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ON THE METRIC DIMENSION OF A CLASS OF PLANAR GRAPHS

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Abstract. Let $H = (V, E)$ be a non-trivial connected graph with vertex set V and edge set E. A set of ordered vertices R_m from $V(H)$ is said to be a resolving set for H if each vertex of H is uniquely determined by its vector of distances to the vertices of R_m. The number of vertices in a smallest resolving set is called the metric dimension of H. In this article, we study the metric dimension for a rotationally symmetric family of planar graphs, each of which is shown to have an independent minimum resolving set of cardinality three.

Keywords: Resolving set, metric dimension, rotationally symmetric plane graph, independent set.

AMS Subject Classification: 05C12, 05C90.

1. Introduction

The minimum resolving set in a connected graph is a well-studied topic in combinatorics and graph theory, as well as in several computer science applications. The theory relating to it is full with remarkable results and unanswered questions. Slater [21] and Harary and Melter [9] were the first to introduce the concept of metric dimension in the mid-1970s. Slater referred to a metric dimension (minimum resolving set) of a non-trivial connected graph as its location number (reference set). Harary and Melter also studied these concepts, but using the terms metric dimension and resolving sets instead of location number and locating sets, respectively. Since then, the metric dimension has been frequently used in robotics, chemistry, graph theory, biology, and in several other fields.

Obtaining the minimum resolving set for different families of graph products and operations, as well as characterizing the graphs with n vertices having a specific metric dimension are fascinating problems and attracts the attention of many researchers. For generic graphs, the problem of determining metric dimension is NP-Complete, but for trees, a polynomial-time algorithm can be used [10]. It is obvious that for a connected

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graph H on n vertices, we have $1 \leq dim(H) \leq n-1$. In [10], Khuller et al. proved that graph H is a path P_n iff $dim(H) = 1$. In [4], Chartrand et al. proved that graph H is complete graph K_n $(n \geq 2)$ iff $dim(H) = n-1$. Recently, Sharma and Bhat also obtained the metric dimension of the line graph of the subdivision graph of the graph of convex polytope [17]. For a comparative study of graph parameters and metric dimension of more algebraic flavor, see [1] by Cameron and Bailey. For a survey on some variations and metric dimension, see Chartrand and Zhang [6] and Saenpholphat and Zhang [13].

The metric dimension can be widely used in a variety of fields, including robot navigation [10], geographical routing protocols [12], telecommunications [3], combinatorial optimization [14], network discovery and verification [3] etc. Moreover, [4] Chartrand et al. associate the metric dimension of graph with pharmacological activity and drug discovery. The metric dimension of Hamming graphs leads Chvátal to the study of mastermind games and allows academics to examine the use of metric dimension in complicated digital games [7].

It might also be applied to more abstract tasks such as detecting a source of misinformation in a social network, classifying chemical structures, comparing network topology, or quantitatively expressing symbolic data. The metric dimension of various important graph classes has been studied: Cayley digraphs [8], Kneser and Johnson graphs [1], Grassmann graphs [2], circulant graph [19], convex polytopes [15, 16] etc. The metric dimension of lexicographic product graphs, cartesian product graphs, corona product graphs, hierarchical product graphs, and strong product graphs was also studied in the recent past [11].

The planar graphs, which are rotationally symmetric, are essential in intelligent networks due to the uniform rate of data transformation to all vertices. In Chapter 18 of *Parallel and Distributed Computing Handbook*, Stojmenovi [18] posed several open problems for various interconnection networks. One of them involves the conception of new architectural structures. Towards this, we intend to construct and define new architectures. We start our exploration in the field of planar geometry. The metric dimension have been studied in recent years for various families of planar graphs. We discovered several types of planar graphs with undefined distance-based graph parameters such as, metric dimension and edge metric dimension during our analysis of the literature. As a result, we commence this research by considering a family of planar graphs (viz., Ł$_n$) and study its metric dimension.

In this paper, we consider a family $\{$Ł$_n\}$ of planar graphs (see Fig. 1), for which we determine its metric dimension. We also prove that the graphs Ł$_n$ possess an independent minimum resolving set of cardinality three, that is, only three vertices are the minimum requirement for the unique identification of all vertices in the planar graphs Ł$_n$. This article is organized as follows. In Section 2, we recall some existing results related to the metric dimension of graphs. In Section 3, we compute the metric dimension of the plane graph Ł$_n$. Finally, the conclusion and future work of this paper is presented in Section 4.

2. Preliminaries

In this section, we provide some basic results and definitions from the literature which are used in order to obtain our findings in subsequent sections.

Suppose H is a connected, simple, and finite graph with edge set $E(H)$ and vertex set

$V(H)$. We write V instead of $V(H)$ and E instead of $E(H)$ throughout the paper, when there is no scope for ambiguity. The topological distance (geodesic) between two vertices u and v in a simple connected graph H, denoted by $d(u,v)$, is the length of a shortest $u-v$ path between the vertices u and v in H.

Degree of a vertex: The number of edges that are incident to a vertex of a graph H is known as its degree (or valency).

Independent set: [15] An independent set is a set of vertices in H, in which no two vertices are adjacent.

