0.0042	0.0146	0.0041	0.0148	0.0041	0.0151	0.0041	0.0149	0.0041	0.0147	0.0041
τ_1	0.70236		0.70103		0.69985		0.70113		0.70053		0.70069	
τ_2	0.49520		0.49110		0.48840		0.49066		0.48991		0.48950	
c_1	0.02444		0.01543		0.00820		0.00160		0.00082		0.00040	
c_2	0.02485		0.01492		0.00829		0.00155		0.00077		0.00039	
25% of censoring												
Re.Bias	-0.0126	0.2926	-0.0128	0.3569	-0.0126	0.3280	-0.0126	0.3417	-0.0122	0.334	-0.0126	0.3247
RMSE	0.0147	0.0043	0.0149	0.0041	0.0146	0.0042	0.0146	0.0040	0.0142	0.0041	0.0146	0.0041
τ_1	0.69894		0.69874		0.70112		0.70002		0.70018		0.70029	
τ_2	0.44299		0.43503		0.44424		0.44622		0.44462		0.44530	
c_1	0.02306		0.01406		0.00691		0.00147		0.00073		0.00038	
c_2	0.02331		0.01521		0.00749		0.00147		0.00072		0.00036	

TABLE 3. Moments estimator performance based on Gumbel survival copula generated from 1000 replications with Pareto margins and shape parameter 0.3. Re.Bias and RMSE of the estimators are calculated for different censoring values and strong dependence.

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