Metric Dimension: [21] If for any three vertices x, y, $z \in V(H)$, we have $d(x,z) \neq d(y,z)$, then the vertex z is said to resolve (distinguish) the pair of vertices x, y $(x \neq y)$ in $V(H)$. If this condition of resolvability is fulfilled by some vertices comprising a subset $R_m \subseteq V(H)$ i.e., every pair of different vertices in the given undirected graph H is resolved by at least one element of R_m, then R_m is said to be a *resolving set* (*metric generator*) of H. The *metric dimension* of the given graph H is the minimum cardinality of a resolving set R_m, and is usually denoted by $dim(H)$. The metric generator R_m with minimum cardinality is the metric basis for H. For an ordered subset of vertices $R_m = \{z_1, z_2, z_3, ..., z_k\}$, the k-code (representation or coordinate) of vertex x in $V(H)$ is;

$$\zeta(x|R_m) \quad = \quad (d(z_1,x), d(z_2,x), ..., d(z_k,x))$$

In this respect, the set R_m is a resolving set for H, if $\zeta(y|R_m) \neq \zeta(x|R_m)$, for any pair of vertices $x, y \in V(H)$ with $y \neq x$.

Independent resolving set (IRS): [5] A set of distinct ordered vertices R_m in H is said to be an IRS for H if R_m is both independent and resolving set.

In [10], Khuller et al. proved the following result.

Proposition 2.1. *Let $A \subseteq V(H)$ be the metric basis for the connected graph H of cardinality two i.e., $|A| = 2$, and say $A = \{x, z\}$. Then, the followings are true:*

- *Between the vertices z and x, there exists a unique shortest path P_m.*
- *The valencies of the vertices x and z can never exceed* 3.
- *The valency of any other vertex on P_m can never exceed* 5.

3. The Plane Graph Ł$_n$

In the present section, we construct a family of the plane graph, denoted by Ł$_n$, and for which we compute its metric dimension.

Graph Ł$_n$: For a positive integer n, the planar graph Ł$_n$ is rotationally symmetric of order $12n$ and size $19n$. It has $2n$ triangular regions, $2n$ regions whose boundary is a 4-cycle, $2n$ regions whose boundary is a 6-cycle, n regions whose boundary is a 8-cycle, and two regions whose boundary is a $2n$-cycle. Moreover, Ł$_n$ has $2n$ vertices of degree four and 10 vertices of degree three. We represent the set of edges and vertices of Ł$_n$ by $E($Ł$_n)$ and $V($Ł$_n)$, respectively. Thus, we have $V($Ł$_n) = \{p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i, x_i, y_i, z_i, a_i : 1 \leq i \leq n\}$

and

$E(\text{Ł}_n) = \{p_iq_i, p_ir_i, q_is_i, r_it_i, t_iv_i, v_iw_i, s_iu_i, u_iw_i, w_ix_i, x_iy_i, y_iz_i, y_ia_i, r_is_i, z_ia_i : 1 \leq i \leq n\} \cup \{q_ip_{i+1}, u_it_{i+1}, x_iv_{i+1}, a_iz_{i+1}, w_iv_{i+1} : 1 \leq i \leq n\}$

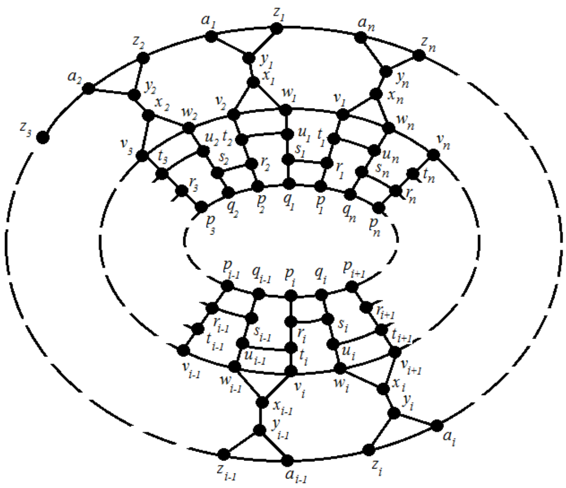

FIGURE 1. Ł_n

We name the vertices $\{p_i, q_i : 1 \leq i \leq n\}$ in Ł_n, as the vertices of pq-cycle, the vertices $\{r_i, s_i : 1 \leq i \leq n\}$ in Ł_n, as the rs-vertices, the vertices $\{t_i, u_i : 1 \leq i \leq n\}$ in Ł_n, as the tu-vertices, the vertices $\{v_i, w_i : 1 \leq i \leq n\}$ in Ł_n, as the vertices of vw-cycle, the vertices $\{x_i : 1 \leq i \leq n\}$ in Ł_n, as the x-vertices, the vertices $\{y_i : 1 \leq i \leq n\}$ in Ł_n, as the y-vertices, and the vertices $\{z_i, a_i : 1 \leq i \leq n\}$ in Ł_n, as the vertices of za-cycle. For convenience, we take $p_{n+1} = p_1$, $q_{n+1} = q_1$, $r_{n+1} = r_1$, $s_{n+1} = s_1$, $t_{n+1} = t_1$, $u_{n+1} = u_1$, $v_{n+1} = v_1$, $w_{n+1} = w_1$, $x_{n+1} = x_1$, $y_{n+1} = y_1$, $z_{n+1} = z_1$, and $a_{n+1} = a_1$. Next, we compute the minimum resolving number for Ł_n.

Now, the graph Ł_n has the vertex set $V(\text{Ł}_n) = \{p_i, q_i, r_i, s_i, t_i, u_i, v_i, w_i, x_i, y_i, z_i, a_i : 1 \leq i \leq n\}$. So, by P, Q,..., Z, and A, we denote the set of codes for the vertices p_i, q_i, ..., z_i, and a_i respectively, for all $1 \leq i \leq n$.

Theorem 3.1. *For the positive integer $n \geq 6$, we have $dim(\text{Ł}_n) = 3$.*

Proof. We divide our proof into two cases depending upon n i.e., when $n \equiv 1\ (mod\ 2)$ and $n \equiv 0\ (mod\ 2)$.

Case(I) $n \equiv 0\ (mod\ 2)$.
Then, $n = 2w$, where $w \geq 3$ and $w \in \mathbb{N}$. Suppose $R_m = \{z_1, z_3, q_1\} \subset V(\text{Ł}_n)$. To complete the proof for this case, we show that the graph Ł_n is resolved by the set R_m. For this, we assign metric codes to every vertex in Ł_n with respect to the set R_m.

For the vertices of pq-cycle $\{p_i, q_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(p_1|R_m) = (7, 10, 1)$; $\zeta(p_2|R_m) = (6, 8, 1)$; $\zeta(p_3|R_m) = (8, 7, 3)$; $\zeta(q_1|R_m) = (6, 9, 0)$; $\zeta(q_2|R_m) = (7, 7, 2)$; $\zeta(q_3|R_m) = (9, 6, 4)$ and for rest of the vertices, we have

$\zeta(p_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(p_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+2, 2i-2, 2i-3)$
$\zeta(p_i\|R_m)$: $(i = w+2)$	$(2w+4, 2w+2, 2w-1)$
$\zeta(p_i\|R_m)$: $(i = w+3)$	$(4w-2i+8, 2w+4, 4w-2i+3)$
$\zeta(p_i\|R_m)$: $(w+4 \leq i \leq 2w)$	$(4w-2i+8, 4w-2i+12, 4w-2i+3)$

and

$\zeta(q_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(q_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+3, 2i-1, 2i-2)$
$\zeta(q_i\|R_m)$: $(i = w+2)$	$(4w-2i+7, 2w+3, 4w-2i+2)$
$\zeta(q_i\|R_m)$: $(w+3 \leq i \leq 2w)$	$(4w-2i+7, 4w-2i+11, 4w-2i+2)$

For the vertices $\{r_i, s_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(r_1|R_m) = (6, 9, 2)$; $\zeta(r_2|R_m) = (5, 7, 2)$; $\zeta(r_3|R_m) = (7, 6, 4)$; $\zeta(s_1|R_m) = (5, 8, 1)$; $\zeta(s_2|R_m) = (6, 6, 3)$; $\zeta(s_3|R_m) = (8, 5, 5)$ and for rest of the vertices, we have

$\zeta(r_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(r_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+1, 2i-3, 2i-2)$
$\zeta(r_i\|R_m)$: $(i = w+2)$	$(2w+3, 2w+1, 2w)$
$\zeta(r_i\|R_m)$: $(i = w+3)$	$(4w-2i+7, 2w+3, 4w-2i+4)$
$\zeta(r_i\|R_m)$: $(w+4 \leq i \leq 2w)$	$(4w-2i+7, 4w-2i+11, 4w-2i+4)$

and

$\zeta(s_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(s_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+2, 2i-2, 2i-1)$
$\zeta(s_i\|R_m)$: $(i = w+2)$	$(4w-2i+6, 2w+2, 4w-2i+3)$
$\zeta(s_i\|R_m)$: $(w+3 \leq i \leq 2w)$	$(4w-2i+6, 4w-2i+10, 4w-2i+3)$

For the vertices $\{t_i, u_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(t_1|R_m) = (5, 8, 3)$; $\zeta(t_2|R_m) = (4, 6, 3)$; $\zeta(t_3|R_m) = (6, 5, 5)$; $\zeta(u_1|R_m) = (4, 7, 2)$; $\zeta(u_2|R_m) = (5, 5, 4)$; $\zeta(u_3|R_m) = (7, 4, 6)$ and for rest of the vertices, we have

$\zeta(t_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(t_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i, 2i-4, 2i-1)$
$\zeta(t_i\|R_m)$: $(i = w+2)$	$(2w+2, 2w, 2w+1)$
$\zeta(t_i\|R_m)$: $(i = w+3)$	$(4w-2i+6, 2w+2, 4w-2i+5)$
$\zeta(t_i\|R_m)$: $(w+4 \leq i \leq 2w)$	$(4w-2i+6, 4w-2i+10, 4w-2i+5)$

and

$\zeta(u_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(u_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+1, 2i-3, 2i)$
$\zeta(u_i\|R_m)$: $(i = w+2)$	$(4w-2i+5, 2w+1, 4w-2i+4)$
$\zeta(u_i\|R_m)$: $(w+3 \leq i \leq 2w)$	$(4w-2i+5, 4w-2i+9, 4w-2i+4)$

For the vertices of vw-cycle $\{v_i, w_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(v_1|R_m) = (4, 7, 4)$; $\zeta(v_2|R_m) = (3, 5, 4)$; $\zeta(v_3|R_m) = (5, 4, 6)$; $\zeta(w_1|R_m) = (3, 6, 3)$; $\zeta(w_2|R_m) = (4, 4, 5)$; $\zeta(w_3|R_m) = (6, 3, 7)$ and for rest of the vertices, we have

$\zeta(v_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(v_i\|R_m)$: $(4 \le i \le w+1)$	$(2i-1, 2i-5, 2i)$
$\zeta(v_i\|R_m)$: $(i = w+2)$	$(2w+1, 2w-1, 2w+2)$
$\zeta(v_i\|R_m)$: $(i = w+3)$	$(4w-2i+7, 2w+1, 4w-2i+6)$
$\zeta(v_i\|R_m)$: $(w+4 \le i \le 2w)$	$(4w-2i+7, 4w-2i+9, 4w-2i+6)$

and

$\zeta(w_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(w_i\|R_m)$: $(4 \le i \le w+1)$	$(2i, 2i-4, 2i+1)$
$\zeta(w_i\|R_m)$: $(i = w+2)$	$(4w-2i+4, 2w, 4w-2i+5)$
$\zeta(w_i\|R_m)$: $(w+3 \le i \le 2w)$	$(4w-2i+4, 4w-2i+8, 4w-2i+5)$

For the vertices $\{x_i : 1 \le i \le n\}$, the metric codes are as follows: $\zeta(x_1|R_m) = (2,5,4)$; $\zeta(x_2|R_m) = (4,3,6)$ and for rest of the vertices, we have

$\zeta(x_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(x_i\|R_m)$: $(3 \le i \le w)$	$(2i, 2i-4, 2i+2)$
$\zeta(x_i\|R_m)$: $(i = w+1)$	$(2w+1, 2w-2, 2w+3)$
$\zeta(x_i\|R_m)$: $(i = w+2)$	$(4w-2i+3, 2w, 4w-2i+5)$
$\zeta(x_i\|R_m)$: $(w+3 \le i \le 2w)$	$(4w-2i+3, 4w-2i+7, 4w-2i+5)$

For the vertices $\{y_i : 1 \le i \le n\}$, the metric codes are as follows: $\zeta(y_i|R_m) = \zeta(x_i|R_m) + (-1,-1,1)$ for $1 \le i \le n$. Finally, for the vertices of za-cycle $\{z_i, a_i : 1 \le i \le w\}$, the metric codes are as follows: $\zeta(z_1|R_m) = (0,4,6)$; $\zeta(z_2|R_m) = (2,2,7)$; $\zeta(z_3|R_m) = (4,0,9)$; $\zeta(a_1|R_m) = (1,3,6)$; $\zeta(a_2|R_m) = ((3,1,8)$ and for rest of the vertices, we have

$\zeta(z_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(z_i\|R_m)$: $(4 \le i \le w+1)$	$(2i-2, 2i-6, 2i+3)$
$\zeta(z_i\|R_m)$: $(i = w+2)$	$(4w-2i+2, 2w-2, 4w-2i+7)$
$\zeta(z_i\|R_m)$: $(w+3 \le i \le 2w)$	$(4w-2i+2, 4w-2i+6, 4w-2i+7)$

and

$\zeta(a_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(a_i\|R_m)$: $(3 \le i \le w)$	$(2i-1, 2i-5, 2i+4)$
$\zeta(a_i\|R_m)$: $(i = w+1)$	$(4w-2i+1, 2w-3, 4w-2i+6)$
$\zeta(a_i\|R_m)$: $(i = w+2)$	$(4w-2i+1, 2w-1, 4w-2i+6)$
$\zeta(a_i\|R_m)$: $(w+3 \le i \le 2w-1)$	$(4w-2i+1, 4w-2i+5, 4w-2i+6)$
$\zeta(a_i\|R_m)$: $(i = 2w)$	$(4w-2i+1, 4w-2i+5, 7)$

From these metric codes, we find that $|P| = |Q| = |R| = ... = |Z| = |A| = n$ and $P \cap Q \cap R \cap ... \cap Z \cap A = \emptyset$, implying R_m to be an ordered vertex resolving set for the graph $Ł_n$ and so $dim(Ł_n) \le 3$. Now, to complete the proof for this case, we have to show that $dim(Ł_n) \ge 3$. We prove that $Ł_n$ has no resolving set R_m with $|R_m| = 2$. On the contrary, assume that $dim(Ł_n) = 2$. Now, by C_1, C_2, C_3,...,C_{12}, we denote the set of vertices as $C_1 = \{p_i : 1 \le i \le n\}$, $C_2 = \{q_i : 1 \le i \le n\}$,...,$C_{12} = \{a_i : 1 \le i \le n\}$. By Proposition 2.1, we find that the degree of basis vertices can be at most three. But except the vertices in the sets C_7 and C_8, all other vertices of $Ł_n$ have a degree less than or equal to three. Therefore, we have the following to be considered.

A. When both of the vertices are from the set C_l; $l = 1, 2, ...6, 9, 10, ..., 12$.

- Suppose $R_m = \{p_1, p_g\}$, p_g $(2 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $2 \leq g \leq w$; $\zeta(q_1|R_m) = \zeta(q_n|R_m)$ when $g = w+1$, a contradiction.
- Suppose $R_m = \{q_1, q_g\}$, q_g $(2 \leq g \leq n)$. Then $\zeta(s_1|R_m) = \zeta(p_1|R_m)$, for $2 \leq g \leq w$; $\zeta(p_1|R_m) = \zeta(p_2|R_m)$ when $g = w+1$, a contradiction.
- Suppose $R_m = \{r_1, r_g\}$, r_g $(2 \leq g \leq n)$. Then $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, for $2 \leq g \leq w$; $\zeta(q_1|R_m) = \zeta(q_n|R_m)$ when $g = w+1$, a contradiction.
- Suppose $R_m = \{s_1, s_g\}$, s_g $(2 \leq g \leq n)$. Then $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, for $2 \leq g \leq w-1$; $\zeta(t_2|R_m) = \zeta(p_1|R_m)$, when $g = w$; $\zeta(p_1|R_m) = \zeta(p_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{t_1, t_g\}$, t_g $(2 \leq g \leq n)$. Then $\zeta(x_n|R_m) = \zeta(w_n|R_m)$, for $g = 2$; $\zeta(p_1|R_m) = \zeta(s_1|)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{u_1, u_g\}$, u_g $(2 \leq g \leq n)$. Then $\zeta(x_n|R_m) = \zeta(w_n|R_m)$, for $g = 2$; $\zeta(q_1|R_m) = \zeta(r_2|R_m)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{x_1, x_g\}$, x_g $(2 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $2 \leq g \leq w$; $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{y_1, y_g\}$, y_g $(2 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $2 \leq g \leq w$; $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{z_1, z_g\}$, z_g $(2 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $2 \leq g \leq w$; $\zeta(a_1|R_m) = \zeta(a_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{a_1, a_g\}$, a_g $(2 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $2 \leq g \leq w$; $\zeta(z_1|R_m) = \zeta(z_2|R_m)$, when $g = w+1$, a contradiction.

B. When one vertex is in the set C_1 and other lies in the set C_l; $l = 2, 3, ..., 6, 9, ..., 12$.

- Suppose $R_m = \{p_1, q_g\}$, q_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_n|R_m)$, for $1 \leq g \leq w$; $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{p_1, r_g\}$, r_g $(1 \leq g \leq n)$. Then $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, for $1 \leq g \leq w$; $\zeta(q_1|R_m) = \zeta(q_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{p_1, s_g\}$, s_g $(1 \leq g \leq n)$. Then $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, for $1 \leq g \leq w-1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $g = w$; $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{p_1, t_g\}$, t_g $(1 \leq g \leq n)$. Then $\zeta(q_n|R_m) = \zeta(q_1|R_m)$, for $g = 1, w+1$; $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, when $g = 2$; $\zeta(r_1|R_m) = \zeta(q_n|R_m)$, when $3 \leq g \leq w$, a contradiction.
- Suppose $R_m = \{p_1, u_g\}$, u_g $(1 \leq g \leq n)$. Then $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, for $g = 1, w+1$; $\zeta(r_1|R_m) = \zeta(q_n|R_m)$, when $2 \leq g \leq w$, a contradiction.
- Suppose $R_m = \{p_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $1 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{p_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, for $1 \leq g \leq w$; $\zeta(t_1|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{p_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, for $1 \leq g \leq w$; $\zeta(t_1|R_m) = \zeta(p_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{p_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(q_1|R_m) = \zeta(r_1|R_m)$, for $1 \leq g \leq w$; $\zeta(t_1|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.

C. When one vertex is in the set C_2 and other lies in the set C_l; $l = 3, ..., 6, 9, ..., 12$.

- Suppose $R_m = \{q_1, r_g\}$, r_g $(1 \leq g \leq n)$. Then $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, for $g = 1$; $\zeta(s_1|R_m) = \zeta(p_1|R_m)$, $2 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{q_1, s_g\}$, s_g $(1 \leq g \leq n)$. Then $\zeta(p_2|R_m) = \zeta(p_1|R_m)$, for $g = 1, w+1$; $\zeta(s_1|R_m) = \zeta(p_1|R_m)$, when $2 \leq g \leq w$, a contradiction.

- Suppose $R_m = \{q_1, t_g\}$, t_g $(1 \leq g \leq n)$. Then $\zeta(x_n|R_m) = \zeta(w_n|R_m)$, for $1 \leq g \leq w$; $\zeta(s_1|R_m) = \zeta(p_1|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{q_1, u_g\}$, u_g $(1 \leq g \leq n)$. Then $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, for $1 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{q_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(x_n|\{q_1, x_g\}) = \zeta(w_n|\{q_1, x_g\})$, for $1 \leq g \leq w-1$; $\zeta(v_2|\{q_1, x_g\}) = \zeta(u_2|\{q_1, x_g\})$, when $w \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{q_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(x_n|\{q_1, y_g\}) = \zeta(w_n|\{q_1, y_g\})$, for $1 \leq g \leq w-1$; $\zeta(v_2|\{q_1, y_g\}) = \zeta(u_2|\{q_1, y_g\})$, when $w \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{q_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(w_2|R_m) = \zeta(w_n|R_m)$, for $g = 1$; $\zeta(q_2|R_m) = \zeta(r_2|R_m)$, when $g = 2$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{q_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(x_n|R_m) = \zeta(w_2|R_m)$, for $g = 1$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $2 \leq g \leq w$; $\zeta(w_2|R_m) = \zeta(t_3|R_m)$, we have $g = w+1$, a contradiction.

D. When one vertex is in the set C_3 and other lies in the set C_l; $l = 4, 5, 6, 9, ..., 12$.

- Suppose $R_m = \{r_1, s_g\}$, s_g $(1 \leq g \leq n)$. Then $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, for $1 \leq g \leq w-1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $w \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{r_1, t_g\}$, t_g $(1 \leq g \leq n)$. Then $\zeta(x_n|R_m) = \zeta(w_n|R_m)$, for $1 \leq g \leq 2$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{r_1, u_g\}$, u_g $(1 \leq g \leq n)$. Then $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, for $1 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{r_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $g = 1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $2 \leq g \leq w$; $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{r_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $g = 1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $2 \leq g \leq w$; $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{r_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $1 \leq g \leq w$; $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{r_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, for $1 \leq g \leq w$; $\zeta(p_n|R_m) = \zeta(s_n|R_m)$, when $g = w+1$, a contradiction.

E. When one vertex is in the set C_4 and other lies in the set C_l; $l = 5, 6, 9, ..., 12$.

- Suppose $R_m = \{s_1, t_g\}$, t_g $(1 \leq g \leq n)$. Then $\zeta(u_1|R_m) = \zeta(q_1|R_m)$, for $g = 1$; $\zeta(v_2|R_m) = \zeta(r_2|R_m)$, when $g = 2$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{s_1, u_g\}$, u_g $(1 \leq g \leq n)$. Then $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, $1 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{s_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $g = 1$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{s_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $g = 1$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$ when $g = w+1$, a contradiction.
- Suppose $R_m = \{s_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $1 \leq g \leq 2$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $3 \leq g \leq w+1$, a contradiction.

- Suppose $R_m = \{s_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(r_1|R_m) = \zeta(q_1|R_m)$, for $g = 1$; $\zeta(t_2|R_m) = \zeta(w_1|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$; when $g = w+1$, a contradiction.

F. When one vertex is in the set C_5 and other lies in the set C_l; $l = 6, 9, ..., 12$.

- Suppose $R_m = \{t_1, u_g\}$, u_g $(1 \leq g \leq n)$. Then $\zeta(w_n|R_m) = \zeta(x_n|R_m)$, for $1 \leq g \leq w$; $\zeta(u_n|R_m) = \zeta(v_1|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{t_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(w_n|R_m) = \zeta(x_n|R_m)$, for $1 \leq g \leq w-1$; $\zeta(x_n|R_m) = \zeta(s_1|R_m)$, when $g = w$, $\zeta(u_n|R_m) = \zeta(v_1|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{t_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(w_n|R_m) = \zeta(x_n|R_m)$, for $1 \leq g \leq w-1$; $\zeta(x_n|R_m) = \zeta(s_1|R_m)$, when $g = w$; $\zeta(u_n|R_m) = \zeta(v_1|R_m)$when $g = w+1$, a contradiction.
- Suppose $R_m = \{t_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(w_1|R_m) = \zeta(x_n|R_m)$, for $g = 1$; $\zeta(r_1|R_m) = \zeta(u_n|R_m)$, when $g = 2$; $\zeta(x_1|R_m) = \zeta(v_2|R_m)$, when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{t_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(w_n|R_m) = \zeta(s_1|R_m)$, for $g = 1$; $\zeta(x_1|R_m) = \zeta(v_2|R_m)$, when $2 \leq g \leq w$; $\zeta(u_n|R_m) = \zeta(v_1|R_m)$, when $g = w+1$, a contradiction.

G. When one vertex is in the set C_6 and other lies in the set C_l; $l = 9, ..., 12$.

- Suppose $R_m = \{u_1, x_g\}$, x_g $(1 \leq g \leq n)$. Then $\zeta(z_1|R_m) = \zeta(a_1|R_m)$, for $g = 1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{u_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(z_1|R_m) = \zeta(a_1|R_m)$, $g = 1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{u_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(w_2|R_m) = \zeta(w_n|R_m)$, $g = 1$; $\zeta(x_2|R_m) = \zeta(z_1|R_m)$, when $g = 2$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$ when $3 \leq g \leq w+1$, a contradiction.
- Suppose $R_m = \{u_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(w_2|R_m) = \zeta(x_n|R_m)$, $g = 1$; $\zeta(w_1|R_m) = \zeta(t_2|R_m)$, when $2 \leq g \leq w$; $\zeta(s_n|R_m) = \zeta(p_n|R_m)$, when $g = w+1$, a contradiction.

H. When one vertex is in the set C_9 and other lies in the set C_l; $l = 10, 11, 12$.

- Suppose $R_m = \{x_1, y_g\}$, y_g $(1 \leq g \leq n)$. Then $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, $1 \leq g \leq w$; $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{x_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(w_2|R_m) = \zeta(t_2|R_m)$, for $g = 1$; $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, when $2 \leq g \leq w$; $\zeta(z_n|R_m) = \zeta(y_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{x_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, for $g = 1$; $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, when $2 \leq g \leq w$; $\zeta(y_n|R_m) = \zeta(a_2|R_m)$, when $g = w+1$, a contradiction.

I. When one vertex is in the set C_{10} and other lies in the set C_l; $l = 11, 12$.

- Suppose $R_m = \{y_1, z_g\}$, z_g $(1 \leq g \leq n)$. Then $\zeta(w_2|R_m) = \zeta(t_2|R_m)$, for $g = 1$; $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, when $2 \leq g \leq w$; $\zeta(z_n|R_m) = \zeta(y_2|R_m)$, when $g = w+1$, a contradiction.
- Suppose $R_m = \{y_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, for $g = 1$; $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, when $2 \leq g \leq w$; $\zeta(y_n|R_m) = \zeta(a_2|R_m)$, when $g = w+1$, a contradiction.

J. When one vertex is in the set C_{11} and other lies in the set C_{12}.

- Suppose $R_m = \{z_1, a_g\}$, a_g $(1 \leq g \leq n)$. Then $\zeta(z_n|R_m) = \zeta(y_n|R_m)$, $1 \leq g \leq w-1$; $\zeta(v_1|R_m) = \zeta(u_1|R_m)$, when $g = w$; $\zeta(w_1|R_m) = \zeta(v_2|R_m)$, when $g = w+1$, a contradiction.

From this, we conclude that $Ł_n$ has no resolving set R_m with $|R_m| = 2$, implying $dim(Ł_n) = 3$ for this case.

Case(II) $n \equiv 1$ $(mod\ 2)$.
Then, $n = 2w+1$, where $w \geq 3$ and $w \in \mathbb{N}$. Suppose $R = \{z_1, z_3, q_1\} \subset V(Ł_n)$. To complete the proof for this case, we show that the graph $Ł_n$ is resolved by the set R. For this, we assign metric codes to every vertex in $Ł_n$ with respect to the set R.

For the vertices of pq-cycle $\{p_i, q_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(p_1|R) = (7, 10, 1)$; $\zeta(p_2|R) = (6, 8, 1)$; $\zeta(p_3|R) = (8, 7, 3)$; $\zeta(q_1|R) = (6, 9, 0)$; $\zeta(q_2|R) = (7, 7, 2)$; $\zeta(q_3|R) = (9, 6, 4)$ and for rest of the vertices, we have

$\zeta(p_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(p_i\|R_m)$: $(4 \leq i \leq w+2)$	$(2i+2, 2i-2, 2i-3)$
$\zeta(p_i\|R_m)$: $(i = w+3)$	$(4w-2i+10, 2w+4, 4w-2i+5)$
$\zeta(p_i\|R_m)$: $(w+4 \leq i \leq 2w+1)$	$(4w-2i+10, 4w-2i+14, 4w-2i+5)$

and

$\zeta(q_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(q_i\|R_m)$: $(4 \leq i \leq w+1)$	$(2i+3, 2i-1, 2i-2)$
$\zeta(q_i\|R_m)$: $(i = w+2)$	$(4w-2i+9, 2w+3, 4w-2i+4)$
$\zeta(q_i\|R_m)$: $(i = w+3)$	$(4w-2i+9, 2w+5, 4w-2i+4)$
$\zeta(q_i\|R_m)$: $(w+4 \leq i \leq 2w+1)$	$(4w-2i+9, 4w-2i+13, 4w-2i+4)$

For the vertices $\{r_i, s_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(r_1|R) = (6, 9, 2)$; $\zeta(r_2|R) = (5, 7, 2)$; $\zeta(r_3|R) = (7, 6, 4)$; $\zeta(s_1|R) = (5, 8, 1)$; $\zeta(s_2|R) = (6, 6, 3)$; $\zeta(s_3|R) = (8, 5, 5)$ and rest of vertices, we have

$\zeta(r_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(r_i\|R_m)$:$(4 \leq i \leq w+2)$	$(2i+1, 2i-3, 2i-2)$
$\zeta(r_i\|R_m)$:$(i = w+3)$	$(4w-2i+9, 2w+3, 4w-2i+6)$
$\zeta(r_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+9, 4w-2i+13, 4w-2i+6)$

and

$\zeta(s_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(s_i\|R_m)$:$(4 \leq i \leq w+1)$	$(2i+2, 2i-2, 2i-1)$
$\zeta(s_i\|R_m)$:$(i = w+2)$	$(4w-2i+8, 2w+2, 4w-2i+5)$
$\zeta(s_i\|R_m)$:$(i = w+3)$	$(4w-2i+8, 2w+4, 4w-2i+5)$
$\zeta(s_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+8, 4w-2i+12, 4w-2i+5)$

For the vertices $\{t_i, u_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(t_1|R) = (5, 8, 3)$; $\zeta(t_2|R) = (4, 6, 3)$; $\zeta(t_3|R) = (6, 5, 5)$; $\zeta(u_1|R) = (4, 7, 2)$; $\zeta(u_2|R) = (5, 5, 4)$; $\zeta(u_3|R) = (7, 4, 6)$ and for rest of vertices, we have

$\zeta(t_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(t_i\|R_m)$:$(4 \leq i \leq w+2)$	$(2i, 2i-4, 2i-1)$
$\zeta(t_i\|R_m)$:$(i = w+3)$	$(4w-2i+8, 2w+2, 4w-2i+7)$
$\zeta(t_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+8, 4w-2i+12, 4w-2i+7)$

and

$\zeta(u_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(u_i\|R_m)$:$(4 \leq i \leq w+1)$	$(2i+1, 2i-3, 2i)$
$\zeta(u_i\|R_m)$:$(i = w+2)$	$(4w-2i+7, 2w+1, 4w-2i+6)$
$\zeta(u_i\|R_m)$:$(i = w+3)$	$(4w-2i+7, 2w+3, 4w-2i+6)$
$\zeta(u_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+7, 4w-2i+11, 4w-2i+6)$

For the vertices of vw-cycle $\{v_i, w_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(v_1|R_m) = (4,7,4)$; $\zeta(v_2|R_m) = (3,5,4)$; $\zeta(v_3|R_m) = (5,4,6)$; $\zeta(w_1|R_m) = (3,6,3)$; $\zeta(w_2|R_m) = (4,4,5)$; $\zeta(w_1|R_m) = (6,3,7)$ and for rest of vertices, we have

$\zeta(v_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(v_i\|R_m)$:$(4 \leq i \leq w+2)$	$(2i-1, 2i-5, 2i)$
$\zeta(v_i\|R_m)$:$(i = w+3)$	$(4w-2i+7, 2w+1, 4w-2i+8)$
$\zeta(v_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+7, 4w-2i+11, 4w-2i+8)$

and

$\zeta(w_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(w_i\|R_m)$:$(4 \leq i \leq w+1)$	$(2i, 2i-4, 2i+1)$
$\zeta(w_i\|R_m)$:$(i = w+2)$	$(4w-2i+6, 2w, 4w-2i+7)$
$\zeta(w_i\|R_m)$:$(i = w+3)$	$(4w-2i+6, 2w+2, 4w-2i+7)$
$\zeta(w_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+6, 4w-2i+10, 4w-2i+7)$

For the vertices $\{x_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(x_1|R) = (2,5,4)$; $\zeta(x_2|R) = (4,3,6)$ and for rest of vertices, we have

$\zeta(x_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(x_i\|R_m)$:$(3 \leq i \leq w+1)$	$(2i, 2i-4, 2i+2)$
$\zeta(x_i\|R_m)$:$(i = w+2)$	$(4w-2i+5, 2w, 4w-2i+7)$
$\zeta(x_i\|R_m)$:$(i = w+3)$	$(4w-2i+5, 2w+2, 4w-2i+7)$
$\zeta(x_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+5, 4w-2i+9, 4w-2i+7)$

For the vertices $\{y_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(y_i|R_m) = \zeta(x_i|R_m) + (-1,-1,1)$ for $1 \leq i \leq n$. Finally, for the vertices of za-cycle $\{z_i, a_i : 1 \leq i \leq n\}$, the metric codes are as follows: $\zeta(z_1|R) = (0,4,6)$; $\zeta(z_2|R) = (2,2,7)$; $\zeta(z_3|R) = (4,0,9)$; $\zeta(a_1|R) = (1,3,6)$; $\zeta(a_2|R) = (3,1,8)$ and for rest of vertices, we have

$\zeta(z_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(z_i\|R_m)$:$(3 \leq i \leq w+1)$	$(2i-2, 2i-6, 2i+3)$
$\zeta(z_i\|R_m)$:$(i = w+2)$	$(4w-2i+4, 2w-2, 4w-2i+9)$
$\zeta(z_i\|R_m)$:$(i = w+3)$	$(4w-2i+4, 2w, 4w-2i+9)$
$\zeta(z_i\|R_m)$:$(w+4 \leq i \leq 2w+1)$	$(4w-2i+4, 4w-2i+8, 4w-2i+9)$

and

$\zeta(a_i\|R_m)$	$R_m = \{z_1, z_3, q_1\}$
$\zeta(a_i\|R_m)$:$(3 \leq i \leq w+1)$	$(2i-1, 2i-5, 2i+4)$
$\zeta(a_i\|R_m)$:$(i = w+2)$	$(4w-2i+3, 2w-1, 4w-2i+8)$
$\zeta(a_i\|R_m)$:$(w+3 \leq i \leq 2w)$	$(4w-2i+3, 4w-2i+7, 4w-2i+8)$
$\zeta(a_i\|R_m)$:$(i = 2w+1)$	$(4w-2i+3, 4w-2i+7, 7)$

Again, from these metric codes, we find that $|P| = |Q| = |R| = ... = |Z| = |A| = n$ and $P \cap Q \cap R \cap ... \cap Z \cap A = \emptyset$, implying R_m to be an ordered vertex resolving set for the graph $Ł_n$ and so $dim(Ł_n) \leq 3$. Now, to complete the proof for this case, we have to show that $dim(Ł_n) \geq 3$. For this, we follow the same pattern as we used in Case(I) and the contradictions can be obtained accordingly. Thus, $dim(Ł_n) = 3$ for this case as well, which conclude the theorem. □

In terms of independent resolving set, we have the following result

Theorem 3.2. *The independent resolving number of $Ł_n$ is three, for every $n \geq 6$.*

Proof. To show that for the plane graph $Ł_n$, there exists a minimum independent resolving set R_m^i of cardinality three, we follow the same technique as used in Theorem 3.1.

Suppose $R_m^i = \{z_1, z_3, q_1\} \subset V(Ł_n)$. Now, by applying the same techniques and following the same pattern as used in Theorem 3.1, it is simple to show that the set of vertices $R_m^i = \{z_1, z_3, q_1\}$ is the independent resolving set for $Ł_n$, which concludes the theorem.

□

4. Conclusions

In this study, we have studied the metric dimension of a rotationally symmetric plane graph $Ł_n$. For this, we proved that $dim(Ł_n) = 3$. We also observed that the minimum resolving set R_m is independent for the graph $Ł_n$. In future, we will try to obtain the edge metric dimension and the mixed metric dimension [20] for the graph $Ł_n$.

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